# Maximum values of gas-dynamic flux densities

A. Roshko
*California Institute of Technology, Pasadena, California 91125*

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A general result valid for any compressible fluid is noted. It gives the maximum values of the flux densities of mass, momentum, and kinetic energy in steady and unsteady flows which are expanding isentropically from a reservoir.

It is well known that, in steady isentropic expansion of a gas from a reservoir, the maximum mass flux density, $\rho u$, occurs at Mach number $M \equiv u/a = 1$ ($\rho$, $u$, and $a$ are the local values of density, velocity, and speed of sound, respectively; $\rho_0$, 0, and $a_0$ are corresponding reservoir values.) The proof is usually given for the special case of a perfect gas and it does not seem to be well known that the result can be obtained in generality for any compressible fluid, as shown by Landau and Lifshitz.[1] Extending their approach, the conditions for maximum flux density of momentum, $\rho u^2$, and of kinetic energy, $\frac{1}{2}\rho u^3$, can also be generally and simply obtained, for steady as well as unsteady flow.

Consider the flux density $\rho u^n$, where $n = 1, 2$, or 3, and its variation with increasing flow velocity $u$. The value of $\rho u^n$ is zero at $u = 0$, where the pressure and density have their maximum values $p_0$ and $\rho_0$. As $p$ (and $\rho$) decrease, $u$ increases in either steady or unsteady flow, attaining a maximum value $u_m$ at $p = 0$, where the density $\rho$ has its minimum value (zero for a perfect gas). Thus $\rho u^n$ may have a maximum at some velocity between 0 and $u_m$. This can be found by setting $(d/du)(\rho u^n) = 0$, i.e.,

$$n\rho u^{n-1} + u^n \frac{d\rho}{du} = 0. \tag{1}$$

The density $\rho$ is related thermodynamically to the pressure $p$ and entropy $s$ by the relation

$$dp = a^2\, d\rho + \left(\frac{\partial p}{\partial s}\right)_\rho ds,$$

where $a^2 \equiv (\partial p/\partial \rho)_s$ is the square of the speed of sound. For the isentropic flows we are considering, $d\rho = dp/a^2$ and Eq. (1) can be written

$$n\rho u^{n-1} + \frac{u^n}{a^2}\frac{dp}{du} = 0. \tag{2}$$

We now need a pressure-velocity $(p-u)$ relation, and this is different in steady and unsteady flows.

For *steady* flow, along a streamtube of varying cross-sectional area, the $(p-u)$ relation comes from the momentum equation

$$dp = -\rho u\, du, \tag{3}$$

which, in integral form, is the Bernoulli equation

$$\frac{u^2}{2} + \int_{p_0}^{p} \frac{dp}{\rho} = 0, \tag{4}$$

and, since the flow is isentropic, is equivalent to the energy equation

$$(u^2/2) + h = h_0. \tag{5}$$

Putting (3) into (2), the velocity for a maximum is found from

$$\rho u^{n-1}\left(n - \frac{u^2}{a^2}\right) = 0;$$

$u/a \equiv M$ is the Mach number. Thus the maxima in a steady isentropic expansion occur at $M^2 = n$, i.e., at $M = 1, \sqrt{2}$, and $\sqrt{3}$ for mass, momentum, and energy, respectively.

For *unsteady* isentropic flow in a streamtube of constant area we have, instead of Eq. (3), the $p-u$ relation

$$dp = -\rho a\, du \tag{6}$$

which comes from the Riemann invariant

$$u + \int_{p_0}^{p} \frac{dp}{\rho a} = 0. \tag{7}$$

This is the pressure–velocity relation in a one-dimensional (plane) simple unsteady wave.

The condition for a maximum is found by putting (6) into (2), which gives

$$\rho u^{n-1}\left(n - \frac{u}{a}\right) = 0.$$

Thus, the maxima in an unsteady expansion occur at $M = n$, i.e., $M = 1$, 2, and 3, respectively.

The values of the maxima depend on the fluid properties. For a thermally and calorically *perfect gas*, with $\gamma \equiv c_p/c_v =$ the ratio of specific heats, they can be evaluated by making use of the integral equations (5) and (7) for steady and unsteady flows, respectively. With $h = a^2/(\gamma-1)$ and $dp/\rho a = [2/(\gamma-1)]da$, Eqs. (5) and (7) take the forms

$$\frac{u^2}{2} + \frac{a^2}{\gamma-1} = \frac{a_0^2}{\gamma-1} \quad \text{(steady)},$$

and

$$\frac{u}{2} + \frac{a}{\gamma-1} = \frac{a_0}{\gamma-1} \quad \text{(unsteady)}.$$

Rewritten in dimensionless forms,

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$$\frac{a_0^2}{a^2}=1+\frac{\gamma-1}{2}\frac{u^2}{a^2}=1+\frac{\gamma-1}{2}M^2 \quad \text{(steady)},$$

$$\frac{a_0}{a}=1+\frac{\gamma-1}{2}\frac{u}{a}=1+\frac{\gamma-1}{2}M \quad \text{(unsteady)}.$$

Finally, for the perfect gas,

$$\frac{\rho}{\rho_0}=\left(\frac{T_0}{T}\right)^{-1/(\gamma-1)}=\left(\frac{a_0}{a}\right)^{-2/(\gamma-1)}$$

and the fluxes are obtained as the following functions of Mach number:

$$\frac{\rho u^n}{\rho_0 a_0^n}=\left(1+\frac{\gamma-1}{2}M^2\right)^{-[1/(\gamma-1)+(n/2)]}(M^2)^{n/2} \quad \text{(steady)}$$

and

$$\left(1+\frac{\gamma-1}{2}M\right)^{[2/(\gamma-1)+(n/2)]}(M)^n \quad \text{(unsteady)}.$$

Their maximum values are

$$\left(1+\frac{\gamma-1}{2}n\right)^{-[1/(\gamma-1)+(n/2)]}(n^{n/2}) \quad \text{(steady)}$$

and

$$\left(1+\frac{\gamma-1}{2}\quad n\right)^{-[2/(\gamma-1)+n]}(n^n) \quad \text{(unsteady)}.$$

TABLE I. Numerical maximum values of $\rho u^n/\rho_0 a_0^n$ for $\gamma=7/5$ and $5/3$.

| $n=$ | 1 | | 2 | | 3 | |
|---|---|---|---|---|---|---|
| $\gamma=$ | 7/5 | 5/3 | 7/5 | 5/3 | 7/5 | 5/3 |
| Steady | 0.5787 | 0.5625 | 0.6160 | 0.5577 | 0.7929 | 0.6495 |
| Unsteady | 0.3349 | 0.3164 | 0.3795 | 0.3110 | 0.6286 | 0.4219 |

Numerical maximum values of $\rho u^n/\rho_0 a_0^n$ for $\gamma=7/5$ and $5/3$ are listed in Table I. The dynamic pressure is often expressed in the dimensionless form $\frac{1}{2}\rho u^2/p_0$, which, for perfect gas, is equivalent to $\frac{1}{2}\gamma\rho u^2/\rho_0 a_0^2$ and thus can be obtained from the above values for $n=2$ by multiplying by $\frac{1}{2}\gamma$. Similarly, the energy flux can be put in the dimensionless form $(\frac{1}{2}\rho u^3/\rho_0 h_0 a_0)$, where $h_0$ is the reservoir enthalpy; for a perfect gas this is equivalent to $[(\gamma-1)/2](\rho u^3/\rho_0 a_0^3)$ and the maximum values can be obtained by multiplying the tabulated values for $n=3$ by $(\gamma-1)/2$.

[1] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon, London, 1987), pp. 316–318.