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D_k Gravitational Instantons and Nahm Equations

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Abstract

We construct D_k Asymptotically Locally Flat gravitational instantons as moduli spaces of solutions of Nahm equations. This allows us to find their twistor spaces and Kähler potentials.

1 Introduction

Gravitational instantons are four-dimensional manifolds with a self-dual curvature tensor. They can be compact (K3 and T^4) or noncompact. Asymptotically Locally Euclidean (ALE) instantons are gravitational instantons that approach $\mathbf{R}^4/\Gamma$ at infinity, where Γ is a finite subgroup of $SU(2)$. These were constructed by Gibbons and Hawking [1] for $\Gamma = \mathbf{Z}_k$ and by Kronheimer [2] for all possible Γ.

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The purpose of this paper is to present a construction of a class of Asymptotically Locally Flat (ALF) spaces. At infinity the metric on these spaces approaches

$$ds^2 = dr^2 + \sigma_1^2 + r^2(\sigma_2^2 + \sigma_3^2), \tag{1}$$

where σ_j are left-invariant one-forms on $\mathbf{S}^3/\Gamma$ for some finite subgroup Γ of $SU(2)$. Thus these spaces are classified by a choice of Γ which can be one of the following groups: $\mathbf{Z}_k$ (cyclic), $\mathbf{D}_k$ (binary dihedral), $\mathbf{T}$ (binary tetrahedral), $\mathbf{O}$ (binary octahedral), $\mathbf{I}$ (binary icosahedral). The first two are infinite classes parametrized by an integer k. It is known [2] that in the ALE case all of these possibilities are realized. The corresponding spaces are called A_{k-1}, D_k, E_6, E_7, and E_8 ALE spaces, respectively. The names come from the McKay correspondence between finite subgroups of $SU(2)$ and simple Lie algebras. These names also reflect the topology of the respective spaces, as the intersection matrix of compact two cycles of such a space is given by the negative of the Cartan matrix of the corresponding Lie algebra.

An example of the A_k ALF space is described by the multi-Taub-NUT metric with $k+1$ centers. In this paper we present the construction of D_k ALF spaces. Namely, we construct their twistor spaces, find all real holomorphic sections of the latter, and compute the Kähler potential.

Our motivation comes from string theory. Let us consider M theory compactified on the D_k ALF space. As explained by Sen [3], in IIA string theory this corresponds to an orientifold 6-plane with k parallel D6-branes. We can probe this background with a D2-brane parallel to the 6-branes. The theory on the D2-brane is described at low energies by an N=4 $SU(2)$ Yang-Mills with k fundamental hypermultiplets. Coordinates on the Coulomb branch are the vevs of the Higgs fields and the dual photon. From the M-theory point of view, the D2-brane is a membrane of M-theory, Higgs fields correspond to the position of the membrane in the three-space transverse to the six-branes, and the dual photon corresponds to the position of the membrane on the circle of M-theory. This allows one to identify a point on the ALF space at which the membrane is positioned with a vacuum of the gauge theory on the D2-brane. So the Coulomb branch of the $D=3, N=4$ $SU(2)$ gauge theory with k fundamental hypermultiplets should be a D_k ALF space. One can also see this from the gauge theory analysis [4].

Remarkably, there is another realization of these gauge theories in type IIB string theory [5], as shown in Figure 1.

In the infrared limit the theory on the internal D3-branes reduces to a $2+1$ dimensional theory. Namely, it is a $D=3, N=4$ $U(2)$ Yang-Mills theory with k fundamentals. As discussed in Ref. [6], every vacuum of the

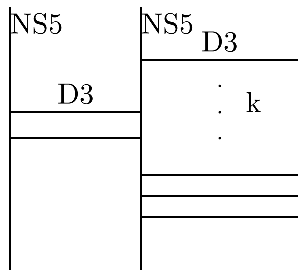

Figure 1: NS5-branes are parallel to the $x^0, x^1, x^2, x^3, x^4, x^5$, and D3-branes are parallel to the x^0, x^1, x^2, x^6 directions. x^6 is the horizontal direction on the figure.

gauge theory on the internal D3-branes corresponds to a singular $SU(2)$ two-monopole on the NS5-branes with k singularities. These singularities look like pointlike Dirac monopoles embedded in $SU(2)$. By virtue of Nahm transform, monopoles correspond to solutions of Nahm equations (the reduction of self-duality equations to one dimension).

Another way to see how Nahm equations appear is to look for supersymmetric vacua of the $D = 4, N = 4$ theory on the D3-branes [7]. The Higgs fields in a vacuum configuration depend on the x^6 coordinate (which we shall call s) and take values in the adjoint of $U(2)$ on the interval between the two NS5-branes and in the adjoint of $U(k)$ on the semiinfinite interval. The equations that Higgs fields satisfy are nothing but Nahm equations [7].

Thus from string theory considerations it follows that the moduli space $\mathcal{M}$ of solutions of Nahm equations corresponding to Figure 1 is a D_k ALF gravitational instanton.

The advantage of having a description in terms of Nahm equations is that they can be interpreted as a moment map of a hyperkähler quotient (see e.g., Ref. [8]). This guarantees that their moduli space $\mathcal{M}$ is hyperkähler. Furthermore, all three complex structures of $\mathcal{M}$ become apparent.

The paper is organized as follows: In Section 2 we formulate the relevant Nahm data. In Section 3 we exhibit the moduli space $\mathcal{M}$ of Nahm data as a complex manifold with respect to one of its complex structures. In order to extract all the information about the hyperkähler manifold $\mathcal{M}$ (e.g., its metric) we need to find a whole 2-sphere of complex structures. In Section 4 we explore the dependence of the construction on the choice of the complex structure and find the twistor space Z of $\mathcal{M}$. In order to find the metric on $\mathcal{M}$ one needs to find a decomposition $Z = \mathbf{P}^1 \times \mathcal{M}$. This is done in Section 5. In Section 6 we obtain the Kähler potential for the D_k ALF metric. The final result agrees with a conjecture of Chalmers [9].

2 Nahm equations

As explained in Refs. [7, 6] the configuration of D3-branes in Figure 1 can be described by certain matrix data called Nahm data.

The Nahm data will be a set of four functions T_0, T_j, $j = 1, 2, 3$ of real coordinate $s \in [-h, \infty)$ taking values in $u(2)$ for $s < 0$ and in $u(k)$ for $s > 0$. The matching condition at $s = 0$ is

$$T_j(s>0) = \begin{pmatrix} -i\rho_j/2s + O(1) & s^{(k-3)/2}p_j + O(s^{(k-1)/2}) \\ -s^{(k-3)/2}\bar{p}_j^T + O(s^{(k-1)/2}) & T_j(0-) + O(s) \end{pmatrix}, \tag{2}$$

where ρ_j are $k-2$ dimensional representations of Pauli sigma matrices σ_j, p_j are $(k-2)\times 2$ matrices. Near $s = -h$ we require $T_j(s) = -\frac{i}{2}\frac{\sigma_j}{s+h} + O(s+h)$, and at $s \to \infty$ $T_j \to diag(x_1^{(j)}, x_2^{(j)}, \ldots, x_k^{(j)})$. The parameter h is the distance between the NS5-branes in Figure 1. We will see that h is inversely proportional to the radius of the circle at the asymptotic infinity of the ALF space.

Nahm data are acted upon by a gauge group G. G is defined as a group whose elements are functions $g(s)$ valued in $U(2)$ for $s \in [-h, 0]$ and in $U(k)$ for $s > 0$, and satisfying $g(-h) = 1$, $g(+\infty) = 1$, and

$$g(0+) = \begin{pmatrix} 1 & 0 \\ 0 & g(0-) \end{pmatrix}. \tag{3}$$

We subject the Nahm data to Nahm equations

$$\frac{dT_i}{ds} + [T_0, T_i] = \frac{1}{2}\epsilon_{ijk}[T_j, T_k]. \tag{4}$$

The moduli space of such Nahm data modulo gauge transformations is the hyperkähler manifold $\mathcal{M}$ we are interested in.

If we consider all quadruplets T_0, T_1, T_2, T_3 valued in $u(2)$ on $[-h, 0)$ and in $u(k)$ on $[0, \infty)$, we can define a norm

$$||T||^2 = Tr\int_{-h}^{\infty}\left(T_0^2 + T_1^2 + T_2^2 + T_3^2\right)ds. \tag{5}$$

The metric corresponding to this norm is hyperkähler. This becomes obvious if we think of a quadruplet $T_0, \ldots, T_3$ as a matrix of quaternions

$$T = T_0 + iT_1 + jT_2 + kT_3.$$

Then the Nahm equations Eq. (4) are moment maps for the action of G

$$T_0 \to g^{-1}T_0 g + g^{-1}dg/ds,\ T_j = g^{-1}T_j g. \tag{6}$$

We construct the moduli space $\mathcal{M}$ in the following way: Let us first consider the restriction of the Nahm data to the interval $[0, \infty)$ modulo the gauge transformations with $g(0) = 1$ and call the resulting manifold $\mathcal{M}_+$. Then consider the restriction of the Nahm data to $[-h, 0]$ modulo the gauge transformations with $g(0) = g(-h) = 1$ and call this manifold $\mathcal{D}^{12}$. We may relax the condition $g(-h) = 1$ and consider the gauge transformations with $g(-h) \sim 1$. This gives a triholomorphic action of $U(1)$ on $\mathcal{D}^{12}$. We may think of this $U(1)$ as the group of gauge transformations "localized" at $s = -h$. The group of all gauge transformations modulo those with $g(0) = 1$ is $U(2)$. This $U(2)$ can be thought of as "localized" at $s = 0$. It acts triholomorphically on both $\mathcal{M}_+$ and $\mathcal{D}^{12}$. The manifold $\mathcal{M}$ is a hyperkähler quotient of $\mathcal{M}_+ \times \mathcal{D}^{12}$ by $U(1) \times U(2)$ at zero level.

3 The complex structure of $\mathcal{M}$

Following Donaldson [10] we define

$$\alpha = \frac{1}{2}\left(T_0 - iT_1\right),\ \beta = \frac{1}{2}\left(T_2 + iT_3\right). \tag{7}$$

Then the Nahm equations can be written as a pair of equations. We will need only one of them called the "complex" equation:

$$\frac{d\beta}{ds} + 2[\alpha, \beta] = 0. \tag{8}$$

Denote by $\mathcal{M}$ the space of Nahm data satisfying Nahm equations modulo the group G of gauge transformations. Let $\tilde{\mathcal{M}}$ be the space of solutions of the complex equation Eq. (8) modulo the complexified group of gauge transformations $G^{\mathbf{C}}$. One can show [10] that $\mathcal{M}$ as a comlpex variety is the same as $\tilde{\mathcal{M}}$. As we will see it is rather easy to solve Eq. (8). This allows us to describe $\mathcal{M}$ as a complex manifold.

First let us look at $\mathcal{M}_+$. Let $a = s\alpha, b = s\beta, s = e^t$; then the complex equation Eq. (8) is equivalent to

$$\frac{db}{dt} + 2[a, b] = b. \tag{9}$$

Fixing the gauge at $t \to -\infty$ so that a is independent of t we can solve Eq.(9) as follows:

$$a = \begin{pmatrix} -\rho_1/4 & 0 \\ 0 & 0 \end{pmatrix}, \qquad b = e^t \begin{pmatrix} e^{\rho_1 t/2} & 0 \\ 0 & 1 \end{pmatrix} \hat{b} \begin{pmatrix} e^{-\rho_1 t/2} & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

Here the matrix $\hat{b}$ is independent of t. Comparing with the boundary condition at $t \to -\infty$ following from Eq. (2), we find

$$\alpha = \begin{pmatrix} -\frac{1}{4s}\rho_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} -\frac{i}{4s}(\rho_2 + i\rho_3) + A & (-2is)^{(k-3)/2}\tilde{p} \\ (-2is)^{(k-3)/2}\tilde{q} & B \end{pmatrix}, \tag{11}$$

where the constant matrices $A, \tilde{p}, \tilde{q}$ satisfy $[\rho_1, A] = 0$, $\rho_1 \tilde{p} = (k-3)\tilde{p}$, $\tilde{q}\rho_1 = -(k-3)\tilde{q}$. It follows that A is a diagonal $(k-2) \times (k-2)$ matrix (in the basis in which ρ_1 is diagonal). We call its eigenvalues $A_1, \ldots, A_{k-2}$. Now recall that $\tilde{p}$ and $\tilde{q}$ are $(k-2) \times 2$ and $2 \times (k-2)$ matrices. Letting a vector v be the highest weight of the representation ρ (that is $\rho_1 v = (k-3)v$) with $|v| = 1$, we see that the two columns of $\tilde{p}$ are $p_1 v$ and $p_2 v$ where $p_1, p_2 \in \mathbf{C}$. So $\tilde{p}$ is parametrized by a vector $p \in \mathbf{C}^2$ with components (p_1, p_2). Similarly, $\tilde{q}$ can be parametrized by a vector $q \in \mathbf{C}^2$.

Boundary conditions at $s \to +\infty$ restrict the eigenvalues of β to be $z_a = (x_a^{(2)} + ix_a^{(3)})/2$. Thus the equation $\det(\beta - z) = 0$ has roots z_a. On the other hand, a direct computation of the determinant yields

$$\det(\beta - z) = \det(B - z)\left(\det(A - z) - p^T(B - z)^{-1}q\right). \tag{12}$$

Now let us turn our attention to $\mathcal{D}^{12}$. The solutions of Nahm equations on $s \in [-h, 0]$ modulo gauge transformations with precisely the right boundary conditions were described by Dancer [11, 12]. As a complex manifold, $\mathcal{D}^{12}$ is a set of pairs (B, w) with B being a 2×2 matrix and $w \in \mathbf{C}^2$, such that w and Bw are linearly independent.

So far we described the solutions of the complex equation Eq. (8) modulo complexified gauge transformations $g(s)$ satisfying $g(0) = g(-h) = 1$. Every such solution is given by a set (A, B, p, q, w) where A is a diagonal matrix with eigenvalues $A_1, \ldots, A_{k-2}$, B is a 2×2 matrix, and $p, q, w \in \mathbf{C}^2$. Now we have to take a symplectic quotient by the complexified groups $U(1)^{\mathbf{C}} = \mathbf{C}^*$ and $U(2)^{\mathbf{C}} \simeq \mathbf{C}^* \times SL(2, \mathbf{C})$. The moment map for the action of $U(1)^{\mathbf{C}}$ is $Tr\ B$, so we require $Tr\ B = 0$.

An element g of $SL(2, \mathbf{C})$ acts on these data as

$$B \to gBg^{-1}, \quad w \to gw, \quad p^T \to p^T g^{-1}, \quad q \to gq, \tag{13}$$

and the two $\mathbf{C}^*$ actions (we will call them $\mathbf{C}^*_\lambda$ and $\mathbf{C}^*_\kappa$) are given by

$$B \to B, \quad w \to w, \quad p^T \to p^T\lambda^{-1}, \quad q \to \lambda q \quad , \tag{14}$$

$$B \to B, \quad w \to \kappa w, \quad p^T \to p^T, \quad q \to q. \tag{15}$$

The $\mathbf{C}^*_\kappa$ action and the fact that w and Bw are linearly independent allows us to put

$$w^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Bw = 1.$$

This gauge-fixes $\mathbf{C}^*_\kappa$. Now we can define $SL(2,\mathbf{C})$ invariants x_1, x_2, y_1, and y_2 by

$$p^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = x_1 w^T B^T + x_2 w^T, \tag{16}$$

$$q = -y_1 Bw + y_2 w. \tag{17}$$

$x_1, x_2, y_1, y_2, \det B$ and $A_1, \ldots, A_{k-2}$ form the full set of invariants with respect to $SL(2,\mathbf{C})$.

The residual $\mathbf{C}^*_\lambda$ acts by $B \to B, x_i \to \lambda^{-1}x_i, y_i \to \lambda y_i$. Thus the invariants of $\mathbf{C}^*_\lambda \times SL(2,\mathbf{C})$ are given by

$$\begin{aligned} \eta_2 &= -\det B \\ \Psi_1 &= x_2 y_2 \\ \Psi_2 &= x_1 y_1 \\ \Psi_3 &= x_1 y_2 \\ \Psi_4 &= x_2 y_1, \end{aligned} \tag{18}$$

and A_b, $b = 1, \ldots, k-2$. These invariants satisfy additional relations

$$\Psi_1\Psi_2 = \Psi_3\Psi_4 \tag{19}$$

$$\left(z_a^2 - \eta_2\right) \prod_{b=1}^{k-2} (A_b - z_a) + z\left(\Psi_3 - \Psi_4\right) + \Psi_1 - \Psi_2\eta_2 = 0 \tag{20}$$

for $a = 1, \ldots, k$. The first of these follows from the definition of the invariants, and the second one is a consequence of Eq. (12).

Let us denote symmetric polynomials of order m by S_m. For example $S_1(z) = z_1 + z_2 + \ldots + z_k$, and $S_{k-2}(A) = A_1 A_2 \ldots A_{k-2}$. One can rewrite Eqs. (19,20) as

$$\begin{aligned} S_1(z) &= S_1(A) \\ S_2(z) &= S_2(A) - \eta_2 \end{aligned}$$

$$\begin{aligned}
S_3(z) &= S_3(A) - \eta_2 S_1(A) \\
&\vdots \\
S_m(z) &= S_m(A) - \eta_2 S_{m-2}(A) \\
&\vdots \\
S_{k-2}(z) &= S_{k-2}(A) - \eta_2 S_{k-4}(A) \\
\Psi_1 &= \eta_2 \Psi_2 + F(\eta_2) \\
\Psi_4 &= \Psi_3 + G(\eta_2)
\end{aligned}$$

$$\eta_2 \Psi_2^2 + \Psi_2 F(\eta_2) = \Psi_3^2 + \Psi_3 G(\eta_2), \tag{21}$$

where the polynomials $F(\eta_2)$ and $G(\eta_2)$ are defined as

$$\begin{aligned}
F(\eta_2) &= S_k(z) + S_{k-2}(z)\eta_2 + \ldots + S_{k-2l}(z)\eta_2^l + \ldots \\
G(\eta_2) &= S_{k-1}(z) + S_{k-3}(z)\eta_2 + \ldots + S_{k-1-2l}(z)\eta_2^l + \ldots .
\end{aligned} \tag{22}$$

The above equations define our complex manifold $\mathcal{M}$ as a subvariety in $\mathbf{C}^{k+3}$. From Eqs. (21,22) one can see that $\mathcal{M}$ develops a D_k-type singularity when all z_a are set to zero.

4 The Twistor Space of $\mathcal{M}$

In the previous section we described $\mathcal{M}$ as a complex variety by picking a particular complex structure on the space of Nahm data. In reality there is a whole 2-sphere of such complex structures. To obtain the twistor space of $\mathcal{M}$ we need to trace the dependence on the choice of the complex structure.

Let ζ be an affine parameter on the $\mathbf{CP}^1$ of complex structures. We define

$$\begin{aligned}
\alpha^0 &= \frac{1}{2}\left(T_0 + iT_1 + \zeta\left(T_3 - iT_2\right)\right), \\
\beta^0 &= \frac{i}{2}\Big(T_3 + iT_2 + 2i\zeta T_1 + \zeta^2\left(T_3 - iT_2\right)\Big)
\end{aligned} \tag{23}$$

for $\zeta \neq \infty$ and

$$\begin{aligned}
\alpha^1 &= \frac{1}{2}\left(T_0 - iT_1 - \frac{1}{\zeta}\left(T_3 + iT_2\right)\right), \\
\beta^1 &= \frac{1}{2}\left(T_2 + iT_3 - 2\frac{1}{\zeta}T_1 + \frac{1}{\zeta^2}\left(-T_2 + iT_3\right)\right)
\end{aligned} \tag{24}$$

for $\zeta \neq 0$. Both pairs (α, β) satisfy the complex Nahm equation

$$\frac{d\beta}{ds} + 2[\alpha, \beta] = 0. \tag{25}$$

The relation between them is given by

$$\alpha^1 = \alpha^0 + \frac{i}{\zeta}\beta^0, \quad \beta^1 = \frac{1}{\zeta^2}\beta^0. \tag{26}$$

Following the same steps as in the Section 3 and fixing the gauge so that $s\alpha^0, s\alpha^1$ are constant, we obtain

$$\alpha^0 = \begin{pmatrix} \frac{1}{4s}\left(\rho_1 - 2\zeta\rho_+\right) & 0 \\ 0 & 0 \end{pmatrix}, \tag{27}$$

$$\beta^0 = \begin{pmatrix} \frac{i}{2s}\left(\rho_- + \zeta\rho_1 - \zeta^2\rho_+\right) + e^{-\zeta\rho_+}A^0(\zeta)e^{\zeta\rho_+} & s^{(k-3)/2}e^{-\zeta\rho_+}p^0(\zeta) \\ s^{(k-3)/2}q^0(\zeta) & B^0(\zeta) \end{pmatrix}$$

and

$$\alpha^1 = \begin{pmatrix} -\frac{1}{4s}\left(\rho_1 + \frac{2}{\zeta}\rho_-\right) & 0 \\ 0 & 0 \end{pmatrix}, \tag{28}$$

$$\beta^1 = \begin{pmatrix} -\frac{i}{2s}\left(\rho_+ - \frac{1}{\zeta}\rho_1 - \frac{1}{\zeta^2}\rho_-\right) + e^{-\rho_-/\zeta}A^1(\zeta)e^{\rho_-/\zeta} & s^{(k-3)/2}p^1(\zeta) \\ s^{\frac{k-3}{2}}q^1(\zeta)e^{\frac{\rho_-}{\zeta}} & B^1(\zeta) \end{pmatrix}.$$

In these formulas $\rho_+ = \frac{1}{2}(\rho_2 + i\rho_3), \rho_- = \frac{1}{2}(\rho_2 - i\rho_3)$,

$$[A^0(\zeta), \rho_1] = 0, \ [A^1(\zeta), \rho_1] = 0 \tag{29}$$

$$\rho_1 p_0(\zeta) = -(k-3)p_0(\zeta), \ q_0(\zeta)\rho_1 = (k-3)q_0(\zeta) \tag{30}$$

$$\rho_1 p_1(\zeta) = (k-3)p_0(\zeta), \ q_1(\zeta)\rho_1 = -(k-3)q_1(\zeta). \tag{31}$$

Equation (26) relates (α^0, β^0) and (α^1, β^1), but in a gauge different from that in Eqs. (27,28). Namely the gauge transformation that makes $s\alpha^1 = s\left(\alpha^0 + \frac{i}{\zeta}\beta^0\right)$ independent of s is

$$g = \exp\left(-\frac{2i}{s\zeta}\beta^0\right)\exp\left(-\frac{1}{\zeta}\rho_- - \rho_1 + \zeta\rho_+\right). \tag{32}$$

Comparing the two expressions for β^1 we can find the transition functions between (A^0, p^0, q^0, B^0) and (A^1, p^1, q^1, B^1). If we let

$$R = e^{\rho_-/\zeta}e^{-\zeta\rho_+}e^{\rho_-/\zeta} \tag{33}$$

then we get

$$A^1 = RA^0R^{-1}, \quad p^1 = \frac{1}{\zeta^2}Rp^0, \quad q_1 = \frac{1}{\zeta^2}q^0R^{-1}, \quad B^1 = \frac{1}{\zeta^2}B^0. \tag{34}$$

This means that the variables A, B, p, q are sections of the following bundles:

$$\begin{array}{rll} B & \text{of} & Mat(2) \otimes \mathcal{O}(2), \\ A_b & \text{of} & \mathcal{O}(2), \\ p & \text{of} & \mathcal{O}(k-1) \times \mathcal{O}(k-1), \\ q & \text{of} & \mathcal{O}(k-1) \times \mathcal{O}(k-1). \end{array} \tag{35}$$

Here $\mathcal{O}(n)$ is defined as the line bundle on $\mathbf{P}^1$ with the transition function ζ^{-n}. Furthermore, from Dancer's analysis [12] it follows that $w^1 = \zeta e^{2ihB^0/\zeta} w^0$. From the above one can find the dependence of the invariants $\Psi_1, \ldots, \Psi_4$ on ζ. This completely determines the twistor space $\mathcal{Z}$ of $\mathcal{M}$.

It turns out that the invariants Ψ_i are not taking values in any nice fibrations, but one can define certain combinations of them that do. Let $\eta = \sqrt{-\eta_2}$, and define

$$\begin{aligned} \mu &= \Psi_1 + \eta^2\Psi_2 - i\eta(\Psi_3 - \Psi_4), \\ \nu &= \Psi_1 + \eta^2\Psi_2 + i\eta(\Psi_3 - \Psi_4), \\ \rho &= \Psi_1 - \eta^2\Psi_2 + i\eta(\Psi_3 + \Psi_4), \\ \xi &= \Psi_1 - \eta^2\Psi_2 - i\eta(\Psi_3 + \Psi_4). \end{aligned} \tag{36}$$

These are two-valued functions of ζ, because $\eta(\zeta) = \sqrt{-\eta_2(\zeta)}$, but they have simple transformation properties:

$$\tilde{\mu} = \zeta^{-2k}\mu, \ \tilde{\nu} = \zeta^{-2k}\nu, \tag{37}$$

$$\tilde{\rho} = \zeta^{-2k} e^{2h\eta/\zeta} \rho, \tag{38}$$

$$\tilde{\xi} = \zeta^{-2k} e^{-2h\eta/\zeta} \xi. \tag{39}$$

Furthemore, from Eqs. (21) we get

$$\mu = \prod_{a=1}^{k} \left(z_a(\zeta) + i\eta\right), \tag{40}$$

$$\nu = \prod_{a=1}^{k} \left(z_a(\zeta) - i\eta\right), \tag{41}$$

and

$$\rho\, \xi = \prod_{a=1}^{k} \left(z_a^2(\zeta) + \eta^2\right). \tag{42}$$

From this description of $\mathcal{Z}$ one can see that $\mathcal{Z}$ is exactly the twistor space of the centered moduli space of two monopoles with k singularities computed in Ref. [13]. In the direct image sheaf construction of Ref. [13]

$\rho_0\xi_0, \rho_1\xi_1, \rho_0\xi_1$, and $\rho_1\xi_0$ correspond to Ψ_1, Ψ_2, Ψ_3, and Ψ_4. This establishes the isometry between the moduli space $\mathcal{M}$ of Nahm data and the centered moduli space of singular monopoles of nonabelian charge two. As explained in the Introduction, the equivalence of the two moduli spaces follows from string theory.

The twistor space $\mathcal{Z}$ is nothing but $\mathbf{P}^1 \times \mathcal{M}$, so the fiber of $\mathcal{Z}$ over ζ is our manifold $\mathcal{M}$ with the complex structure determined by ζ. In order to find the Riemannian metric of $\mathcal{M}$ we need to pick two local complex coordinates on $\mathcal{M}$ that depend on ζ holomorphically. For example, locally we can pick (η, ρ) or (η, ξ) as such coordinates. There is a natural two-form $\omega = 2d\eta \wedge d\xi/\xi = -2d\eta \wedge d\rho/\rho$ on $\mathcal{Z}$. This two-form is necessary to recover the metric on $\mathcal{M}$ from the twistor space $\mathcal{Z}$ [14]. It is degenerate along the ζ direction and satisfies $\tilde{\omega} = \zeta^{-2}\omega$. Let us rewrite it as

$$\omega = d\eta \wedge d \log \frac{\xi}{\rho}. \tag{43}$$

We can introduce a new coordinate

$$\chi = \frac{1}{\eta} \log \frac{\rho}{\xi} = \frac{1}{\eta} \log \frac{\Psi_1 - \eta^2\Psi_2 - i\eta(\Psi_3 + \Psi_4)}{\Psi_1 - \eta^2\Psi_2 + i\eta(\Psi_3 + \Psi_4)}, \tag{44}$$

which does not depend on the choice of the branch of the square root in $\eta = \sqrt{-\eta_2}$. In terms of η_2 and χ, $\omega = d\eta_2 \wedge d\chi$. To compute the metric from the twistor data we will employ the generalized Legendre transform of Ref. [15, 16]. This technique yields directly the Kähler potential of the metric. But first we have to find χ as a function on ζ. This is the subject of the next section.

5 Real sections of $\mathcal{Z}$

Let us recall that $\eta_2 = -\det B$ and $\tilde{B} = \zeta^{-2}B$, so η_2 is a quartic polynomial in ζ. Also η_2 should satisfy $\eta_2(-1/\bar{\zeta}) = \bar{\zeta}^{-4}\bar{\eta}_2(\zeta)$. Thus

$$\eta_2 = z + v\zeta + w\zeta^2 - \bar{v}\zeta^3 + \bar{z}\zeta^4, \tag{45}$$

where $z, v \in \mathbf{C}, w \in \mathbf{R}$. Consider a curve S given by $\eta^2 + \eta_2(\zeta) = 0$. It covers the $\mathbf{P}^1$ parametrized by ζ twice, and has genus one. We will think of it as a torus $\mathbf{C}/\mathbf{Z}^2$ with a holomorphic coordinate u.

To establish the connection between η, ζ and the coordinate u let us make a change of variables

$$\zeta = \frac{a\hat{\zeta} + b}{-\bar{b}\hat{\zeta} + \bar{a}}, \qquad \eta = \frac{\hat{\eta}}{(-\bar{b}\hat{\zeta} + \bar{a})^2}, \qquad |a|^2 + |b|^2 = 1, \tag{46}$$

so that the equation for S takes the canonical form

$$\hat{\eta}^2 = 4k_1^2\left(\hat{\zeta}^3 - 3k_2\hat{\zeta}^2 - \hat{\zeta}\right), k_1 > 0, k_2 \in \mathbf{R}. \tag{47}$$

From this form we see that one period of the torus, ω_r, is real, and the other one, ω_i, is purely imaginary. So $S = \mathbf{C}/\Omega$ where Ω is a rectangular lattice spanned by ω_r, ω_i. Every such curve S is parametrized by an $SU(2)$ matrix

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

and two real numbers k_1 and k_2. The relation of $\hat{\eta}$ and $\hat{\zeta}$ with u is given by $\hat{\eta} = k_1\mathcal{P}'(u), \hat{\zeta} = \mathcal{P}(u) + k_2$, where $\mathcal{P}(u)$ is the Weierstrass elliptic function.

Now we regard ρ and ξ as functions of u. The equation

$$\rho\,\xi = \prod_{\alpha=1}^{k}\left(z_a^2(\zeta) + \eta^2\right)$$

and the fact that ρ and ξ are interchanged by the change of the sign of η implies that ρ vanishes at the points u_a, u_a' in u plane at which $\eta = z_a$, and ξ vanishes at the points v_a, v_a' at which $\eta = -z_a$.

Let us denote the two preimages of $\zeta = 0$ in the u-plane by u_0 and u_0', and the two preimages of $\zeta = \infty$ by u_∞ and u_∞'. Then ρ/ξ is meromorphic for $u \neq u_\infty, u_\infty'$ with zeros at u_a, u_a' and poles at v_a, v_a', while

$$\frac{\tilde{\rho}}{\tilde{\xi}} = e^{4h\eta/\zeta}\frac{\rho}{\xi}$$

is meromorphic for $u \neq u_0, u_0'$.

Let us cover S with two charts $u \neq u_\infty, u_{\infty'}$ and $u \neq u_0, u_0'$. In the first chart we find

$$\frac{\rho}{\xi} = \exp\left(-4hk_1\left(\zeta_w(u+u_\infty) + \zeta_w(u-u_\infty)\right) - 2cu\right)\prod_a \frac{\sigma(u-u_a)\sigma(u-u_a')}{\sigma(u-v_a)\sigma(u-v_a')}. \tag{48}$$

Here $\zeta_W(u)$ and $\sigma(u)$ are Weierstrass quasielliptic functions (see e.g. Ref. [17] for definitions), and c is a constant to be determined. The monodromy properties of the Weierstrass functions ζ_W and σ are

$$\sigma(u+\omega_r) = -\sigma(u)e^{2\eta_r(u+\omega_r/2)},\ \sigma(u+\omega_i) = -\sigma(u)e^{2\eta_i(u+\omega_i/2)}, \tag{49}$$

$$\zeta_W(u + m\omega_r + n\omega_i) = \zeta_W(u) + 2m\eta_r + 2n\eta_i. \tag{50}$$

where $\eta_r = \zeta_W(\omega_r/2)$ and $\eta_i = \zeta_W(\omega_i/2)$. Using the above relations one can find the transformation law of $\log \frac{\rho}{\xi}$ when u changes by a period of the lattice Ω. In fact, ρ and ξ have to be well-defined functions on the torus S, therefore the change in their logarithms should be an integer multiple of $2\pi i$ on every period of Ω. This imposes a constraint on S and allows us to determine the constant c. Considering $\log \rho$ yields

$$\eta_r \left(-2hk_1 - \sum_a (u_a + u'_a)\right) - c_\rho \omega_r = \pi i n_r, \tag{51}$$

$$\eta_i \left(-2hk_1 - \sum_a (u_a + u'_a)\right) - c_\rho \omega_i = \pi i n_i, \tag{52}$$

where n_i and n_r are integers, and c_ρ is a constant to be determined. Using Legendre's relation $\eta_r\omega_i - \eta_i\omega_r = \pi i$ we have

$$-2hk_1 - \sum_a (u_a + u'_a) = n_r\omega_i - n_i\omega_r. \tag{53}$$

Using the same reasoning for $\log \xi$, we obtain

$$-2hk_1 + \sum_a (v_a + v'_a) = m_r\omega_i - m_i\omega_r. \tag{54}$$

In order to find the integers n_r, n_i, m_r, and m_i we consider the limit of large k_1 which corresponds to asymptotic infinity on our space $\mathcal{M}$. In this limit the roots of the equation $\eta_2 = z_a^2$ tend to the roots of $\eta_2 = 0$, consequently $Re(u_a + u'_a) \to \omega_r$, $Re(v_a + v'_a) \to \omega_r$. The condition that Nahm data be nonsingular inside the interval $s \in (-h, 0)$ requires that there are no solutions to the above equations for any smaller parameter h. This yields

$$m_i = k - 1, \qquad n_i = -k - 1, \qquad m_r = -k, \qquad n_r = k. \tag{55}$$

Then the transformation properties of $\log \frac{\rho}{\xi}$ become

$$\log \frac{\rho(u+\omega_r)}{\xi(u+\omega_r)} = \log \frac{\rho(u)}{\xi(u)}, \qquad \log \frac{\rho(u+\omega_i)}{\xi(u+\omega_i)} = \log \frac{\rho(u)}{\xi(u)} + 4\pi i. \tag{56}$$

Adding Eqs. (53) and (54) together we obtain a constraint that the curve S has to satisfy:

$$-4hk_1 + \sum_{a=1}^{k} (v_a + v'_a - u_a - u'_a) = 2\omega_r. \tag{57}$$

One can also determine c, but we do not need it here.

Recall that the curve S was parametrized by two complex numbers a, b satisfying $|a|^2 + |b|^2 = 1$, and two real numbers k_1, k_2, which makes a total of five parameters. Eq. (57) reduces the number of independent parameters to four. Thus we obtain a four-parameter family of real holomorphic sections of $\mathcal{Z}$. The space of parameters is nothing but $\mathcal{M}$. In the next section we compute the Kähler potential for the metric on $\mathcal{M}$.

6 Kähler potential

Having described the twistor space $\mathcal{Z}$ of $\mathcal{M}$ and its holomorphic sections we now find the Kähler potential of $\mathcal{M}$. As mentioned before, $\eta_2(\zeta)$ and $\chi(\zeta)$ (see Eq. (44)) can be thought of as local complex coordinates on $\mathcal{M}$. Furthermore, $\eta_2 = z + v\zeta + w\zeta^2 - \bar{v}\zeta^3 + \bar{z}\zeta^4$ depends on ζ holomorphically, so we can apply the generalized Legendre transform construction of Ref. [15, 16]. We briefly outline the construction here. We must compare the coordinate χ holomorphic in the neighborhood of $\zeta = 0$ with the coordinate $\tilde{\chi}$ holomorphic in the neighborhood of $\zeta = \infty$. To this end we define a function $\hat{f}$ and a contour C by the equation

$$\oint_C \frac{d\zeta}{\zeta^n}\hat{f} = \oint_0 \frac{d\zeta}{\zeta^{n-2}}\chi - \oint_\infty \frac{d\zeta}{\zeta^n}\tilde{\chi}, \tag{58}$$

which has to hold for all integer n. Further, we define a function $G(\eta_2, \zeta)$ by $\partial G/\partial \eta_2 = \hat{f}/\zeta^2$ and consider a contour integral

$$F = \frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta^2} G. \tag{59}$$

F is effectively a function of the coefficients z, v and w of the polynomial η_2. According to Ref. [16], to obtain the Kähler potential K one has to impose a constraint $\partial F/\partial w = 0$ and perform Legendre transform on F with respect to v and $\bar{v}$:

$$K(z, \overline{z}, t, \overline{t}) = F(z, \overline{z}, v, \overline{v}, w) - tv - \overline{t}\overline{v}, \quad \frac{\partial F}{\partial v} = t, \frac{\partial F}{\partial \overline{v}} = \overline{t}. \tag{60}$$

Since we know the function χ explicitly as a function of u, it is more convenient to work with contour integrals in the u-plane, rather than in the ζ-plane. Let as rewrite Eq.(58) as

$$\oint_C \frac{d\zeta}{\zeta^n}\hat{f} = \left(\oint_0 \frac{d\zeta}{\zeta^{n-2}}\chi - \oint_\infty \frac{d\zeta}{\zeta^{n-2}}\chi\right) + \left(\oint_\infty \frac{d\zeta}{\zeta^n}\left(\zeta^2\chi - \tilde{\chi}\right)\right). \tag{61}$$

The first step is to compute

$$\oint_0 \frac{d\zeta}{\zeta^{n-2}}\chi - \oint_\infty \frac{d\zeta}{\zeta^{n-2}}\chi = \frac{1}{2k_1}\oint_{\Gamma_1} \frac{du}{\zeta^{n-2}(u)}\log\frac{\rho(u)}{\xi(u)}, \tag{62}$$

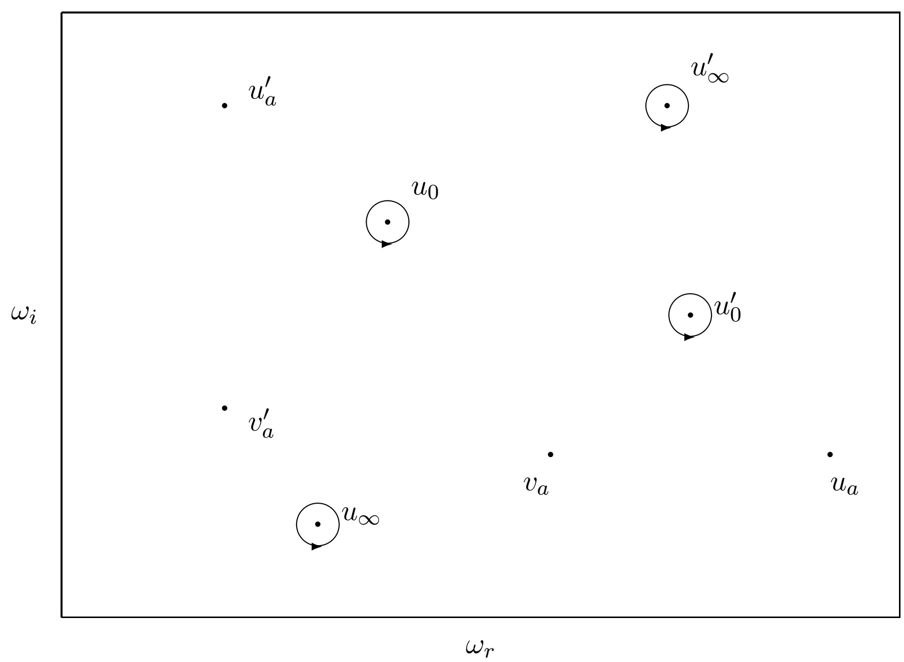

Figure 2: Contour Γ_1.

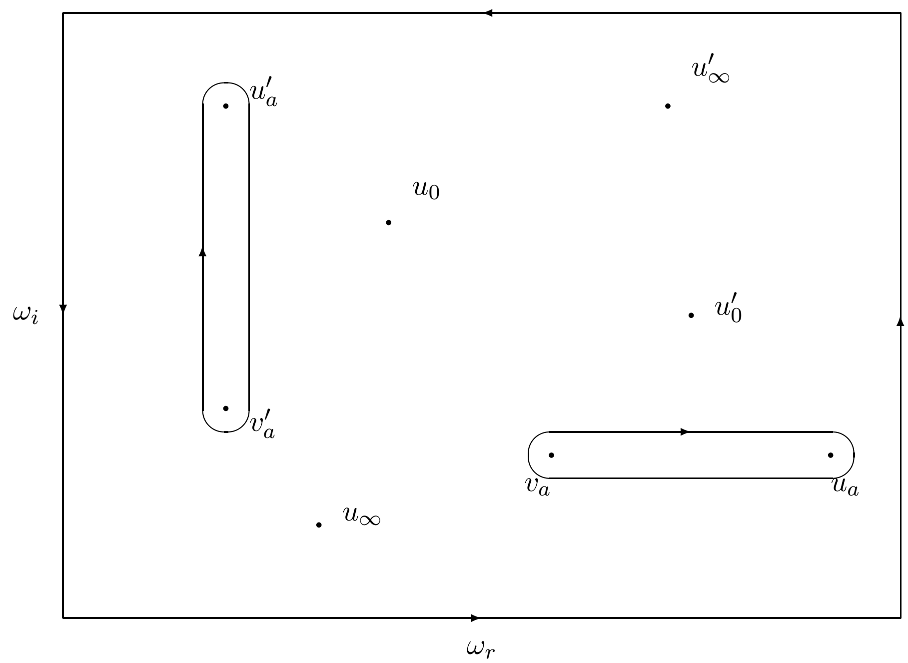

Figure 3: Contour Γ_2

where the contour Γ_1 is shown in Figure 2.

We can deform Γ_1 to the contour Γ_2 in Figure 3. Using the transformation properties Eq. (56) the integral along the boundary of the rectangle is easily computed to be

$$-4\pi i \int_{\omega_r} \frac{du}{\zeta^{n-2}(u)}, \tag{63}$$

where $\int_{\omega_r}$ denotes the integral along the real period of S. Integrals along the contours surrounding the pairs (u_a, v_a) or (u'_a, v'_a) get contributions only from the $\log\left(\sigma(u-u_a)\sigma(u-u'_a)/(\sigma(u-v_a)\sigma(u-v'_a))\right)$ part of $\log\frac{\varrho}{\xi}$. If one replaces the latter expression with

$$\log\left(\sigma(u-u_a)\sigma(u-u'_a)\sigma(u-v_a)\sigma(u-v'_a)\right)$$

and simultaneously changes the contours to the figure-eight shaped contours in Figure 4, the value of the integral is not changed. Finally the integral over the figure-eight shaped contour Γ_a in Figure 4 is equal to

$$\oint_{\Gamma_a} \frac{du}{\zeta^{n-2}(u)} \log\left(\eta(u) - z_a(\zeta)\right). \tag{64}$$

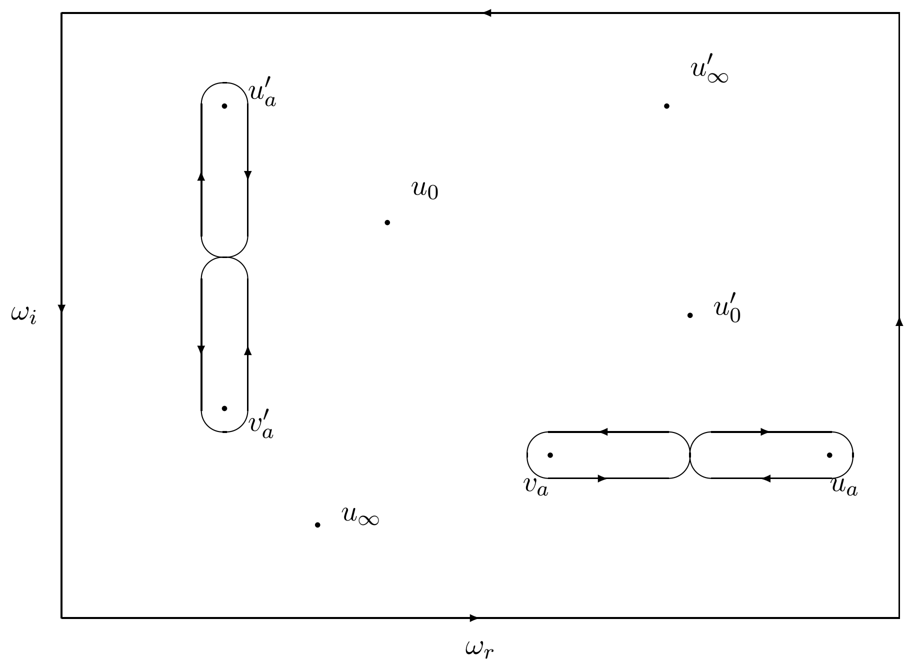

Figure 4: Contour Γ_a

We must also evaluate the last term in Eq. (61). This is easily done using the definition of $\chi, \tilde{\chi}$ and the transformation rules Eqs. (38,39). Putting all this together we find

$$\begin{aligned}\oint_C \frac{d\zeta}{\zeta^n}\hat{f}(\eta,\zeta) &= 4h\oint_0 \frac{d\zeta}{\zeta^{n-1}} - 2\pi i\oint_{\omega_r}\frac{d\zeta}{\eta}\zeta^{-n+2} \\ &+\frac{1}{2}\sum_a \oint_{C_a}\frac{d\zeta}{\eta}\zeta^{-n+2}\log(\eta - z_a(\zeta)). \qquad (65)\end{aligned}$$

Here $\eta = \sqrt{-\eta_2}$, and the contour C_a in the ζ plane comes from the contour Γ_a in the u plane. This implies

$$\begin{aligned}F &= \frac{1}{2\pi i}\oint_0 d\zeta \frac{4h\eta_2}{\zeta^3} + 2\oint_{\omega_r} d\zeta \frac{\sqrt{-\eta_2}}{\zeta^2} - \\ &\sum_a \frac{1}{2\pi i}\oint_{C_a}\frac{d\zeta}{\zeta^2}(\sqrt{-\eta_2} - z_a(\zeta))\log(\sqrt{-\eta_2} - z_a(\zeta)). \qquad (66)\end{aligned}$$

The Legendre transform of this function with respect to $v, \bar{v}$ is the Kähler potential for the metric on $\mathcal{M}$. Eq. (66) agrees with the conjecture of Chalmers in Ref. [9]. To see that the metric is indeed ALF, one must consider the behaviour of F at infinity, which corresponds to taking k_1 to infinity. It is rather easy to see that in this limit the quartic polynomial $\eta_2(\zeta)$ tends to $-(P(\zeta))^2$, where $P(\zeta)$ is a quadratic polynomial in ζ (this is a consequence of Eq. (57)). Substituting this into the formula for F one can see that F takes the Taub-NUT form [16], with h being inversely proportional to the radius of the Taub-NUT circle at infinity. Thus the metric is indeed ALF.

7 Conclusions and open problems

String-theoretic arguments indicate that gravitational instantons can be identified as the moduli spaces of Nahm equations. We used this relationship to construct D_k ALF gravitational instantons and find their twistor spaces and Kähler potentials. Finding the Kähler potential explicitly requires solving a transcendental constraint $\partial F/\partial w = 0$ and performing Legendre transform. In the case of the Atiyah-Hitchin manifold (D_0 ALF space in our notation) this constraint can be solved [16]. Solving the constraint in the general case seems to be hard.

In fact for $k \leq 4$ a description of D_k ALF spaces as finite hyperkähler quotients of known hyperkähler manifolds (Eguchi-Hanson manifold, Dancer's manifold [11] and $\mathbf{R}^4$) was presented in Ref. [6]. It would be interesting to compare the two descriptions and to see how in these cases the constraint $\partial F/\partial w = 0$ is effectively solved.

In the limit $h \to 0$ our D_k ALF metrics become ALE. On the other hand, Kronheimer constructed the same metrics as hyperkähler quotients [2]. Apparently, in Kronheimer's description the difficulty of solving the transcendental constraint translates into the difficulty of solving a system of algebraic equations.

Hyperkähler manifolds which have two compact directions at infinity are also of physical interest. It remains to be seen whether the string-theoretic approach can lead to a construction of such manifolds.

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