

**Charged-particle decay at finite temperature**Andrzej Czarnecki,<sup>1</sup> Marc Kamionkowski,<sup>2,3</sup> Samuel K. Lee,<sup>2,3</sup> and Kirill Melnikov<sup>3</sup><sup>1</sup>*Department of Physics, University of Alberta, Edmonton, AB, Canada T6G 2G7*<sup>2</sup>*California Institute of Technology, Mail Code 350-17, Pasadena, California 91125, USA*<sup>3</sup>*Department of Physics and Astronomy, Johns Hopkins University, Baltimore, Maryland 21218, USA*

(Received 17 October 2011; published 26 January 2012)

Radiative corrections to the decay rate of charged fermions caused by the presence of a thermal bath of photons are calculated in the limit when temperatures are below the masses of all charged particles involved. The cancellation of finite-temperature infrared divergences in the decay rate is described in detail. Temperature-dependent radiative corrections to a two-body decay of a hypothetical charged fermion and to electroweak decays of a muon  $\mu \rightarrow e\nu_\mu \bar{\nu}_e$  are given. We touch upon possible implications of these results for charged particles in the early Universe.

DOI: 10.1103/PhysRevD.85.025018

PACS numbers: 12.20.-m, 14.60.Ef, 95.30.Cq, 98.80.Cq

**I. INTRODUCTION**

Precise predictions for decay rates of charged particles might be of interest in a variety of cosmological contexts that introduce long-lived particles with electric charge. These include scenarios for modified big-bang nucleosynthesis [1] and small-scale-power suppression [2] and mechanisms for dark-matter detection wherein a charged quasistable heavier particle is produced [3].

In the early Universe, decays of charged particles occur in a thermal bath whose very presence seems to affect decay rates in a peculiar way. Indeed, consider the decay of a hypothetical charged particle  $\psi$  to two lighter particles  $\chi$  and  $\phi$ . The leading-order Feynman diagram is shown in Fig. 1. In a thermal bath of photons, the process  $\gamma\psi \rightarrow \chi\phi$  also occurs and modifies the vacuum decay rate. Hence, the vacuum decay rate of a particle  $\psi$  must be augmented in the cosmological context by the inclusion of the rate for the “induced” decay wherein the unstable particle absorbs a thermal photon (see Fig. 2). It is easy to see that a naive computation of the diagrams in Fig. 2 leads to a divergent result. This divergence is of the infrared type—it appears when the energy of the absorbed photon becomes very small. In this paper we discuss in detail how, when all possible processes that modify the vacuum decay rate are taken into account, the infrared divergences cancel out. This is a finite-temperature analog of the celebrated cancellation of infrared divergences in QED pointed out by Bloch and Nordsieck long ago [4].

Although thermal effects have been computed for static thermodynamic quantities such as the effective potential, the free energy, the pressure, and so on [5], sometimes to very high orders in the perturbative expansion in QCD and QED (for recent reviews, see Refs. [6]), less is known about finite-temperature corrections to cross sections and decay rates. Radiative corrections to dynamical scattering and decay processes at a finite temperature  $T$  are peculiar for three reasons. First, as pointed out already, if  $T \neq 0$ , new processes involving absorption and emission of

particles from the heat bath contribute to cross sections and decay rates. The second complication is that the preferred reference frame defined by the heat bath spoils Lorentz invariance. Third, thermal averages and loop integrals over Bose-Einstein distributions introduce infrared divergences that are powerlike, rather than logarithmic.

Pioneering studies of radiative corrections to neutron  $\beta$ -decays at finite temperature were first described in Ref. [7], in the context of big-bang nucleosynthesis. In Ref. [8], the finite-temperature decay rate of a neutral Higgs boson into two charged leptons was first computed. These and subsequent papers [9] illustrated the cancellation of infrared divergences and clarified many important features of radiative corrections. They also discussed the issue of radiative corrections that are enhanced by the logarithms of small masses of final-state charged particles. Such terms are known to cancel in total decay rates at zero temperature [10], but the situation at finite temperature is less clear.

Most of the papers just described dealt only with a neutral initial state; in such a case, the problem of an infinite decay rate induced by absorption of very soft photons by the initial state does not occur. Here we discuss the calculation of radiation corrections for a charged initial state, where this issue can not be avoided. For simplicity, we begin by considering a toy model of charged-fermion

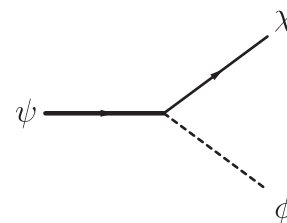


FIG. 1. The diagram for the decay  $\psi \rightarrow \chi\phi$ . Note that we shall distinguish between the  $\psi$  and  $\chi$  particles in the diagrams below by representing them with thick and thin solid lines, respectively.

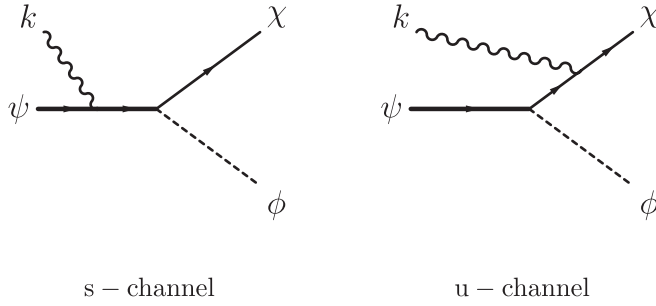


FIG. 2. The two diagrams via which absorption of a photon can lead to induced  $\psi$  decay (or  $\chi$  and  $\phi$  production).

decay and focus on the low-temperature case. We introduce the toy model in Sec. II and calculate the decay rate induced by real-radiation scattering processes in the thermal bath (i.e., absorption and emission of photons), showing how infrared divergences arise. In Sec. III, we compute the virtual radiative corrections to the decay rate. In Sec. IV, we sum the real and virtual corrections to find the total finite-temperature decay rate, and demonstrate the cancellation of the divergences at first order in perturbation theory. We carry out an analogous analysis for muon decay in Sec. V. We conclude and consider the implications for charged particles in the early Universe in Sec. VI.

## II. TOY MODEL

We shall first discuss a simple model to illustrate the nature of the infrared divergences and their cancellation. Consider the process  $\psi \rightarrow \chi\phi$ , the decay of a heavy charged fermion  $\psi$  to a light charged fermion  $\chi$  and a massless neutral scalar  $\phi$  via the interaction

$$\mathcal{L} \supset g\phi\bar{\psi}L\chi + \text{H.c.}, \quad (1)$$

depicted in Fig. 1. Here  $L = (1 - \gamma_5)/2$ . We shall assume that the charge of both fermions is the elementary charge  $e = \sqrt{4\pi\alpha}$ , and that the mass ratio  $\epsilon \equiv m_\chi/m_\psi$  is small.

At  $T = 0$ , the tree-level amplitude for the decay  $\psi \rightarrow \chi\phi$  is given by

$$\mathcal{M}_{\text{tr}} = g\bar{u}_\chi L u_\psi. \quad (2)$$

This amplitude gives the  $\mathcal{O}(g^2\alpha^0)$  zero-temperature decay rate,

$$\tilde{\Gamma}_0 = \Gamma_0(1 - \epsilon^4), \quad (3)$$

where we have defined  $\Gamma_0 \equiv \frac{g^2}{32\pi}m_\psi$ . We will state our subsequent results for the temperature-dependent decay rate in terms of  $\Gamma_0$ .

On account of radiative corrections and finite-temperature effects, the decay rate can be written as a triple series in  $\tau = T/m_\psi$ ,  $\epsilon = m_\chi/m_\psi$ , and the fine-structure constant  $\alpha$ . Unless explicitly stated otherwise, we work in the low-temperature approximation  $\tau \ll \epsilon \ll 1$ . Our goal is to compute relative corrections to the decay rate that

scale as  $\alpha\tau^2$ . In those terms, we shall set  $\epsilon \rightarrow 0$ . Note that terms of the form  $\tau/\epsilon$  do not appear in the total decay rate.

### A. Photon absorption

We now consider the process  $\gamma\psi \rightarrow \chi\phi$ , the induced decay of  $\psi$  in a thermal bath of photons. This process can occur via the two diagrams shown in Fig. 2. For a photon with 4-momentum  $k = (\omega, \mathbf{k})$ , where  $\omega = |\mathbf{k}|$ , the tree-level amplitudes for these channels are

$$\begin{aligned} \mathcal{M}_{\text{abs,s}} &= \frac{eg}{s - m_\psi^2} \bar{u}_\chi L (\not{p}_\psi + \not{k} + m_\psi) \not{\epsilon}_\gamma u_\psi, \\ \mathcal{M}_{\text{abs,u}} &= \frac{eg}{u - m_\chi^2} \bar{u}_\chi \not{\epsilon}_\gamma (\not{p}_\chi - \not{k} + m_\chi) L u_\psi, \end{aligned} \quad (4)$$

giving the total amplitude  $\mathcal{M}_{\text{abs}} = \mathcal{M}_{\text{abs,s}} + \mathcal{M}_{\text{abs,u}}$ . We use the amplitude to compute the cross section for  $\gamma\psi \rightarrow \chi\phi$  by the standard procedure. We assume that the  $\psi$  particle is at rest with respect to the photon bath and express the result in terms of the energy of the photon  $w \equiv \omega/m_\psi$ .<sup>1</sup> We find

$$\begin{aligned} \sigma_{\text{abs}}(w) &\equiv \frac{1}{2m_\psi} \frac{1}{2|\omega|} \int d\text{LIPS}_{\chi\phi} (2\pi)^4 \\ &\quad \times \delta^4(p_\psi + k - p_\chi - p_\phi) \langle |\mathcal{M}_{\text{abs}}(k)|^2 \rangle \\ &= \Gamma_0 \frac{\alpha\pi}{m_\psi^3 w^3} \rho(w), \end{aligned} \quad (5)$$

where

$$\rho(w) \equiv (1 + 2w + 2w^2) \ln \frac{1 + 2w}{\epsilon^2} - (2 + 4w + 3w^2), \quad (6)$$

at leading order in  $\epsilon$ . Note that  $\rho \propto w^0$  and  $\sigma_{\text{abs}} \propto w^{-3}$  as  $w \rightarrow 0$ .

To compute corrections to the decay rate of a particle  $\psi$  due to the absorption of thermal photons from the heat bath, we need to integrate the cross section  $\sigma_{\text{abs}}(w)$  in Eq. (5) multiplied with the average occupation number for thermal photons. We find

$$\begin{aligned} \Gamma_{\text{abs}}^T &= \int dw \frac{dn_\gamma}{dw}(w) \sigma_{\text{abs}}(w) = g_\gamma \int \frac{d^3\mathbf{k}}{(2\pi)^3} f_B(\omega) \sigma_{\text{abs}}(w) \\ &= \frac{\alpha}{\pi} \Gamma_0 \int_0^\infty \frac{dw}{w} f_B(\omega) \rho(w), \end{aligned} \quad (7)$$

where  $f_B(\omega) = 1/(e^{\omega/T} - 1) = 1/(e^{w/\tau} - 1)$  is the Bose-Einstein distribution function and  $g_\gamma = 2$  is the number of independent photon polarizations. Since  $f_B \propto w^{-1}$  and

<sup>1</sup>We are implicitly considering a cosmological scenario in which the heavy  $\psi$  particle has decoupled from the thermal bath and is out of equilibrium. Furthermore, we assume that the massless  $\phi$  particle is also not thermalized in the bath (which may be the case if the coupling  $g$  is very weak and  $\psi$  decay is the dominant mode of  $\phi$  production, for example). Similar assumptions will also be made in the case of muon decay discussed in Sec. V.

$\rho \propto w^0$  as  $w \rightarrow 0$ , we see that the integrand in Eq. (7) is proportional to  $w^{-2}$  as  $w \rightarrow 0$ , and hence  $\Gamma_{\text{abs}}^T$  indeed has a powerlike infrared divergence, implying an infinite decay rate.

As in the case of infrared divergences at zero temperature, the infinite rate is unphysical. We arrived at this unphysical result because we considered only the photon absorption process  $\gamma\psi \rightarrow \chi\phi$  in the calculation of the finite-temperature  $\psi$  decay rate. However, as we shall demonstrate, we cannot consider this absorption process independently of other processes that also result in  $\psi$  decay. In particular, we must also take into account the finite-temperature rates of the radiative decay process  $\psi \rightarrow \gamma\chi\phi$  and the decay process  $\psi \rightarrow \chi\phi$ . At finite temperature, the emission of photons in the first process is stimulated by the presence of photons in the thermal bath. At the same order in  $\alpha$ , the second process is affected by  $T$ -dependent additions to the virtual-photon propagator. When all of these processes are included in the calculation, all  $T$ -dependent infrared divergences cancel to yield a finite rate. The nature of this cancellation is similar to zero-temperature cancellations of infrared divergences in QED, as described in a classic paper by Bloch and Nordsieck [4]. We shall now demonstrate this cancellation to  $\mathcal{O}(g^2\alpha)$ .

### B. Photon emission

We begin by considering the photon-emitting radiative decay process  $\psi \rightarrow \gamma\chi\phi$ . The two contributing diagrams are shown in Fig. 3. Comparing these diagrams to those in Fig. 2, we see that the amplitudes for emission are formally equivalent to those for absorption given in Eq. (4) if we make the substitutions  $k \leftrightarrow -k$  and  $\epsilon_\gamma \leftrightarrow \epsilon_\gamma^*$ . This crossing symmetry gives the photon-emission amplitude from the photon-absorption amplitude,

$$\langle |\mathcal{M}_{\text{em}}(k)|^2 \rangle = g_\gamma \langle |\mathcal{M}_{\text{abs}}(-k)|^2 \rangle, \quad (8)$$

where the factor of  $g_\gamma$  arises because we do not average over photon polarizations on the left-hand side. The  $\mathcal{O}(g^2\alpha)$   $T$ -dependent part of the rate for this process is then given by

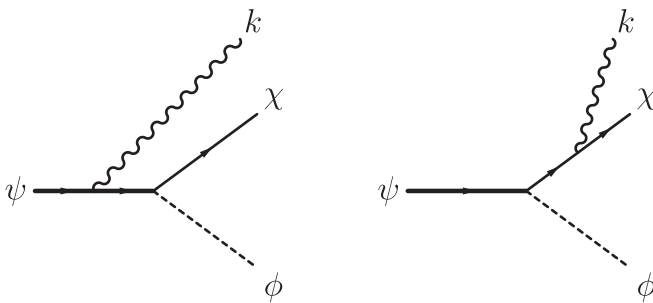


FIG. 3. The diagrams for radiative  $\psi$ -decays. Note that the topologies of the diagrams are identical to those for absorption in Fig. 2, with the exception of the photon line placement. This results in the relation given in Eq. (8).

$$\Gamma_{\text{em}}^T = \frac{1}{2m_\psi} \int d\text{LIPS}_{\gamma\chi\phi} f_B(\omega) (2\pi)^4 \times \delta^4(p_\psi - k - p_\chi - p_\phi) \langle |\mathcal{M}_{\text{em}}(k)|^2 \rangle, \quad (9)$$

where the factor of  $f_B(\omega)$  comes from the  $T$ -dependent part of the  $(1 + f_B)$  Bose-enhancement factor for the final-state photons. Comparing Eq. (9) to Eq. (7), and using Eqs. (5) and (8), it can be shown that the expression for the emission rate is very similar to that for the absorption rate given in Eq. (7),

$$\Gamma_{\text{em}}^T = \frac{\alpha}{\pi} \Gamma_0 \int_0^{1/2} \frac{dw}{w} f_B(\omega) \rho(-w). \quad (10)$$

We see that the only differences are  $\rho(w) \rightarrow \rho(-w)$ , arising from the  $k \rightarrow -k$  substitution used to switch the absorbed photon to an emitted photon, and the limits of integration. The upper limit reflects that the emitted photon is limited by the kinematics to have  $w < 1/2$  in the final state, whereas an absorbed photon is allowed to have any energy in the initial state. Note that this is further reflected in the fact that  $\rho(w)$  is defined only for  $-1/2 < w < \infty$ .

### C. Real-radiation corrections

We are now in position to calculate the total rate of  $\psi$  decay due to processes involving either the absorption or emission of real photons,

$$\Gamma_{\text{real}}^T = \Gamma_{\text{abs}}^T + \Gamma_{\text{em}}^T = \frac{\alpha}{\pi} \Gamma_0 \int_0^\infty \frac{dw}{w} f_B(\omega) \rho_{\text{real}}(w), \quad (11)$$

where  $\rho_{\text{real}}(w) = \rho(w) + \theta(1/2 - w)\rho(-w)$ . Exact integration over  $w$  in the above formula is complicated because of the Bose-Einstein factor. However, if we consider the low-temperature case  $\tau \ll 1$ , then  $f_B(\omega)$  is only non-negligible for  $w \leq \tau \ll 1/2$ . We can thus use the approximation  $\theta(1/2 - w) \rightarrow 1$  and integrate Eq. (11) by expanding  $\rho_{\text{real}}(w)$  in a Taylor series in  $w$ . This approximation picks up all the terms that are suppressed by powers of  $\tau$ , but it misses the exponentially suppressed terms of  $\mathcal{O}(e^{-1/\tau})$ . The  $\mathcal{O}(g^2\alpha)$  result is then

$$\Gamma_{\text{real}}^T = \frac{\alpha}{\pi} \Gamma_0 \{ [-4 - 4 \ln \epsilon + \mathcal{O}(\epsilon^2 \ln \epsilon)] J_{-1} + [-2 - 8 \ln \epsilon + \mathcal{O}(\epsilon^2)] J_1 \tau^2 + \mathcal{O}(\tau^4, e^{-1/\tau}) \}, \quad (12)$$

where we have defined the integrals

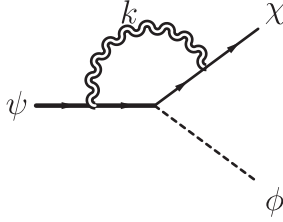
$$J_n \equiv \lim_{x_0 \rightarrow 0} \int_{x_0}^\infty dx x^n f_B(xT) \theta(x - x_0). \quad (13)$$

Note that for  $n > 0$ ,  $J_n = \text{Li}_{n+1}(1) \Gamma(n+1)$  is finite.<sup>2</sup> The infrared-divergent part of the decay rate due to real-radiation processes is thus given by the  $J_{-1}$  term in Eq. (12); we shall now proceed to show that it is canceled

<sup>2</sup>A few terms read  $J_1 = \frac{\pi^2}{6}$ ,  $J_3 = \frac{\pi^4}{15}$ ,  $J_5 = \frac{8\pi^6}{63}$ ,  $J_7 = \frac{8\pi^8}{15}$ .



$$= -2\pi g_{\mu\nu} f_B(|k^0|) \delta(k^2)$$

FIG. 4. The  $T$ -dependent part of the photon propagator.FIG. 5. The diagram contributing to the  $T$ -dependent part of the  $\mathcal{O}(\alpha)$  correction to the vertex.

by corresponding terms in the virtual corrections to the decay rate.

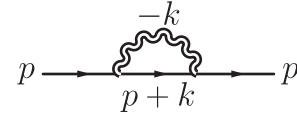
### III. VIRTUAL CORRECTIONS

As previously mentioned, the rate of the decay process  $\psi \rightarrow \chi\phi$  is affected by  $T$ -dependent additions to the virtual corrections that enter at  $\mathcal{O}(g^2\alpha)$ . These affect both the vertex correction shown in Fig. 5 and the charged-fermion self-energies shown in Fig. 6. At finite temperature, the bare propagator for the virtual photons that appear in these diagrams is modified compared to the zero-temperature case,

$$-\frac{ig_{\mu\nu}}{k^2 + i0} \rightarrow -g_{\mu\nu} \left[ \frac{i}{k^2 + i0} + 2\pi f_B(|k^0|) \delta(k^2) \right]. \quad (14)$$

The first term in this equation is the usual  $T = 0$  photon propagator; its effects are accounted for in conventional zero-temperature perturbation theory. The second term in the right-hand side of Eq. (14) leads to temperature-dependent corrections to the decay rate (see Fig. 4). We note that the temperature-dependent contribution to the photon propagator accounts for interactions of real, on-shell photons from the thermal bath with the charged fermions in the initial and final states. In particular, it represents processes in which a photon from the thermal bath is absorbed by either of the fermions, while simultaneously another photon is emitted by either of the fermions with the exact same momentum and polarization as the initial photon.<sup>3</sup>

<sup>3</sup>We can see how the term arises by expanding the photon field in terms of creation and annihilation operators in the usual manner. At  $T = 0$ ,  $aa^\dagger$  terms generate the usual propagator, but the  $a^\dagger a$  terms proportional to the particle number vanish. At finite temperature, the latter terms are instead proportional to the occupation number  $f_B$  and hence are nonvanishing.

FIG. 6. The diagram contributing to the  $T$ -dependent part of the fermion self-energy.

#### A. Vertex correction

We first consider the  $T$ -dependent part of the  $\mathcal{O}(\alpha)$  correction to the vertex, shown in Fig. 5. The  $T$ -dependent part of the relevant amplitude is given by

$$\mathcal{M}_{\text{vert}}^T = -e^2 g \int \frac{d^4 k}{(2\pi)^3} F(k^0, \mathbf{k}) f_B(|k^0|) \delta(k^2), \quad (15)$$

where

$$F(k^0, \mathbf{k}) \equiv \frac{\bar{u}_\chi \gamma^\mu (\not{p}_\chi - \not{k} + m_\chi) L(\not{p}_\psi - \not{k} + m_\psi) \gamma_\mu u_\psi}{[(p_\chi - k)^2 - m_\chi^2][(p_\psi - k)^2 - m_\psi^2]}. \quad (16)$$

We may use the properties of the gamma matrices to simplify this expression. Integration over  $k^0$  removes  $\delta(k^2)$ ,

$$\mathcal{M}_{\text{vert}}^T = -e^2 g \int \frac{d^3 \mathbf{k}}{(2\pi)^3 2\omega} [F(\omega, \mathbf{k}) + F(-\omega, \mathbf{k})] f_B(\omega), \quad (17)$$

where  $\omega = |\mathbf{k}|$ . Now, by changing the variable of integration from  $\mathbf{k} \rightarrow -\mathbf{k}$  in the second term in the sum enclosed in brackets, the sum becomes  $[F(k) + F(-k)]$ . Examining Eq. (16), we see that the numerator of this sum is then independent of  $k$ , since  $k^2 = 0$  and terms that are linear in  $k$  cancel. However, the denominator of the sum retains its quadratic dependence on  $k$ , since it is proportional to  $(p_\chi \cdot k)(p_\psi \cdot k)$ . By counting powers of  $\omega$  and recalling that  $f_B \propto \omega^{-1}$  at small values of  $\omega$ , we observe that the integral in Eq. (17) indeed has a powerlike divergence.

We can then take the interference of this amplitude with the tree-level amplitude given in Eq. (2). The  $T$ -dependent part of the vertex correction to the decay rate then follows by taking the usual spin-sum average and evaluating the resulting integral over  $\mathbf{k}$ , using the appropriate kinematics of the 2-body decay in the rest frame of the  $\psi$  particle. Writing the result in terms of the integrals  $J_n$  as before, we find at  $\mathcal{O}(g^2\alpha)$

$$\Gamma_{\text{vert}}^T = \frac{\alpha}{\pi} \Gamma_0 \{ [4 \ln \epsilon + \mathcal{O}(\epsilon^2 \ln \epsilon)] J_{-1} \}. \quad (18)$$

Comparing to the decay rate due to real-radiation processes in Eq. (12), we see that part of the divergent  $J_{-1}$  term is indeed canceled.

### B. Self-energy corrections

The remaining part of the infrared divergence is canceled by the  $T$ -dependent corrections to the fermion self-energy  $\Sigma^T(p)$ , which enter via the virtual photon propagator in Fig. 6 and lead to  $T$ -dependent contributions to the full (dressed) fermion propagator  $S_F^T(p)$ . At  $T = 0$ , these self-energy contributions are conveniently treated through the mass shift  $\delta m$ , and the wave-function renormalization factor  $Z_2$ . The wave-function renormalization factors are usually obtained as derivatives of the self-energy  $\Sigma$  with respect to  $p$ , evaluated on the mass shell  $p^2 = m^2$ . This treatment relies on the fact that at  $T = 0$ ,  $\Sigma$  depends only on the momentum of the particle  $p$ . Unfortunately, this feature is violated at finite temperature because the thermal bath introduces a preferred reference frame. As a result, the self-energy of a particle at rest and the self-energy of a particle in motion are not related in an obvious way.

We will need expressions for the self-energy of both the  $\psi$  and  $\chi$  fermions. We can consider more generally the  $T$ -dependent part of the self-energy  $\Sigma^T(p)$  of a fermion with electric charge  $e$  and  $T = 0$  physical mass  $m$ . This calculation has been discussed extensively in the literature; below, we loosely follow the formalism laid out in Ref. [11]. Our ultimate goal will be to use the expression for  $\Sigma^T$  to show that, in the limit  $p^2 \rightarrow m_T^2$ , the full finite-temperature fermion propagator takes the form

$$S_F^T(p) = Z_2^T \frac{i \sum_s u_s^T(p) \bar{u}_s^T(p)}{p^2 - m_T^2}. \quad (19)$$

That is, the pole of the propagator is shifted to the finite-temperature physical mass  $m_T$ , and the fermion wave functions are given by  $\Psi_s(p) = \sqrt{Z_2^T/2p^0} u_s^T(p) e^{-ip \cdot x}$ , where  $Z_2^T$  and  $u_s^T(p)$  are the finite-temperature wave-function renormalization factor and the finite-temperature spinor, respectively.<sup>4</sup> This form implies that self-energy corrections to the decay rate will follow from three distinct sources: (1) matrix elements will be multiplied by a factor of  $Z_2^T$  for each external fermion line, (2) the shift in the physical mass, which will effectively modify the fermion phase-space, and (3) the spinor completeness relation will be modified, affecting the evaluation of spin sums.

To this end, we start by finding the self-energy  $\Sigma^T$  at  $\mathcal{O}(\alpha)$ . We take  $p$  to be the off-shell fermion momentum and  $-k$  to be the momentum of the photon in the loop, as shown in Fig. 6. The self-energy reads

$$\Sigma^T(p) = 2e^2 \int \frac{d^4 k}{(2\pi)^3} \frac{\not{p} + \not{k} - 2m}{(p+k)^2 - m^2} f_B(|k^0|) \delta(k^2). \quad (20)$$

It is convenient to decompose  $\Sigma^T(p)$  as

$$\Sigma^T(p) = \not{p} c_B(p) - 2m c_B(p) + \not{K}(p), \quad (21)$$

where

$$\begin{aligned} c_B(p) &\equiv 2e^2 \int \frac{d^4 k}{(2\pi)^3} \frac{f_B(|k^0|) \delta(k^2)}{(p+k)^2 - m^2}, \\ K^\mu(p) &\equiv 2e^2 \int \frac{d^4 k}{(2\pi)^3} k^\mu \frac{f_B(|k^0|) \delta(k^2)}{(p+k)^2 - m^2} \\ &\xrightarrow{p^2=m^2} \frac{\alpha}{\pi} J_1 \frac{T^2}{|\mathbf{p}|} \left( L_p \frac{\mathbf{p}}{|\mathbf{p}|} \left[ \frac{p^0}{|\mathbf{p}|} L_p - 2 \right] \right), \end{aligned} \quad (22)$$

and  $L_p = \ln \frac{p^0 + |\mathbf{p}|}{p^0 - |\mathbf{p}|}$ . We can now use these results for  $\Sigma^T$  to find the full (dressed) finite-temperature fermion propagator  $S_F^T(p)$  at  $\mathcal{O}(\alpha)$  in the usual way,

$$\begin{aligned} S_F^T(p) &= \frac{i}{\not{p} - m - \Sigma^T} \\ &= \frac{i[\not{p}(1 - c_B) + m(1 - 2c_B) - \not{K}]}{p^2(1 - 2c_B) - m^2(1 - 4c_B) - 2p \cdot K + \mathcal{O}(\alpha^2)}. \end{aligned} \quad (23)$$

Examining the denominator of Eq. (23), we see that it can be simplified by defining

$$\delta m_T^2 \equiv 2e^2 \int \frac{d^4 k}{(2\pi)^3} f_B(|k^0|) \delta(k^2) = \frac{2\pi}{3} \alpha T^2. \quad (24)$$

Also, since  $-2p \cdot k = -[(p+k)^2 - m^2] + (p^2 - m^2) + k^2$ , we can write

$$-2p \cdot K = -\delta m_T^2 + (p^2 - m^2) c_B. \quad (25)$$

Finally, we can expand in  $(p^2 - m^2)$  around the on-shell momentum  $\hat{p}$  (which satisfies  $\hat{p}^2 = m^2$ ) by writing

$$c_B(p) = \widehat{c}_B + (p^2 - m^2) \widehat{c}'_B + \mathcal{O}[(p^2 - m^2)^2], \quad (26)$$

where  $\widehat{c}_B \equiv c_B(\hat{p}) = 0$  and

$$\begin{aligned} \widehat{c}'_B &\equiv \frac{dc_B}{dp^2}(\hat{p}) \\ &= -2e^2 \int \frac{d^4 k}{(2\pi)^3} \frac{f_B(|k^0|) \delta(k^2)}{[(\hat{p}+k)^2 - m^2]^2} \left[ 1 + \frac{d(2p \cdot k)}{dp^2}(\hat{p}) \right] \\ &= -\frac{\alpha J_{-1}}{\pi m^2}. \end{aligned} \quad (27)$$

We note that the vanishing of the  $\widehat{c}_B$  coefficient follows from the antisymmetry of the integrand for  $c_B$  in Eq. (22) at  $p = \hat{p}$  w.r.t.  $k \rightarrow -k$ ; for the same reason, the derivative term in the integral in Eq. (27) vanishes as well.

We are now in position to recover the form of the propagator advertised in Eq. (19), by using Eqs. (25) and (26) in Eq. (23), and keeping only  $\mathcal{O}(\alpha)$  terms there. We find the result

<sup>4</sup>Note that we assume that renormalization has already been carried out at  $T = 0$ .

$$S_F^T(p) = (1 - 2m^2\widehat{c}_B') \frac{i(\not{p} + m - \not{K})}{p^2 - m^2 - \delta m_T^2}. \quad (28)$$

Comparing this expression with Eq. (19) and using Eqs. (24) and (27), we obtain

$$Z_2^T = 1 - 2m^2\widehat{c}_B' = 1 + 2\frac{\alpha}{\pi}J_{-1}, \quad (29)$$

$$m_T^2 = m^2 + \delta m_T^2 = m^2 + \frac{2\pi}{3}\alpha T^2, \quad (30)$$

$$\sum_s u_s^T(p)\bar{u}_s^T(p) = \not{p} + m - \not{K}(p), \quad (31)$$

where in the last expression the momentum-dependent results of Eq. (22) are to be used in evaluating spin sums. These finite-temperature relations affect the decay rate in the three aforementioned ways; we shall now calculate each of their contributions separately.

First, the finite-temperature wave-function renormalization factor  $Z_2^T$  simply affects the tree-level decay rate  $\tilde{\Gamma}_0$  of Eq. (3) as an overall multiplicative factor; one factor enters for each external fermion line. This yields the  $\mathcal{O}(g^2\alpha)$  temperature-dependent contribution

$$\Gamma_{Z_2}^T = \frac{\alpha}{\pi}\Gamma_0\{[4 + \mathcal{O}(\epsilon^4)]J_{-1}\}. \quad (32)$$

Combining this with Eqs. (12) and (18), we see that this contribution cancels the remaining infrared-divergent  $J_{-1}$  part of the total decay rate.

Second, Eq. (30) results in a shift of the pole of the fermion propagator to  $p^2 = m_T^2$ . Since the pole masses of the fermions define the leading-order rate, the mass shifts  $\Delta m_i \equiv m_{T,i} - m_i \approx \delta m_{T,i}^2/2m_i$  lead to the following  $\mathcal{O}(g^2\alpha)$  ‘‘phase-space’’ correction

$$\Gamma_{\text{ph}}^T = \frac{\partial\tilde{\Gamma}_0}{\partial m_\psi}\Delta m_\psi + \frac{\partial\tilde{\Gamma}_0}{\partial m_\chi}\Delta m_\chi = \frac{\alpha}{\pi}\Gamma_0\{[2 + \mathcal{O}(\epsilon^2)]J_1\tau^2\}. \quad (33)$$

Finally, the finite-temperature spin-sum relation found in Eq. (31) modifies the calculation of matrix elements. Note from Eq. (22) that the relation is actually momentum-dependent, and must be computed for both the  $\psi$  particle (at rest) and the  $\chi$  particle (with energy  $p^0 = m_\psi(1 + \epsilon^2)/2$ ). The leading-order contribution to the decay rate is then found by using the finite-temperature spin-sum relation in the tree-level calculation, giving the  $\mathcal{O}(g^2\alpha)$  temperature-dependent part,

$$\Gamma_K^T = \frac{\alpha}{\pi}\Gamma_0\{[-2 + 8\ln\epsilon + \mathcal{O}(\epsilon^4)]J_1\tau^2\}. \quad (34)$$

The total  $\mathcal{O}(g^2\alpha)$  self-energy correction is then given by the sum of these three effects [Eqs. (32)–(34)],

$$\begin{aligned} \Gamma_\Sigma^T &= \Gamma_{Z_2}^T + \Gamma_{\text{ph}}^T + \Gamma_K^T \\ &= \frac{\alpha}{\pi}\Gamma_0\{[4 + \mathcal{O}(\epsilon^4)]J_1 + [8\ln\epsilon + \mathcal{O}(\epsilon^2)]J_1\tau^2\}. \end{aligned} \quad (35)$$

#### IV. TOTAL DECAY RATE IN THE TOY MODEL

We are now in position to present the final formula for the decay rate of the hypothetical fermion  $\psi$  in a thermal bath. We consider the low-temperature limit  $T \ll m_\psi, m_\chi$  and include contributions from processes involving both real photons and virtual photons; the latter category includes the vertex correction and corrections arising from the self-energy of charged fermions. The total decay rate is the sum of these contributions given in Eqs. (12), (18), and (35). We remind the reader that our calculation is performed in the approximation  $\tau \ll \epsilon \ll 1$  and that we are interested in the leading  $\mathcal{O}(\alpha\tau^2)$  temperature-dependent correction to the rate. We find

$$\Gamma_{\text{tot}}^T = \Gamma_{\text{real}}^T + \Gamma_{\text{vert}}^T + \Gamma_\Sigma^T = -\alpha\frac{\pi}{3}\tau^2\Gamma_0 + \mathcal{O}(\tau^2\epsilon^4, \tau^4, e^{-1/\tau}). \quad (36)$$

We see that all the infrared-divergent terms proportional to the integrals  $J_{-1}$  cancel out in the total rate. We note that this statement remains valid if exact  $\epsilon$ -dependence of the rate is restored. Furthermore, we find in Eq. (36) that all the terms that contains logarithms of the mass ratio  $\ln\epsilon$  cancel in the correction to the total rate, in contrast to individual contributions in Eqs. (12) and (34). Cancellation of such terms in the zero-temperature case follows from the Kinoshita-Lee-Nauenberg theorem [10]. We are not aware of a general proof of a similar cancellation at a *finite* temperature, so it is important to watch for such terms in explicit  $T \neq 0$  computations.

#### V. MUON DECAY $\mu \rightarrow e\nu_\mu\bar{\nu}_e$

In this section, we present the temperature-dependent correction to the muon decay rate at low temperature. The details of the calculation are similar to the preceding discussion of the toy model. The main difference is that the muon decay is a three-body process, so that integration over the phase-space of final-state particles is more complex.

The muon decay to electron and neutrinos is described by an effective Lagrangian

$$\mathcal{L} \supset \frac{4G_F}{\sqrt{2}}\bar{e}\gamma^\rho L\nu_e\bar{\nu}_\mu\gamma_\rho L\mu, \quad (37)$$

where  $G_F$  is the Fermi constant. The leading-order, zero-temperature decay rate reads

$$\tilde{\Gamma}_0 = \Gamma_0(1 - 8\epsilon^2 - 24\epsilon^4\ln\epsilon + 8\epsilon^6 - \epsilon^8), \quad (38)$$

where now  $\Gamma_0 \equiv \frac{G_F^2 m_\mu^5}{192\pi^3}$  and  $\epsilon \equiv m_e/m_\mu$ . Similar to the toy-model case, the radiative corrections to the rate are given by the sum of real photon emission/absorption corrections, the vertex corrections, and the self-energy corrections. For future reference, we show those corrections separately.

The contribution to the decay rate from real photon emission/absorption reads

$$\Gamma_{\text{real}}^T = \frac{\alpha}{\pi} \Gamma_0 \left[ \left( -\frac{17}{3} - 4 \ln \epsilon \right) J_{-1} + \left( -\frac{70}{3} - 32 \ln \epsilon \right) J_1 \tau^2 \right]. \quad (39)$$

Note that here and below we keep only the leading term in  $\epsilon$  for each power of  $\tau \equiv T/m_\mu$ , and consistently neglect all powers of  $\tau$  beyond  $\tau^2$ . The result for the vertex correction reads

$$\Gamma_{\text{vert}}^T = \frac{\alpha}{\pi} \Gamma_0 \left[ \left( \frac{5}{3} + 4 \ln \epsilon \right) J_{-1} \right]. \quad (40)$$

We note that the temperature dependence in Eq. (40) is exact and that higher-order terms in  $\tau$  do not appear there.

The self-energy correction to the fermion propagator was discussed in the previous section and much of that discussion remains valid. For this reason, we just summarize the result. The total self-energy correction is given by

$$\Gamma_{\Sigma}^T = \frac{\alpha}{\pi} \Gamma_0 \left[ 4J_{-1} + \left( \frac{64}{3} + 32 \ln \epsilon \right) J_1 \tau^2 \right]. \quad (41)$$

The final result for temperature-dependent radiative corrections to the muon decay is given by the sum of the three contributions of Eqs. (39)–(41). Including also the  $T = 0$  radiative corrections [12], we find the final  $\mathcal{O}(\alpha, \tau^2, \epsilon^0)$  result

$$\Gamma_{\mu \rightarrow e \nu \bar{\nu}} = \Gamma_0 \left\{ 1 + \frac{\alpha}{\pi} \left[ \left( \frac{25}{8} - \frac{\pi^2}{2} \right) - \frac{\pi^2}{3} \tau^2 \right] \right\}. \quad (42)$$

We note that  $\mathcal{O}(\alpha \tau^2)$  correction to the rate for the muon decay is identical to the analogous correction to the two-body fermion decay rate in the toy model, suggesting the possibility of deriving and understanding this result in a simpler fashion. We also note that our result Eq. (42) disagrees with the one given in Ref. [13].

Part of the discrepancy can be traced to the issue of mass singularities. Indeed, in Ref. [13], the  $\ln \epsilon$  terms are present even at  $\mathcal{O}(\alpha \tau^2)$  contributions to the rate but, as follows from our analysis, such terms cancel when all contributions are taken into account. Nevertheless, as pointed out already, whether mass singularities cancel in the rate if higher-order terms in  $\tau$  are accounted for is an open question. When we extend the calculation of the muon decay rate to include  $\mathcal{O}(\alpha \tau^4)$  terms, we find that Eq. (42) is modified by

$$\Delta \Gamma_{\mu \rightarrow e \nu \bar{\nu}} = -\frac{\alpha}{\pi} \Gamma_0 \frac{64 \pi^4 \tau^4}{45} \left( 2 \ln \epsilon + \frac{1}{3} \right), \quad (43)$$

which shows the logarithmic sensitivity to the electron-to-muon mass ratio. In the low-temperature regime that we consider in this paper  $\tau \ll \epsilon \ll 1$ , there exist more important corrections to Eq. (42) than the ones displayed in Eq. (43). However, most of the “more relevant”

corrections involve powers of the mass ratio  $\epsilon$ , while Eq. (43) shows logarithmic sensitivity to  $\epsilon$ , a unique feature in the low-temperature regime. It is interesting to point out that by relaxing the relationship between the temperature  $T$  and the mass of the charged particle in the final state (the electron), we obtain new sources of mass logarithms related to the thermal Fermi-Dirac distribution of *fermions* in the heat bath. Complete analysis of the corrections to the muon decay rate for the *intermediate-temperature regime*  $m_e \ll T \ll m_\mu$ —where proper interplay of bosonic and fermionic temperature-dependent corrections becomes important—is beyond the scope of the present paper.

## VI. CONCLUSIONS

Long-lived charged particles appear in a variety of scenarios for early-Universe physics and dark matter. In all cases, these long-lived particles are bathed for a long time in a gas of photons, giving rise to the possibility of induced decays through processes such as those shown in Fig. 2. If the rate of this induced process is large, then the cosmological effects of these long-lived charged particles may be substantially modified.

A naive evaluation of the rate for these induced decays leads to a result which diverges as the frequency of the photon in the heat bath that induces the decay vanishes. As with infrared divergences at zero temperature, a proper calculation of the decay rate requires accounting for all degenerate processes. Once this is done, the infrared divergences cancel, leading to small correction to the decay rate. By considering a simple toy model with a two-body final state and a realistic process—muon decay—with a three-body final state, we found a universal leading finite-temperature correction  $\delta \Gamma / \Gamma_0 = -\frac{\alpha}{\pi} \frac{\pi^2 T^2}{3m^2}$ , where  $m$  is the mass of the decaying particle.

In this paper, we focused on discussing infrared divergences in decays of charged particles in the thermal bath. This issue can be sharply defined by considering temperatures that are small compared to masses of decaying particles and their decay products. An interesting set of questions arises if we depart from the low-temperature regime and consider the intermediate-temperature scenario, where the mass of the decaying particle is large and masses of decay products are small, compared to the heat-bath temperature. In such a case, radiative corrections enhanced by the logarithms of the mass ratios can become numerically important in the context of a variety of scenarios that occur in the early Universe. For example, light gravitinos that arise in theories of supergravity with gauge-mediated supersymmetry breaking may be produced by the decay of short-lived charged staus to taus [14]. Temperatures greater than the tau mass will then fall into the intermediate-temperature scenario. If mass-enhanced corrections are indeed present, large modifications of the stau decay rate and the production rate of light gravitino dark matter become conceivable. Such modifications may

affect the early-Universe thermal history and have implications for collider phenomenology [15].

Furthermore, in some regions of the supersymmetric parameter space, the process of co-annihilation is important in the determination of the dark-matter relic abundance after freeze-out [16]. Freeze-out occurs roughly at temperatures  $T \sim m_{\text{SUSY}}/20$  that may be greater than the masses of some of the products of supersymmetric particle decay, so this indeed presents another intermediate-temperature scenario. Finite-temperature effects and mass singularities may then become important in determining the individual scattering and decay rates for charged particles, which would be important if one is interested in the detailed thermal history of these particles. However, note that the final dark-matter relic abundance is most likely not strongly affected by finite-temperature effects, since only the co-annihilation rates (and not other scattering or decay rates) enter the calculation

[17]. For example, in Ref. [18] it was found that finite-temperature corrections to co-annihilation occur only at the  $10^{-4}$  level. Finite-temperature effects may also affect neutrino decoupling [19]. It has also been suggested that the original calculations of temperature-dependent corrections to neutron decay are incomplete [20]. Clearly, there remains much work to be done concerning finite-temperature effects in the early Universe.

## ACKNOWLEDGMENTS

This research was supported at Alberta by Science and Engineering Research Canada (NSERC), at JHU by the NSF Grant No. PHY-0855365 and JHU startup funds, and at Caltech by DOE DE-FG03-92-ER40701, NASA NNX10AD04G, and the Gordon and Betty Moore Foundation.

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