An analytical solution method for the unsteady, unbounded, incompressible three-dimensional Navier-Stokes equations in Cartesian coordinates using coordinate axis symmetry degeneracy

David Abrecht
Department of Chemical Engineering, California Institute of Technology,
1200 E California Blvd, MC 210-41, Pasadena, CA 91125, USA
Correspondence to the author should be sent to; E-mail: dabrecht@caltech.edu.

December 25, 2011

Abstract

Analytical solutions are developed for the unsteady Navier-Stokes equations for incompressible fluids in unbounded flow systems with external, time-dependent driving pressure gradients using the degeneracy of the $(1\ 1\ 1)$ axis to reduce the inherent non-linearity of the coupled partial differential equations, which is normally performed with boundary conditions. These solutions are then extended to all directions through rotation of the reference axis, yielding a general solution set. While the solutions are self-consistent and developed from a physical understanding of flow systems, they have not been proven unique or applied to experimental data.

Introduction

The Navier-Stokes equations are the fundamental governing statements of momentum conservation for fluid dynamics, and along with a statement of mass conservation, are used to model such varying processes as pipe flow and pump design[1], chemical reactor behavior[2], aircraft

flight component design[3], and even atmospheric events and weather patterns[4]. However, because the equations are inherently non-linear and highly coupled, no general exact solution has yet been discovered and these models are often developed using numerical solutions[5−7]. While these numerical simulations have had success, they are less than ideal for providing fundamental understanding as to the nature of the involved fluid dynamics, and exact solutions are desired to study the underlying physics.

A large number of exact solutions to the Navier-Stokes equations do exist, and have been reviewed elsewhere[8,9]. However, each of these solutions relies on a limiting set of boundary conditions, reference plane assignment, or initial assumptions that simplifies the equations and reduces the non-linearity. This practice leads to an exact solution but reduces the degrees of freedom, thereby limiting the application to special cases which meet the assumed criteria. As a result, the set of exact solutions are a diverse series of special cases from which it is difficult to develop general principles. A more general solution set is desired to reduce the complexity and allow for a more complete understanding of fluid systems.

In this work, the Navier-Stokes equations are examined using the property of symmetry axis degeneracy rather than boundary conditions to reduce the non-linearity of the equations for incompressible, unsteady systems. This method allows for the solution of the unbounded equations in three spatial directions and time, and will hopefully provide insight into the physics of fluid motion.

Unit Analysis and Definitions

To begin, it is useful to perform a unit analysis in order to help define the necessary frame of reference from which to view the equations. The Navier-Stokes equation and the continuity equation, which is the required statement of mass conservation, for incompressible unsteady flow appear in vector form in Equations 1a and 1b, respectively. These equations are derived from momentum and mass flux balances on a control volume of vanishingly small dimensions as a fluid passes across the volume boundaries.

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla\mathbf{P} + \eta\nabla^2\mathbf{v} \tag{1a}$$

$$\rho\nabla\cdot\mathbf{v} = 0 \tag{1b}$$

In these equations **v** is the velocity vector with SI units of $\mathrm{m\ s^{-2}}$, **P** is the pressure vector with units of $\mathrm{kg\ m^{-1}\ s^{-2}}$, ρ is the constant density of the fluid with units of $\mathrm{kg\ m^{-3}}$ and η is the fluid viscosity with units of $\mathrm{kg\ m^{-1}\ s^{-1}}$. The four terms of Equation 1a can be referred to as the unsteady, convective, pressure, and viscous terms, respectively. By common convention,

all conservative body forces acting on the fluid are combined into the pressure term in this representation.

Expanding Equations 1a and 1b to the component forms gives:

$$\rho\left[\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right]=-\frac{\partial P}{\partial x}+\eta\left[\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right] \tag{2a}$$

$$\rho\left[\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right]=-\frac{\partial P}{\partial y}+\eta\left[\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right] \tag{2b}$$

$$\rho\left[\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right]=-\frac{\partial P}{\partial z}+\eta\left[\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right] \tag{2c}$$

$$\rho\frac{\partial v_x}{\partial x}+\rho\frac{\partial v_y}{\partial y}+\rho\frac{\partial v_z}{\partial z}=0 \tag{2d}$$

From Equations 2a-2d it is evident that each term in the Navier-Stokes equations has units of force per volume, $\mathrm{kg\ m^{-2}\ s^{-2}}$, and the Navier-Stokes equations can be interpreted as the force balance on a fluid within an infinitesimally small volume. Each term except the pressure term contains a derivative of the velocity and one of the natural constants ρ or η, and the pressure term to convert between the kinetic energy associated with the force of the flowing fluid and potential energy stored within the volume. When solving for the velocity in these equations, the pressure term should integrate such that the velocity is directly proportional to the pressure field as expected from physical observations. The natural constants provide two ways for the pressure term to reach velocity units, each of which is more convenient in specific circumstances.

Division of Equations 2a-2c by the viscosity η leads to each term having units of $\mathrm{m^{-1}\ s^{-1}}$, and double integration in the spatial dimensions would lead to the pressure term regaining units of velocity for the final solution. However, since there are also time derivatives of the velocity, the pressure term must be independent of time for this to occur. Because integrating twice by the spatial dimensions leads to the more complex mechanical energy balance, this manner of addressing the units of the Navier-Stokes equations would be more practical when a system's pressure field is defined by the gradients in the velocity of the fluid. Integrating once by the spatial derivatives would lead to a "snapshot" expression of the time-independent pressure field defined by chosen gradients in the velocity. Since any pressure field should theoretically be able to be defined by local velocity gradients, this leads to a large number of solutions. However, this viewpoint is most useful for fluids that are compressible (and can thus generate local pressure gradients through changes in the density) and which lack a large external driving pressure that overwhelms the small local variations. Since this paper is concerned with incompressible fluids, this method of addressing the units in the Navier-Stokes equations will not be discussed further.

If the density is instead used to achieve the correct units, division by ρ gives each term in Equations 2a-2c units of $\mathrm{m\ s^{-2}}$, and the pressure terms take on a form similar to the pressure

term from the classic macroscopic Bernoulli's equation for incompressible fluids. However, like the viscosity case the dependence of the pressure term is limited. To recover velocity units, the pressure term must not have spatial integrals, and must only be a function of time. This restriction at first glance seems very limiting, especially given that the pressure term contains spatial gradients of the pressure field. However, consider that the velocity described is the velocity of a fluid element as it enters and exits the defined space of the control volume. In unsteady systems, the fluid element also has an associated position defined as the time integral of the velocity, $\mathbf{x}(x, y, z, t) = \int \mathbf{v}(x, y, z, t)\partial t$, and the fluid travels through many control volumes as it moves to new positions as a function of time if the control volumes are defined from a fixed reference. Because time and position are related through the velocity, the pressure gradients in the Navier-Stokes equations can be viewed as *local* forces from the viewpoint of the moving fluid rather than the fixed reference frame, and from this viewpoint the spatial variations in the pressure are sampled as a function of time as the fluid moves to new positions.

The latter construction, which treats the pressure gradients as instantaneous and independent forces acting on a tracked fluid element, is convenient for treating incompressible flows under external driving pressures and will be the perspective from which the rest of the analysis will be derived.

Isotropic Solution

Any attempt to analytically solve Equations 2a-2d requires reducing or eliminating the non-linearity of the equations in order to produce a more manageable mathematical problem. Common strategies for accomplishing this reduction involve choosing a reference plane and boundary conditions for an assumed system in order to reduce the non-linearity, generating a solution set for a special case. However, this strategy must be avoided if more general solutions are desired. In this analysis, boundary conditions are not assigned and the simplification is produced by examination of the flow conditions along axes of symmetry relative to the reference plane. Because mathematical degeneracy is generated along these special directions, the degrees of freedom in the problem are also reduced, leading to more general solutions.

Consider a case in which flow is driven by a time-dependent, spatially-uniform externally-applied pressure gradient directed along the $(1\ 1\ 1)$ direction relative to an arbitrarily-assigned reference plane in an unbounded flow system. From the perspective of this direction the coordinate axes have eight-fold symmetry, i.e., the x, y, and z assignations can be interchanged without alterating the magnitude of the vector components. The components of the pressure gradient along this direction are equivalent relative to the reference plane, giving $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = \Delta P(t)$. Because the pressure is driving the flow along this direction and the ambiguity of

the directional assignments, the components of the velocity vector from the flow produced by this pressure gradient are also expected to be equivalent, such that $v_x = v_y = v_z = v$.

In this limiting case, Equations 2a-2c collapse into a single equation, and Equation 2d simplifies giving, respectively:

$$\rho\left[\frac{\partial v}{\partial t} + v\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\right)\right] = -\Delta P(t) + \eta\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] \tag{3a}$$

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = 0 \tag{3b}$$

From Equation 3a, it can be seen that for solutions with equivalent velocity components that obey the continuity equation of Equation 3b the convective term $\mathbf{v} \cdot \nabla \mathbf{v}$ *identically vanishes*, leaving only the unsteady, pressure, and viscous terms. This single term vanishing is consistent with the unbounded nature of the system - with no spatial restrictions to force convective acceleration, the acceleration does not occur and the convective force must be zero.

Performing the substitution, using Equation 3b to simplify 3a yields:

$$\rho\frac{\partial v}{\partial t} = -\Delta P(t) + \eta\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] \tag{4}$$

Because the pressure gradient is only a function of time, and there is a single time derivative of the velocity, the pressure term is linearly independent and may be separated out:

$$v(x, y, z, t) = v'(x, y, z, t) - \frac{1}{\rho}\int \Delta P(t)dt$$

$$\rho\frac{\partial v'}{\partial t} = \eta\left[\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2}\right]$$

yielding a separable equation very similar to the second-order heat equation. Assuming the time and spatial dependences are separable gives:

$$v'(x, y, z, t) = V(x, y, z)T(t)$$

$$\frac{\rho}{\eta T(t)}\frac{dT(t)}{dt} = \frac{1}{V(x, y, z)}\left[\frac{\partial^2 V(x, y, z)}{\partial x^2} + \frac{\partial^2 V(x, y, z)}{\partial y^2} + \frac{\partial^2 V(x, y, z)}{\partial z^2}\right] = -\lambda^2 \tag{5}$$

where λ is a constant with units of inverse length and is defined such that decaying solutions correspond to real values of lambda. Three sets of solutions can be obtained from Equation 5, depending on the values of lambda.

Case 1: $\lambda = 0$, trivial solution In the case where $\lambda = 0$ (or, the characteristic length scale of the system is infinite), the time function T(t) is a constant, and the spatial function V(x,y,z) may be solved by linear or constant functions in each of the spatial variables. These solutions must also meet the continuity requirement of Equation 3b and, since this is defined as a special symmetric case, the velocity solution must also maintain symmetry about the $(1\ 1\ 1)$ axis, i.e., the variables x,y, and z must be capable of being exchanged without altering the equation, such that the linear function must be:

$$V(x, y, z) = A(x + y + z) + B$$

where A and B are constants with units of velocity. Since this equation can only satisfy Equation 3b if $A = 0$, the spatial function must also be a constant, giving an overall constant function:

$$v'(x, y, z, t) = B$$

The constant B, which corresponds to the initial velocity profile of the system, must be the same for each component of the velocity vector in order for the convective term to cancel and for the solution to be valid. This is a consequence of the assumptions of incompressibility and unbounded systems - initial velocities that did not align with the direction of the pressure would initially produce both an initial compression of the fluid and shear forces which would require convective acceleration to describe. Adding the pressure term back to v' to recover v, the solution set becomes:

$$v_x(x, y, z, t) = v_y(x, y, z, t) = v_z(x, y, z, t) = v(x, y, z, t)$$

$$v(x, y, z, t) = B - \frac{1}{\rho} \int \Delta P(t) dt \tag{6}$$

This solution has all spatial derivatives equal to zero, and the unsteady term equal to the negative of the pressure gradient for all three directions, and thus satisfies both continuity in Equation 1b and the condensed Navier-Stokes equation in Equation 1a. The solution is similar to that expected for an applied force on a solid object, which is a result of the infinite length scale of the system.

Case 2: λ^2 positive, decaying solutions In the case where λ^2 is a positive constant, the solution to the time equation T(t) is a decaying exponential:

$$T(t) = Ae^{-\frac{\eta}{\rho}\lambda^2 t}$$

The solution to the spatial portion, which has the second derivatives returning the negative of the function, correspond to sine or cosine functions. These solutions must also meet the symmetry requirements described in Case 1, and are expected to produce a velocity of the

highest magnitude along the path of the pressure gradient, i.e., the $(1\ 1\ 1)$ axis. The simplest arguments that satisfy these conditions and the continuity equation in Equation 3b were found by inspection to be:

$$\begin{aligned} V(x,y,z) =& B\left[\sin\left(\frac{\lambda}{\sqrt{2}}(x-y)\right)+\sin\left(\frac{\lambda}{\sqrt{2}}(x-z)\right)+\sin\left(\frac{\lambda}{\sqrt{2}}(y-z)\right)\right] \\ &+C\left[\cos\left(\frac{\lambda}{\sqrt{2}}(x-y)\right)+\cos\left(\frac{\lambda}{\sqrt{2}}(x-z)\right)+\cos\left(\frac{\lambda}{\sqrt{2}}(y-z)\right)\right] \end{aligned}$$

From a mathematical standpoint both sine and cosine will solve the equations, and the arguments may be arbitrarily shifted by a constant phase correction. However, because the velocity is expected to be largest when $x = y = z$ and symmetry must be maintained, the odd function sine solutions are eliminated, giving only the even function cosine solutions that are unchanged on switching any two of the variables:

$$V(x,y,z) = C\left[\cos\left(\frac{\lambda}{\sqrt{2}}(x-y)\right)+\cos\left(\frac{\lambda}{\sqrt{2}}(x-z)\right)+\cos\left(\frac{\lambda}{\sqrt{2}}(y-z)\right)\right]$$

Combining the spatial and temporal equations, absorbing the $\sqrt{2}$ into lambda without loss of generality and adding in the pressure gradient dependence and initial velocity fields gives the solutions:

$$\begin{aligned} v(x,y,z,t) =& Ae^{-2\frac{\eta}{\rho}\lambda^2 t}\left[\cos\left(\lambda(x-y)\right)+\cos\left(\lambda(x-z)\right)+\cos\left(\lambda(y-z)\right)\right] \\ &-\frac{1}{\rho}\int \Delta P(t)dt + B \end{aligned} \tag{7}$$

Substituting the derivatives of Equation 7 into Equation 3b gives:

$$\begin{aligned} \nabla\cdot\mathbf{v} = A\lambda e^{-2\frac{\eta}{\rho}\lambda^2 t}\big[&-\sin\left(\lambda(x-y)\right)-\sin\left(\lambda(x-z)\right) \\ &+\sin\left(\lambda(x-y)\right)-\sin\left(\lambda(y-z)\right) \\ &+\sin\left(\lambda(x-z)\right)+\sin\left(\lambda(y-z)\right)\big] \\ =& 0 \end{aligned}$$

indicating that the solution set obeys continuity and mass is conserved. From Equations 3a and 3b, since continuity is obeyed the convective term of Equation 3a is identically zero. Substituting the derivatives into the unsteady and viscous terms of Equation 3a then gives:

$$\begin{aligned} \frac{\partial v}{\partial t} &= -2A\frac{\eta}{\rho}\lambda^2 e^{-2\frac{\eta}{\rho}\lambda^2 t}\left[\cos\left(\lambda(x-y)\right)+\cos\left(\lambda(x-z)\right)+\cos\left(\lambda(y-z)\right)\right]-\frac{1}{\rho}\Delta P(t) \\ \nabla^2 v &= -2A\lambda^2 e^{-2\frac{\eta}{\rho}\lambda^2 t}\left[\cos\left(\lambda(x-y)\right)+\cos\left(\lambda(x-z)\right)+\cos\left(\lambda(y-z)\right)\right] \end{aligned}$$

and substitution of these expressions into Equation 3a leads to a balanced equation, satisfying the Navier-Stokes equation and momentum balance. These equations describe the dissipation of momentum over time and over a lengthscale dictated by λ. The solution in Equation 7 has a similar form to the solutions for the general Beltrami flows[9], but has non-zero and non-parallel curl and so is not included in that subset.

Case 3: λ^2 negative, exploding solutions If λ^2 is a negative constant (λ imaginary), the solution to the time equation is a growing exponential:

$$T(t) = Ae^{\frac{\eta}{\rho}\lambda^2 t}$$

and the spatial equations are solved by hyperbolic functions. Again, due to the requirement for symmetry from the original argument, only the even solution is chosen, giving:

$$v'(x,y,z) = A\left[\cosh\left(\frac{\lambda}{\sqrt{2}}(x-y)\right) + \cosh\left(\frac{\lambda}{\sqrt{2}}(x-z)\right) + \cosh\left(\frac{\lambda}{\sqrt{2}}(y-z)\right)\right]$$

Combining these solutions in a similar manner to Case 2 with the pressure and integration constant gives:

$$\begin{aligned} v(x,y,z,t) =& Ae^{2\frac{\eta}{\rho}\lambda^2 t}\left[\cosh\left(\lambda(x-y)\right) + \cosh\left(\lambda(x-z)\right) + \cosh\left(\lambda(y-z)\right)\right] \\ &- \frac{1}{\rho}\int \Delta P(t)dt + B \end{aligned} \tag{8}$$

Substituting the derivatives of Equation 8 into Equation 3b gives:

$$\begin{aligned} \nabla\cdot\mathbf{v} = A\lambda e^{2\frac{\eta}{\rho}\lambda^2 t}&\left[\sinh\left(\lambda(x-y)\right) + \sinh\left(\lambda(x-z)\right)\right. \\ &- \sinh\left(\lambda(x-y)\right) + \sinh\left(\lambda(y-z)\right) \\ &\left.- \sinh\left(\lambda(x-z)\right) - \sinh\left(\lambda(y-z)\right)\right] \\ = 0 \end{aligned}$$

indicating that the exploding solution set also obeys continuity. From Equations 3a and 3b, since continuity is obeyed the convective term of Equation 3a is, again, identically zero. Substituting the derivatives of Equation 8 into the unsteady and viscous terms of Equation 3a then gives:

$$\begin{aligned} \frac{\partial v}{\partial t} &= 2A\frac{\eta}{\rho}\lambda^2 e^{2\frac{\eta}{\rho}\lambda^2 t}\left[\cosh\left(\lambda(x-y)\right) + \cosh\left(\lambda(x-z)\right) + \cosh\left(\lambda(y-z)\right)\right] - \frac{1}{\rho}\Delta P(t) \\ \nabla^2 v &= 2A\lambda^2 e^{2\frac{\eta}{\rho}\lambda^2 t}\left[\cosh\left(\lambda(x-y)\right) + \cosh\left(\lambda(x-z)\right) + \cosh\left(\lambda(y-z)\right)\right] \end{aligned}$$

which, when substituted into Equation 3a gives a balanced equation, indicating that this is also a solution to the Navier-Stokes equations. However, because these solutions have the magnitude of the velocity grow over time, they are expected to be non-physical, and will not be included in further analysis.

Extension of the solutions to other directions

For the sake of conciseness, Case 1 and Case 3 examples will not be shown in the remaining analysis. Case 1 is a special condition of Case 2 arrived at by setting λ equal to zero, and Case

3 is expected to be non-physical and may be ignored for those interested in physical systems. However, because they have been shown to be solutions to Equations 3a and 3b, the analysis in the remaining parts of the paper may be equally applied to these cases.

The solutions presented in the preceding analysis are limited to the single case of a driving pressure gradient along the $(1\ 1\ 1)$ axis relative to a reference plane, and a more general set of solutions is desired. This set may be derived from the single solution presented above by taking advantage of a property of Equations 1a and 1b as statements of conservation. Because they describe conserved properties, solutions derived from these equations should satisfy these equations *regardless of the choice of reference plane*. Therefore, by rotation of the reference plane the solution remains valid but is described by new units, giving a solution for a driving pressure directed along an alternative axis

Consider the rotation of the original spatial reference frame defined by the unit vectors $\mathbf{i} = (1\ 0\ 0)$, $\mathbf{j} = (0\ 1\ 0)$ and $\mathbf{k} = (0\ 0\ 1)$ to a generic spatial reference frame defined by the mutually-orthogonal unit vectors $\mathbf{i}'' = (i_1\ \ i_2\ \ i_3)$, $\mathbf{j}'' = (j_1\ \ j_2\ \ j_3)$ and $\mathbf{k}'' = (k_1\ \ k_2\ \ k_3)$, defined such that $\mathbf{i}'' \times \mathbf{j}'' = \mathbf{k}''$, $\mathbf{j}'' \times \mathbf{k}'' = \mathbf{i}''$ and $\mathbf{k}'' \times \mathbf{i}'' = \mathbf{j}''$. The rotation is performed by multiplication of all system vectors by the transformation matrix, given by:

$$\mathbf{R} = \begin{bmatrix} \mathbf{i}\cdot\mathbf{i}'' & \mathbf{j}\cdot\mathbf{i}'' & \mathbf{k}\cdot\mathbf{i}'' \\ \mathbf{i}\cdot\mathbf{j}'' & \mathbf{j}\cdot\mathbf{j}'' & \mathbf{k}\cdot\mathbf{j}'' \\ \mathbf{i}\cdot\mathbf{k}'' & \mathbf{j}\cdot\mathbf{k}'' & \mathbf{k}\cdot\mathbf{k}'' \end{bmatrix} = \begin{bmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{bmatrix}$$

Applying this rotation to the position, pressure, and velocity equations gives the relationships:

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9a}$$

$$\Delta\mathbf{P}''(t) = \begin{bmatrix} \Delta P''_x(t) \\ \Delta P''_y(t) \\ \Delta P''_z(t) \end{bmatrix} = \begin{bmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} \Delta P(t) \\ \Delta P(t) \\ \Delta P(t) \end{bmatrix} \tag{9b}$$

$$\mathbf{v}''(x'', y'', z'', t) = \begin{bmatrix} v''_x(x'', y'', z'', t) \\ v''_y(x'', y'', z'', t) \\ v''_z(x'', y'', z'', t) \end{bmatrix} = \begin{bmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} v(x'', y'', z'', t) \\ v(x'', y'', z'', t) \\ v(x'', y'', z'', t) \end{bmatrix} \tag{9c}$$

where $\Delta\mathbf{P}'$, and $\mathbf{v}'$ are the transformed pressure and velocity vectors, respectively. Equation 9a is multiplied by the inverse transformation matrix to give x,y, and z as a function of x'',y'',and z''.

$$\mathbf{R}^{-1} = \begin{bmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{bmatrix}^{-1} = \begin{bmatrix} j_2k_3 - j_3k_2 & i_3k_2 - i_2k_3 & i_2j_3 - i_3j_2 \\ j_3k_1 - j_1k_3 & i_1k_3 - i_3k_1 & i_3j_1 - i_1j_3 \\ j_1k_2 - j_2k_1 & i_2k_1 - i_1k_2 & i_1j_2 - i_2j_1 \end{bmatrix}$$

However, the components of this matrix are the components of the cross products of the unit vectors, which by definition are equivalent to the components of the third vector in the basis set. Substitution of this relationship reveals that the transpose of the transformation matrix is the inverse:

$$\mathbf{R}^{-1} = \begin{bmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{bmatrix}^{-1} = \begin{bmatrix} i_1 & j_1 & k_1 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{bmatrix}$$

and gives the special relationships:

$$\begin{aligned} & i_n^2 + j_n^2 + k_n^2 = 1 \\ & i_n i_m + j_n j_m + k_n k_m = 0 \\ & n, m \in \{1, 2, 3\}, \quad n \neq m \end{aligned} \tag{10}$$

Multiplying both sides of Equation 9a by the inverse matrix gives:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} i_1 & j_1 & k_1 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$

and subtraction of the rows gives the substitution for the cosine arguments in Equation 7:

$$\begin{bmatrix} x - y \\ x - z \\ y - z \end{bmatrix} = \begin{bmatrix} i_1 - i_2 & j_1 - j_2 & k_1 - k_2 \\ i_1 - i_3 & j_1 - j_3 & k_1 - k_3 \\ i_2 - i_3 & j_2 - j_3 & k_2 - k_3 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$

Substituting these values into Equation 7 gives the transformed velocity component:

$$\begin{aligned} v(x'', y'', z'', t) = Ae^{-2\frac{\eta}{\rho}\lambda^2 t} \big[& \cos\left(\lambda\left((i_1 - i_2)x'' + (j_1 - j_2)y'' + (k_1 - k_2)z''\right)\right) \\ & + \cos\left(\lambda\left((i_1 - i_3)x'' + (j_1 - j_3)y'' + (k_1 - k_3)z''\right)\right) \\ & + \cos\left(\lambda\left((i_2 - i_3)x'' + (j_2 - j_3)y'' + (k_2 - k_3)z''\right)\right)\big] \\ & - \frac{1}{\rho}\int \Delta P(t)dt + B \end{aligned} \tag{11}$$

Substituting Equation 11 into Equation 9c gives the components of the new velocity vector:

$$\begin{aligned} v_x''(x'', y'', z'', t) = & A(i_1 + i_2 + i_3)e^{-2\frac{\eta}{\rho}\lambda^2 t} \big[\cos\left(\lambda\left((i_1 - i_2)x'' + (j_1 - j_2)y'' + (k_1 - k_2)z''\right)\right) \\ & + \cos\left(\lambda\left((i_1 - i_3)x'' + (j_1 - j_3)y'' + (k_1 - k_3)z''\right)\right) \\ & + \cos\left(\lambda\left((i_2 - i_3)x'' + (j_2 - j_3)y'' + (k_2 - k_3)z''\right)\right)\big] \\ & - \frac{i_1 + i_2 + i_3}{\rho}\int \Delta P(t)dt + (i_1 + i_2 + i_3)B \end{aligned} \tag{12a}$$

$$
\begin{aligned}
v_y''(x'',y'',z'',t) =& A(j_1+j_2+j_3)e^{-2\frac{\eta}{\rho}\lambda^2 t}\left[\cos\left(\lambda\left((i_1-i_2)x''+(j_1-j_2)y''+(k_1-k_2)z''\right)\right)\right.\\
&+\cos\left(\lambda\left((i_1-i_3)x''+(j_1-j_3)y''+(k_1-k_3)z''\right)\right)\\
&\left.+\cos\left(\lambda\left((i_2-i_3)x''+(j_2-j_3)y''+(k_2-k_3)z''\right)\right)\right]\\
&-\frac{j_1+j_2+j_3}{\rho}\int\Delta P(t)dt+(j_1+j_2+j_3)B
\end{aligned}
\tag{12b}
$$

$$
\begin{aligned}
v_z''(x'',y'',z'',t) =& A(k_1+k_2+k_3)e^{-2\frac{\eta}{\rho}\lambda^2 t}\left[\cos\left(\lambda\left((i_1-i_2)x''+(j_1-j_2)y''+(k_1-k_2)z''\right)\right)\right.\\
&+\cos\left(\lambda\left((i_1-i_3)x''+(j_1-j_3)y''+(k_1-k_3)z''\right)\right)\\
&\left.+\cos\left(\lambda\left((i_2-i_3)x''+(j_2-j_3)y''+(k_2-k_3)z''\right)\right)\right]\\
&-\frac{k_1+k_2+k_3}{\rho}\int\Delta P(t)dt+(k_1+k_2+k_3)B
\end{aligned}
\tag{12c}
$$

Since these equations no longer have equivalent velocity components, Equations 3a and 3b no longer apply, and the components must be substituted into Equations 1a and 1b to determine if the Navier-Stokes equations are satisfied. Beginning with the continuity equation, Equation 1b, substituting the derivatives of Equations 12a-12c gives:

$$
\begin{aligned}
\nabla\cdot\mathbf{v}'' =& Ae^{-2\frac{\eta}{\rho}\lambda^2 t}\left\{(i_1+i_2+i_3)\left[(i_1-i_2)\cos\left(\lambda\left((i_1-i_2)x''+(j_1-j_2)y''+(k_1-k_2)z''\right)\right)\right.\right.\\
&+(i_1-i_3)\cos\left(\lambda\left((i_1-i_3)x''+(j_1-j_3)y''+(k_1-k_3)z''\right)\right)\\
&\left.+(i_2-i_3)\cos\left(\lambda\left((i_2-i_3)x''+(j_2-j_3)y''+(k_2-k_3)z''\right)\right)\right]\\
&+(j_1+j_2+j_3)\left[(j_1-j_2)\cos\left(\lambda\left((i_1-i_2)x''+(j_1-j_2)y''+(k_1-k_2)z''\right)\right)\right.\\
&+(j_1-j_3)\cos\left(\lambda\left((i_1-i_3)x''+(j_1-j_3)y''+(k_1-k_3)z''\right)\right)\\
&\left.+(j_2-j_3)\cos\left(\lambda\left((i_2-i_3)x''+(j_2-j_3)y''+(k_2-k_3)z''\right)\right)\right]\\
&+(k_1+k_2+k_3)\left[(k_1-k_2)\cos\left(\lambda\left((i_1-i_2)x''+(j_1-j_2)y''+(k_1-k_2)z''\right)\right)\right.\\
&+(k_1-k_3)\cos\left(\lambda\left((i_1-i_3)x''+(j_1-j_3)y''+(k_1-k_3)z''\right)\right)\\
&\left.\left.+(k_2-k_3)\cos\left(\lambda\left((i_2-i_3)x''+(j_2-j_3)y''+(k_2-k_3)z''\right)\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\mathbf{v}'' =& Ae^{-2\frac{\eta}{\rho}\lambda^2 t}\left\{\left[(i_1^2+j_1^2+k_1^2)-(i_2^2+j_2^2+k_2^2)+(i_1i_3+j_1j_3+k_1k_3)\right.\right.\\
&\left.-(i_2i_3+j_2j_3+k_2k_3)\right]\cos\left(\lambda\left((i_1-i_2)x''+(j_1-j_2)y''+(k_1-k_2)z''\right)\right)\\
&+\left[(i_1^2+j_1^2+k_1^2)-(i_3^2+j_3^2+k_3^2)+(i_1i_2+j_1j_2+k_1k_2)\right.\\
&\left.-(i_2i_3+j_2j_3+k_2k_3)\right]\cos\left(\lambda\left((i_1-i_3)x''+(j_1-j_3)y''+(k_1-k_3)z''\right)\right)\\
&+\left[(i_2^2+j_2^2+k_2^2)-(i_3^2+j_3^2+k_3^2)+(i_1i_2+j_1j_2+k_1k_2)\right.\\
&\left.\left.-(i_1i_3+j_1j_3+k_1k_3)\right]\cos\left(\lambda\left((i_2-i_3)x''+(j_2-j_3)y''+(k_2-k_3)z''\right)\right)\right\}
\end{aligned}
$$

By substituting the special relations in Equation 10, the coefficients of the trigonometric functions become zero, and the solutions in Equations 12a-12c are shown to meet the continuity

criteria in Equation 1b.

The convective terms in Equation 1a are best dealt with in vector form, and are given by:

$$\begin{bmatrix} \mathbf{v}'' \cdot \nabla v_x'' \\ \mathbf{v}'' \cdot \nabla v_y'' \\ \mathbf{v}'' \cdot \nabla v_z'' \end{bmatrix} = \begin{bmatrix} v_x'' \frac{\partial v_x''}{\partial x''} + v_y'' \frac{\partial v_x''}{\partial y''} + v_z'' \frac{\partial v_x''}{\partial z''} \\ v_x'' \frac{\partial v_y''}{\partial x''} + v_y'' \frac{\partial v_y''}{\partial y''} + v_z'' \frac{\partial v_y''}{\partial z''} \\ v_x'' \frac{\partial v_z''}{\partial x''} + v_y'' \frac{\partial v_z''}{\partial y''} + v_z'' \frac{\partial v_z''}{\partial z''} \end{bmatrix}$$

Because the components of the new velocity vector differ only by a constant, a rearrangement can be made to simplify the expression. By substitution of the values in Equation 9c and rearrangement of the coefficients, the expression becomes:

$$\begin{bmatrix} \mathbf{v}'' \cdot \nabla v_x'' \\ \mathbf{v}'' \cdot \nabla v_y'' \\ \mathbf{v}'' \cdot \nabla v_z'' \end{bmatrix} = \begin{bmatrix} v_x'' \left(\frac{\partial v_x''}{\partial x''} + \frac{\partial v_y''}{\partial y''} + \frac{\partial v_z''}{\partial z''} \right) \\ v_y'' \left(\frac{\partial v_x''}{\partial x''} + \frac{\partial v_y''}{\partial y''} + \frac{\partial v_z''}{\partial z''} \right) \\ v_z'' \left(\frac{\partial v_x''}{\partial x''} + \frac{\partial v_y''}{\partial y''} + \frac{\partial v_z''}{\partial z''} \right) \end{bmatrix}$$

Since the solution has already been shown to meet the continuity equation, the terms in parentheses are equal to zero, and the convective term again cancels for component direction in the more general case, consistent with no spatial constrictions on the flow.

The viscous terms derived from Equations 12a-12c undergo a similar condensation from the special relationships given in Equation 10 to the continuity equation, achieving a final form of:

$$\begin{aligned} \frac{\partial^2 v_x''}{\partial (x'')^2} + \frac{\partial^2 v_x''}{\partial (y'')^2} + \frac{\partial^2 v_x''}{\partial (z'')^2} = & -2A\lambda^2 (i_1 + i_2 + i_3) e^{-2\frac{\eta}{\rho}\lambda^2 t} * \\ & [\cos(\lambda((i_1 - i_2)x'' + (j_1 - j_2)y'' + (k_1 - k_2)z'')) \\ & + \cos(\lambda((i_1 - i_3)x'' + (j_1 - j_3)y'' + (k_1 - k_3)z'')) \\ & + \cos(\lambda((i_2 - i_3)x'' + (j_2 - j_3)y'' + (k_2 - k_3)z''))] \\ \frac{\partial^2 v_y''}{\partial (x'')^2} + \frac{\partial^2 v_y''}{\partial (y'')^2} + \frac{\partial^2 v_y''}{\partial (z'')^2} = & -2A\lambda^2 (j_1 + j_2 + j_3) e^{-2\frac{\eta}{\rho}\lambda^2 t} * \\ & [\cos(\lambda((i_1 - i_2)x'' + (j_1 - j_2)y'' + (k_1 - k_2)z'')) \\ & + \cos(\lambda((i_1 - i_3)x'' + (j_1 - j_3)y'' + (k_1 - k_3)z'')) \\ & + \cos(\lambda((i_2 - i_3)x'' + (j_2 - j_3)y'' + (k_2 - k_3)z''))] \end{aligned}$$

$$\frac{\partial^2 v_z''}{\partial (x'')^2} + \frac{\partial^2 v_z''}{\partial (y'')^2} + \frac{\partial^2 v_z''}{\partial (z'')^2} = -2A\lambda^2(k_1+k_2+k_3)e^{-2\frac{\eta}{\rho}\lambda^2 t} * \\ [\cos(\lambda((i_1-i_2)x''+(j_1-j_2)y''+(k_1-k_2)z'')) \\ + \cos(\lambda((i_1-i_3)x''+(j_1-j_3)y''+(k_1-k_3)z'')) \\ + \cos(\lambda((i_2-i_3)x''+(j_2-j_3)y''+(k_2-k_3)z''))]$$

which, along with the time derivatives:

$$\frac{\partial v_x''}{\partial t} = -2A\frac{\eta}{\rho}\lambda^2(i_1+i_2+i_3)e^{-2\frac{\eta}{\rho}\lambda^2 t}[\cos(\lambda((i_1-i_2)x''+(j_1-j_2)y''+(k_1-k_2)z'')) \\ + \cos(\lambda((i_1-i_3)x''+(j_1-j_3)y''+(k_1-k_3)z'')) \\ + \cos(\lambda((i_2-i_3)x''+(j_2-j_3)y''+(k_2-k_3)z''))] \\ - \frac{i_1+i_2+i_3}{\rho}\Delta P(t)$$

$$\frac{\partial v_y''}{\partial t} = -2A\frac{\eta}{\rho}\lambda^2(j_1+j_2+j_3)e^{-2\frac{\eta}{\rho}\lambda^2 t}[\cos(\lambda((i_1-i_2)x''+(j_1-j_2)y''+(k_1-k_2)z'')) \\ + \cos(\lambda((i_1-i_3)x''+(j_1-j_3)y''+(k_1-k_3)z'')) \\ + \cos(\lambda((i_2-i_3)x''+(j_2-j_3)y''+(k_2-k_3)z''))] \\ - \frac{j_1+j_2+j_3}{\rho}\Delta P(t)$$

$$\frac{\partial v_z''}{\partial t} = -2A\frac{\eta}{\rho}\lambda^2(k_1+k_2+k_3)e^{-2\frac{\eta}{\rho}\lambda^2 t}[\cos(\lambda((i_1-i_2)x''+(j_1-j_2)y''+(k_1-k_2)z'')) \\ + \cos(\lambda((i_1-i_3)x''+(j_1-j_3)y''+(k_1-k_3)z'')) \\ + \cos(\lambda((i_2-i_3)x''+(j_2-j_3)y''+(k_2-k_3)z''))] \\ - \frac{k_1+k_2+k_3}{\rho}\Delta P(t)$$

Satisfy Equations 1a and 2a-2c, indicating that the general velocity vector described by the component equations in Equations 12a-12c are solutions to the Navier-Stokes Equations for a driving pressure defined by Equation 9b in unbounded space.

Uniaxial flow

In order to demonstrate how the results from the previous section may be used, the solutions for flow along a single axis are derived from Equations 12a-12c. In order to align the above solution with the x-axis of a new reference plane, the direction of the driving pressure must be

shifted from the $(1\ 1\ 1)$ direction to the $(1\ 0\ 0)$ direction. This is accomplished by shifting the x-axis of the original reference plane to a unit vector aligned in the $(1\ 1\ 1)$ direction. Since they no longer contain any additional pressure information, the shifts for y and z may then be assigned arbitrarily without changing the problem, as long as the three new unit vectors remain mutually orthogonal. For this example, y is transformed to the $\left(1\ \overline{1}\ 0\right)$ direction, and z to the $\left(\ 1\ 1\ \overline{2}\right)$ direction. This gives the tranformation matrix:

$$\mathbf{R} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{bmatrix}$$

Applying this transformation matrix in place of the generic matrix in Equation 9b gives the expression for the driving pressure as:

$$\Delta\mathbf{P}''(t) = \begin{bmatrix} \Delta P''_x(t) \\ \Delta P''_y(t) \\ \Delta P''_z(t) \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{3}}\Delta P(t) \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

indicating the pressure is only directed along the new x-axis with a new magnitude relative to the original value. Although this magnitude change does not affect this example because the pressure gradient magnitude is a variable, it becomes important in establishing the correct *ratio* of magnitudes for a shift that is not onto a symmetry axis.

Applying the transformation matrix to Equations 12a-12c gives the velocity solutions for uniaxial flow:

$$\begin{aligned} v''_x(x'', y'', z'', t) =& \frac{3A}{\sqrt{3}} e^{-2\frac{\eta}{\rho}\lambda^2 t} \left[\cos\left(\lambda y''\sqrt{2}\right) + \cos\left(\frac{\lambda}{2}(z''\sqrt{6} + y''\sqrt{2})\right) \right. \\ & \left. + \cos\left(\frac{\lambda}{2}(z''\sqrt{6} - y''\sqrt{2})\right) \right] - \frac{3}{\rho\sqrt{3}} \int \Delta P(t) dt + \frac{3B}{\sqrt{3}} \end{aligned} \tag{14a}$$

$$v''_y(x'', y'', z'', t) = 0 \tag{14b}$$

$$v''_z(x'', y'', z'', t) = 0 \tag{14c}$$

giving, as expected, velocity along only along the direction of the driving pressure. The coefficients of the spatial terms in the trigonometric arguments are scaled precisely by the transformation so that Equations 1a and 1b are balanced.

Conclusions and Future Work

In this work, mathematical solutions have been developed for the unsteady Navier-Stokes equations for incompressible fluids in unbounded, three dimensional flow systems based on and consistent with a theoretical understanding of the underlying physics of fluid motion. These solutions have been developed using the symmetry degeneracy of the $(1\ 1\ 1)$ axis to reduce the inherent non-linearity of the Navier-Stokes equations, and extended to all directions through the independency of these conservation equations on the definition of the reference axis. While these solutions are self-consistent, their uniqueness has not been established, and their consistency with experimental fluid data has not been explored. Future work would require comparison of the results developed here with experimental fluid flow data or established numerical models to verify the application of these equations to real systems.

References

1. Constantinescu, G.S.; Patel, V.C.; *J. Hydraul. Eng.*, **124** (1998): 213-237.

2. Raja, L.L.; Kee, R.J.; Deutschmann, O.; Warnatz, J.; Schmidt, L.D.; *Cat. Today*, **59** (2000): 47-60.

3. Jameson, A.; Martinelli, L.; Pierce, N.A.; *Theor. and Comp. Fluid Dynamics*, **10** (1998): 213-237.

4. Reisner, J.; Wyszogrodzki, A.; Mousseau, V.; Knoll, D.; *J. Comp. Physics*, **189** (2003): 30-44.

5. Feng, J.Q.; Beard, K.V.; *Atmospheric Res.*, **101** (2011): 480-491.

6. Audusse, E.; Bristeau, M.-O.; Pelanti, M.; Sainte-Marie, J.; *J. Comp. Physics*, **230** (2011): 3453-3478.

7. Selim, K.; Logg, A.; Larson, M.G.; *Comp. Methods Appl. Mech. Eng.*, **209-212** (2012): 54-65.

8. Wang, C.Y.; *Appl. Mech. Rev.* **42** (1989): 269-282.

9. Wang, C.Y.: *Ann. Rev. Fluid Mech.* **23** (1991): 159-177.