ON L^p BOUNDS FOR KAKEYA MAXIMAL FUNCTIONS AND THE MINKOWSKI DIMENSION IN $\mathbb{R}^2$

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Abstract

We prove that the bound on the L^p norms of the Kakeya type maximal functions studied by Cordoba [**2**] and Bourgain [**1**] are sharp for $p > 2$. The proof is based on a construction originally due to Schoenberg [**5**], for which we provide an alternative derivation. We also show that $r^2 \log(1/r)$ is the exact Minkowski dimension of the class of Kakeya sets in $\mathbb{R}^2$, and prove that the exact Hausdorff dimension of these sets is between $r^2 \log(1/r)$ and $r^2 \log(1/r)[\log\log(1/r)]^{2+\varepsilon}$.

1. *Introduction*

Consider the following two Kakeya type maximal operators. The first, studied in [**2**], $M_\delta : L^2(\mathbb{R}^2) \mapsto L^2(\mathbb{R}^2)$, is defined for $\delta > 0$ as

$$M_\delta f(x) \stackrel{d}{=} \sup_{x \in R \in \mathfrak{R}_\delta} \frac{1}{R} \int_R |f|, \tag{1}$$

where $\mathfrak{R}_\delta$ is the set of rectangles $R \in \mathbb{R}^2$ of size $1 \times \delta$. The second was introduced by Bourgain in [**1**]. We denote it by $K_\delta : L^p(\mathbb{R}^2) \mapsto L^p(S^1)$, and it is defined as

$$K_\delta f(e) \stackrel{d}{=} \sup_{x \in \mathbb{R}^2} \frac{1}{T_e^\delta(x)} \int_{T_e^\delta(x)} |f|,$$

where $T_e^\delta(x)$ is the $1 \times \delta$ rectangle oriented in the e-direction with x at its centre.

In [**2**, Proposition 1.2], Cordoba proves that for $p \geqslant 2$,

$$\|M_\delta\|_p \lesssim \left(\log \frac{1}{\delta}\right)^{1/p}. \tag{2}$$

In [**1**, (1.5)], Bourgain shows that for $p \geqslant 2$,

$$\|K_\delta\|_p \lesssim \left(\log \frac{1}{\delta}\right)^{1/p}. \tag{3}$$

More precisely, both authors prove their results in the case $p = 2$. The case $p > 2$ then follows from the obvious bounds $|M_\delta f|_\infty \leqslant |f|_\infty$ and $|K_\delta f|_\infty \leqslant |f|_\infty$ and the Marcinkiewicz interpolation theorem.

For the case $p = 2$, these bounds were known to be sharp; for example, consider the function [**3**]

$$f_\delta(x) \stackrel{d}{=} \begin{cases} 1 & |x| < \delta, \\ \delta/|x| & \delta \leqslant |x| \leqslant 1, \\ 0 & |x| > 1. \end{cases}$$

Received 25 November 1997; revised 11 June 1998.

1991 *Mathematics Subject Classification* 42B25, 28A78.

Bull. London Math. Soc. 31 (1999) 213–221

The key to showing that (2) and (3) are sharp lies in a certain 'optimal' construction, due to Schoenberg [**5**], of a thin set which contains a unit length line segment in every direction. Unaware of his result, we came up with a different construction of essentially the same set. This is the content of Theorem 1.

REMARK. For $p \in [1, 2)$, it can be proved, using arguments analogous to those for the case $p = 2$, that

$$\|K_\delta\|_p \lesssim \delta^{1-2/p}, \quad \|M_\delta\|_p \lesssim \delta^{1-2/p}.$$

These are known to be sharp: consider the function $f_\delta(x) \stackrel{d}{=} \chi_{D(0,\delta)}$, where $D(0, \delta)$ is the disc of radius δ about 0.

We need the following notation.

- Let l be a line segment $l = \{(x, ax+b) : x \in [0, 1]\}$. We consider lines with $a(l) \stackrel{d}{=} a \in [0, 1]$ and $b(l) \stackrel{d}{=} b \in [-1, 0]$.
- For $\delta > 0$ and such an l, let $R_\delta(l)$ be the triangle defined by the set of vertices $\{(0, l(0)), (0, l(0)-\delta), (1, l(1))\}$, where $l(x)$ denotes a shorthand for $a(l)x + b(l)$.
- Let $\vec{R}_\delta(l)$ be the triangle obtained by translating $R_\delta(l)$ by $2\sqrt{2}$ along the direction of l.
- For a set $E \subset \mathbb{R}^2$, let $|E|$ denote its Lebesgue measure, and let $E(\delta)$ denote its δ-neighbourhood.
- $x_n \lesssim y_n$ means that there exists a $C > 0$ such that $x_n \leqslant C y_n$. The symbol $\approx$ is short for both $\gtrsim$ and $\lesssim$.

THEOREM 1. *For any n, there exist 2^n line segments $\{l_i^n : i = 0, 1, \ldots, 2^n - 1\}$ with $a(l_i^n) = i2^{-n}$ such that the triangles $R_{2^{-n}}(l_i^n)$ satisfy the following two properties.*

(i) $\left| \bigcup_i R_{2^{-n}}(l_i^n) \right| < \dfrac{1}{n}$.

(ii) *The translated triangles $\vec{R}_{2^{-n}}(l_i^n)$ are disjoint.*

REMARK. Though not mentioned in [**5**], (ii) would follow from Schoenberg's work as well.

Let

$$E_n \stackrel{d}{=} \bigcup_{i=1}^{2^n} R_{2^{-n}}(l_i^n). \tag{4}$$

Then E_n has a unit length line segment with any given slope $a \in [0, 1]$, it is composed of triangles with eccentricity 2^n, and $|E_n| < 1/n$, so we have the following result.

COROLLARY 1. *The bounds* (2) *and* (3) *are sharp for $p > 2$.*

Proof. Let E_n be defined as in (4), and let $f_n \stackrel{d}{=} \chi_{E_n}$. Then by (i) of Theorem 1, $|f_n|_p < (1/n)^{1/p}$. On the other hand, let $\tilde{M}$ be defined as in (1) but with rectangles of size $3\sqrt{2} \times \delta$ instead of $1 \times \delta$. Then one can check that $\tilde{M}_\delta f(x) > C > 0$ for $x \in \bigcup_i \vec{R}_{2^{-n}}(l_i^n)$, and it follows that $|\tilde{M}_{2^{-n}}(f_n)|_p \gtrsim 1$. But $|\tilde{M}_\delta(f)|_p \approx |M_\delta(f)|_p$, therefore the bound in (2) is necessarily sharp. As for $K_{2^{-n}}$, it is not hard to show that $K_{2^{-n}}(\chi_{E_n})(\theta) \geqslant C > 0$ for $\theta \in [0, \pi/4]$, which implies that (3) is sharp for $p \geqslant 2$.

A *Kakeya set* in $\mathbb{R}^2$ is a set of Lebesgue measure 0 which contains a unit length line segment in every direction in the plane.

The triangles mentioned above allow us to constructively prove the following.

Lemma 1. *There exists a (compact) Kakeya set E such that for any $\varepsilon < 1$,*

$$|E(\varepsilon)| \lesssim \frac{1}{\log(1/\varepsilon)}. \tag{5}$$

Since the reversed inequality is the rule for Kakeya sets, we can now prove the following.

Theorem 2. *The exact Minkowski dimension of the class of Kakeya sets in $\mathbb{R}^2$ is*

$$h(r) = r^2 \log \frac{1}{r}.$$

Finally, we provide some partial results for the exact Hausdorff dimension of the class of Kakeya sets. Specifically, we show that it is between $r^2 \log(1/r)$ and $r^2 \log(1/r)(\log\log(1/r))^{2+\varepsilon}$ for any $\varepsilon > 0$.

2. *The basic construction*

A few more notations are useful.

- A G-set for us means a compact set $E \subset [0,1] \times \mathbb{R}$, such that for any $a \in [0,1]$ there exists a (unit length) line segment $l_a \subset E$ with slope a.
- By the upper edge of the triangle $R_\delta(l)$ we mean the segment l, and by the lower edge we mean the segment between $(0, l(0)-\delta)$ and $(1, l(1))$. The vertical edge is the third segment.
- For a set $E \subset \mathbb{R}^2$, let $|E|_x$ be the (one-dimensional) Lebesgue measure of its cross-section at x.
- For $k = 0, 1, \ldots, 2^n - 1$, we denote by $\varepsilon_i(k)$ the ith binary digit in the expansion

$$\frac{k}{2^n} = \sum_{i=1}^{n} \varepsilon_i 2^{-i}, \quad \varepsilon_i \in \{0,1\}.$$

Proof of Theorem 1. We first provide the geometric view of the construction which closely follows that of Sawyer [**4**] and Wolff [**6**]. Start with a triangle with vertices at $(0,0)$, $(0,-1)$, $(1,0)$. Cut it into two triangles by adding a vertex at $(0,-1/2)$, and then slide the lower triangle upward until the vertical edges of the two triangles overlap completely. At the kth stage ($k = 1, 2, \ldots, n-1$), you have 2^k triangles. Cut each of these into two triangles by adding a vertex in the middle of the vertical edge. For each of these newly created pairs, slide the lower triangle upward until the upper edges of the two triangles intersect at $x = k/n$. This construction leaves us with 2^n triangles of equal area (2^{-n-1}), and it is obvious that the union of these is a G-set. We next show that this construction satisfies (i) and (ii) of the theorem.

We define our set of 2^n lines $l_0, \ldots, l_{2^n-1}$ (these correspond to the upper edges of the triangles in the above construction) as follows: l_k has slope

$$a(l_k) \stackrel{d}{=} \frac{k}{2^n},$$

and with $\varepsilon_i \stackrel{d}{=} \varepsilon_i(a(l_k))$,

$$b(l_k) \stackrel{d}{=} -\sum_1^n \varepsilon_i 2^{-i} + \sum_1^n \varepsilon_i \left(1-\frac{i-1}{n}\right) 2^{-i} = \sum_1^n \frac{1-i}{n} \varepsilon_i 2^{-i}.$$

Note that $\sum \varepsilon_i (1-(i-1)/n) 2^{-i}$ is the total upward translation that was applied to the kth line (triangle) in our construction. It is, at times, convenient to index our lines by their strictly increasing slopes: $\{l_a : a = 0, 1/2^n, 2/2^n, \ldots, (2^n-1)/2^n\}$. With this notation,

$$l_a(x) = \sum_{i=1}^n \left(x + \frac{1-i}{n}\right) \varepsilon_i 2^{-i},$$

where $\varepsilon_i = \varepsilon_i(a)$. To prove (ii), it suffices to show that for $a > \tilde{a}$, $l_a(1) \geqslant l_{\tilde{a}}(1)$. There exists a $k \in \{1, \ldots, n\}$ such that $\varepsilon_i = \tilde{\varepsilon}_i$ for $i \in \{1, \ldots, k-1\}$, and $\varepsilon_k = 1 > 0 = \tilde{\varepsilon}_k$, so

$$\begin{aligned} l_a(1) - l_{\tilde{a}}(1) &= \frac{n+1-k}{n} 2^{-k} + \sum_{k+1}^n \frac{n+1-i}{n} (\varepsilon_i - \tilde{\varepsilon}_i) 2^{-i} \\ &\geqslant \frac{n+1-k}{n} 2^{-k} - \sum_{k+1}^n \frac{n+1-i}{n} 2^{-i} > 0. \end{aligned}$$

To prove (i), it suffices to show that for any $x \in [0, 1]$,

$$\left| \bigcup_{i=0}^{2^n-1} R_{2^{-n}}(l_i) \right|_x < \frac{1}{n}. \tag{6}$$

For $k = 1, 2, \ldots, n$, we show that (6) holds in $I_k \stackrel{d}{=} [(k-1)/n, k/n]$, by grouping the lines into 2^{k-1} sets of lines determined by the first $k-1$ binary digits of their slopes. The triangles corresponding to each of these sets contribute at most $(2^{1-k} - 2^{-n})/n$ to the measure of the cross-section at any $x \in I_k$. Since there are 2^{k-1} such sets, (6) follows. More precisely, let $k \in \{1, 2, \ldots, n\}$. For $j = 0, 1, \ldots, 2^{k-1} - 1$, we define

$$L_j \stackrel{d}{=} \left\{ l_a : \varepsilon_i(a) = \varepsilon_i\left(\frac{j}{2^{k-1}}\right) \text{ for } i = 1, 2, \ldots, k-1 \right\}.$$

Let $l_a \in L_j$ and, with $\varepsilon_i = \varepsilon_i(a)$, let $r \stackrel{d}{=} \sum_1^{k-1} \varepsilon_i 2^{-i}$ (or $r = j/2^{k-1}$). Then

$$l_a(x) = \sum_1^{k-1} \left(x + \frac{1-i}{n}\right) \varepsilon_i 2^{-i} + \sum_k^n \left(x + \frac{1-i}{n}\right) \varepsilon_i 2^{-i},$$

so for $x \in I_k$,

$$\begin{aligned} l_a(x) &= l_r(x) + \sum_k^n \left(x + \frac{1-i}{n}\right) \varepsilon_i 2^{-i} \\ &\leqslant l_r(x) + \left(x + \frac{1-k}{n}\right) \varepsilon_k 2^{-k} \\ &\leqslant l_r(x) + \left(x + \frac{1-k}{n}\right) 2^{-k} \\ &= l_{r+2^{-k}}(x). \end{aligned}$$

Similarly,

$$\begin{aligned} l_a(x) &\geqslant l_r(x) + \sum_{k+1}^{n} \left(x + \frac{1-i}{n}\right) \varepsilon_i 2^{-i} \\ &\geqslant l_r(x) + \sum_{k+1}^{n} \left(x + \frac{1-i}{n}\right) 2^{-i} \\ &= l_{r+2^{-k}-2^{-n}}(x). \end{aligned}$$

Thus, for any $j \in \{0, 1, \ldots, 2^{k-1}-1\}$ and with $r = j/2^{k-1}$, the set of triangles $\{R_{2^{-n}}(l) : l \in L_j\}$ is bounded, for $x \in I_k$, from above by the line $l_{r+2^{-k}}(x)$, and from below by $l_{r+2^{-k}-2^{-n}}(x) - 2^{-n}(1-x)$, the latter being the lower edge of $R_{2^{-n}}(l_{r+2^{-k}-2^{-n}})$. Hence

$$\begin{aligned} \left| \bigcup_{l \in L_j} R_{2^{-n}}(l) \right|_x &\leqslant l_{r+2^{-k}}(x) - [l_{r+2^{-k}-2^{-n}}(x) - 2^{-n}(1-x)] \\ &= l_{2^{-k}}(x) - [l_{2^{-k}-2^{-n}}(x) - 2^{-n}(1-x)]. \end{aligned}$$

But the lines $l_{2^{-k}}(x)$ and $l_{2^{-k}-2^{-n}}(x) - 2^{-n}(1-x)$ are parallel, so

$$\begin{aligned} \left| \bigcup_{l \in L_j} R_{2^{-n}}(l) \right|_x &\leqslant l_{2^{-k}}\left(\frac{k-1}{n}\right) - \left[l_{2^{-k}-2^{-n}}\left(\frac{k-1}{n}\right) - 2^{-n}\left(1 - \frac{k-1}{n}\right)\right] \\ &= 0 - \left[\sum_{k+1}^{n} \frac{k-i}{n} 2^{-i} - 2^{-n}\left(1 - \frac{k-1}{n}\right)\right] \\ &= \frac{2^{1-k} - 2^{-n}}{n}. \end{aligned}$$

Hence

$$\left| \bigcup_{l} R_{2^{-n}}(l) \right|_x \leqslant 2^{k-1} \frac{2^{1-k} - 2^{-n}}{n} < \frac{1}{n}.$$

3. *The exact Minkowski dimension*

Let F be a subset of $\mathbb{R}^2$. For a monotone increasing function f on $\mathbb{R}$ and $\delta > 0$, we define

$$\mathfrak{M}_f(F, \delta) \stackrel{d}{=} \inf\left\{ N \cdot f(r) : \bigcup_{i=1}^{N} D(x_i, r) \supset F \text{ and } r < \delta \right\}.$$

Let $\mathfrak{M}_f(F) \stackrel{d}{=} \sup_\delta \mathfrak{M}_f(F, \delta)$. By the exact Minkowski dimension for the class of Kakeya sets, we mean a monotone increasing function h such that:

- for any Kakeya set E, $\mathfrak{M}_h(E) > 0$;
- there exists a Kakeya set E with $\mathfrak{M}_h(E) < \infty$.

CLAIM 3.1. For any n, there exists a G-set, G^n, such that

$$|G^n(2^{-n})| \lesssim \frac{1}{\log 2^n}.$$

Proof. Consider the set of triangles $E_n = \bigcup_i R_{2^{-n}}(l_i^n)$ that was constructed in the proof of Theorem 1. Let I be the identity map on $\mathbb{R}^2$. Then by (i) of the theorem,

$$|6I(E_n)| = \left| \bigcup_i 6I(R_{2^{-n}}(l_i^n)) \right| < \frac{36}{n}. \tag{7}$$

Let $a \stackrel{d}{=} a(l_i^n) \in [0,1]$ and $b \stackrel{d}{=} b(l_i^n) \in [-1,0]$. We define the triangle $\hat{R}_i^n$ by its vertices as follows:

$$V(\hat{R}_j^n) \stackrel{d}{=} \{(1, a+6b-2\cdot 2^{-n}), (1, a+6b-3\cdot 2^{-n}), (2, 2a+6b-2\cdot 2^{-n})\}.$$

Since $V(R_{2^{-n}}(l_i^n)) = \{(0,b), (0, b-2^{-n}), (1, a+b)\}$, it is easy to verify that $\hat{R}_i^n$ is a translation of $R_{2^{-n}}(l_i^n)$, and that

$$\hat{R}_i^n(2^{-n}) \subset 6I(R_{2^{-n}}(l_i^n)).$$

Hence $|\bigcup_i \hat{R}_i^n(2^{-n})| < 36/n$, and translating the triangles $\hat{R}_i^n$ to the left gives our G-set.

REMARKS. The set G^n constructed in the above claim is contained in $[0,1] \times [-6,6]$.

When $\delta = 2^{-n}$, we shall also refer to G^n as G^δ.

Proof of Lemma 1. The proof is an adaptation of a standard limiting argument (for example, Lemma 1.3 and Corollary 1.4 in [**6**]). Let $\varepsilon_n \stackrel{d}{=} 2^{-2^n}$. Then it suffices to prove that (5) holds for ε_n. Suppose that there exists a sequence of G-sets, F_n, such that

(i) $F_n(\varepsilon_n) \subset F_{n-1}(\varepsilon_{n-1})$,

(ii) $|\overline{F_n(2\varepsilon_n)}| \lesssim 2^{-n}$.

Let $E \stackrel{d}{=} \bigcap_n \overline{F_n(\varepsilon_n)}$. Then by (i), E is a G-set. Moreover,

$$E(\varepsilon_n) \subset (F_n(\varepsilon_n))(\varepsilon_n) = F_n(2\varepsilon_n),$$

hence (ii) proves the lemma. Next, we inductively construct the sequence F_n.

Start with, say, $F_0 = G^{1/2}$. Given F_n, we define F_{n+1} so that (i) and (ii) will be satisfied. Since F_n is a G-set, it contains a unit length line segment l_{m_j} for slopes $m_j = j\delta$, where δ is short for $\delta_{n+1} \stackrel{d}{=} \varepsilon_n/256 = 2^{-2^n-8}$, and $j = 0, 1, \ldots, \delta^{-1}-1$. Let $A_j^\delta : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $A_j^\delta((x,y)) \stackrel{d}{=} (x, l_{m_j}(x) + \delta y)$. Note that A_j^δ affinely maps $[0,1] \times [-6,6]$ onto the parallelogram $S_j^\delta \stackrel{d}{=} \{(x,y) : x \in [0,1] \text{ and } |y - l_{m_j}(x)| \leqslant 6\delta\}$.

Let η stand for $\eta_{n+1} \stackrel{d}{=} 2^{-2^n+11}$, and define

$$F_{n+1} \stackrel{d}{=} \bigcup_j A_j^\delta(G^\eta).$$

Since A_j^δ maps segments with slope μ to segments with slope $\delta\mu + m_j$, F_{n+1} is a G-set. Since $\delta = \varepsilon_n/256$, for each j,

$$[A_j^\delta(G^\eta)](\varepsilon_{n+1}) \subset (l_{m_j})(12\delta + \varepsilon_{n+1}) \subset F_n(\varepsilon_n),$$

and (i) follows.

As for (ii), note that with $\delta \in (0,1]$ and $m \in [0,1]$,

$$(x_1 - x_2)^2 + [m(x_1 - x_2) + \delta(y_1 - y_2)]^2 < \delta^2 \rho^2$$

implies

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 < 5\rho^2.$$

Hence

$$[A_j^\delta(G^\eta)]\left(\frac{\delta\eta}{4}\right) \subset A_j^\delta[G^\eta(\eta)],$$

and so as $2\varepsilon_{n+1} = \delta\eta/4$,

$$F_{n+1}(2\varepsilon_{n+1}) = \bigcup_j [A_j^\delta(G^\eta)]\left(\frac{\delta\eta}{4}\right) \subset \bigcup_j A_j^\delta[G^\eta(\eta)].$$

Since A_j^δ reduces areas by a factor of δ, by Claim 3.1,

$$|A_j^\delta[G^\eta(\eta)]| \leqslant C\delta \frac{1}{\log \eta^{-1}},$$

which implies that

$$|F_{n+1}(2\varepsilon_{n+1})| \leqslant \sum_j C\delta \frac{1}{\log \eta^{-1}} = \frac{C}{\log \eta^{-1}}.$$

The proof is now completed by observing that

$$\log \frac{1}{2\varepsilon_{n+1}} \approx 2^{n+1} \approx \log \frac{1}{\eta_{n+1}}.$$

Proof of Theorem 2. For any $r > 0$ and a covering of a Kakeya set E by N_r discs of radius r, we have $N_r r^2 \gtrsim |E(r)|$, so by (3),

$$N_r h(r) = N_r r^2 \log \frac{1}{r} \gtrsim |E(r)| \log \frac{1}{r} \gtrsim 1.$$

Thus $\mathfrak{M}_h(E,\delta) \gtrsim 1$, and so $\mathfrak{M}_h(E) > 0$. On the other hand, let E be the Kakeya set obtained from the construction in Lemma 1. For any $\delta > 0$, there exists a covering of E by $N_\delta \approx |E(\delta)|/\delta^2$ discs of radius δ. With this covering and by Lemma 1, we have

$$\mathfrak{M}_g(E,\delta) \lesssim N_\delta \delta^2 \log \frac{1}{\delta} \lesssim |E_\delta| \log \frac{1}{\delta} \lesssim 1.$$

As for the exact Hausdorff dimension of the class of Kakeya sets in $\mathbb{R}^2$, our results are not sharp. You can borrow the lower bound of $h \geqslant r^2 \log(1/r)$ from the analysis of the Minkowski dimension, but the upper bound we currently have is strictly larger.

CLAIM 3.2. Let E be a Kakeya set, and for $\varepsilon > 0$, let

$$h_\varepsilon(r) \stackrel{d}{=} r^2 \log \frac{1}{r} \left(\log \log \frac{1}{r}\right)^{2+\varepsilon}.$$

Then there exists a $C_\varepsilon > 0$ such that for any covering of E by $\bigcup_i D(x_i, r_i)$ with $r_i < \delta$, $\sum_i h_\varepsilon(r_i) \geqslant C_\varepsilon$.

Proof. The proof is a variation on Lemma 2.15 in [**1**]. Let

$$J_k \stackrel{d}{=} \{j : 2^{-2^k} \leqslant r_j \leqslant 2^{-2^{k-1}}\},$$

and let $v_k \stackrel{d}{=} |J_k|$. Since, for small r and $c > 1$, $h(cr) < c^2 h(r)$, we can assume without loss of generality that $r_i = m_i 2^{-2^k}$ with $m_i \in \{1, 2, \ldots, 2^{2^{k-1}}\}$. Each disc $D(x, m \cdot 2^{-2^k})$ can be covered by $\lesssim m^2$ discs of radius 2^{-2^k}, and since

$$\frac{h(m \cdot 2^{-2^k})}{m^2 h(2^{-2^k})} \gtrsim \frac{\log 2^{2^{k-1}} [\log\log(2^{2^{k-1}})]^{2+\varepsilon}}{\log 2^{2^k} [\log\log(2^{2^k})]^{2+\varepsilon}} \approx \frac{1}{2},$$

we can assume, without loss of generality, that $r_j = 2^{-2^k}$ for all $j \in J_k$.

Retaining the notation in [**6**], denote $D(x_j, r_j)$ by D_j, and let

$$E_k \stackrel{d}{=} E \cap (\bigcup_{j \in J_k} D_j), \quad \tilde{D}_j \stackrel{d}{=} D(x_j, 2r_j), \quad \tilde{E}_k \stackrel{d}{=} \bigcup_{j \in J_k} \tilde{D}_j.$$

Let $e \in S^1$. Since E is a Kakeya set, there exists a unit length line segment in the e-direction, l_e, contained in E. Suppose that $|l_e \cap E_k| > C/k^{1+\varepsilon}$ for some $C > 0$. Then, as explained in [**6**], $K_{2^{-2^k}}(\chi_{\tilde{E}_k})(e) > C/k^{1+\varepsilon}$, thus

$$\left|\left\{e \in S^1 : K_{2^{-2^k}}(\chi_{\tilde{E}_k})(e) > \frac{C}{k^{1+\varepsilon}}\right\}\right| \geqslant \left|\left\{e \in S^1 : |l_e \cap E_k| > \frac{C}{k^{1+\varepsilon}}\right\}\right|_*,$$

where $|F|_*$ is the outer measure of F. Note that $|\tilde{E}_k| \lesssim v_k (2^{-2^k})^2$, so (3) with $p = 2$ yields

$$v_k h(2^{-2^k}) \gtrsim \frac{|\tilde{E}_k| \log 2^{2^k}}{(1/k)^{2+\varepsilon}} \gtrsim \left|\left\{e \in S^1 : K_{2^{-2^k}}(\chi_{\tilde{E}_k})(e) > \frac{C}{k^{1+\varepsilon}}\right\}\right|.$$

Summing over k, we find that

$$\sum_j h(r_j) \gtrsim \left|\bigcup_k \left\{e \in S^1 : |l_e \cap E_k| > \frac{C}{k^{1+\varepsilon}}\right\}\right|_*.$$

But for each $e \in S^1$, $\sum_k |l_e \cap E_k| = 1$, so if we let $C \stackrel{d}{=} (\sum_k 1/k^{1+\varepsilon})^{-1}$, then by the pigeonhole principle, the union is S^1, and therefore $\sum_j h(r_j) \gtrsim 1$.

Acknowledgement. I should like to express my gratitude to Tom Wolff for his invaluable advice.

References

1. J. Bourgain, 'Besicovitch type maximal operators and applications to Fourier analysis', *Geom. Funct. Anal.* 1 (1991) 147–187.
2. A. Cordoba, 'The Kakeya maximal function and spherical summation multipliers', *Amer. J. Math.* 99 (1977) 1–22.
3. D. Müller, 'A note on the Kakeya maximal function', *Arch. Math.* 49 (1987) 66–71.
4. E. Sawyer, 'Families of plane curves having translates in a set of measure zero', *Mathematika* 34 (1987) 69–76.

5. I. J. SCHOENBERG, 'On the Besicovitch–Perron solution of the Kakeya problem', *Studies in mathematical analysis and related topics* (Stanford University Press, 1962) 359–363.

6. T. WOLFF, 'Recent work connected with the Kakeya problem', *Prospects in mathematics: Invited talks on the occasion of the 250th Anniversary of Princeton University* (ed. Hugo Rossi, Amer. Math. Soc., Providence, RI, 1998) 129–162.

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