AUTOMORPHIC EQUIVALENCE WITHIN GAPPED PHASES OF QUANTUM LATTICE SYSTEMS

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Abstract. Gapped ground states of quantum spin systems have been referred to in the physics literature as being 'in the same phase' if there exists a family of Hamiltonians $H(s)$, with finite range interactions depending continuously on $s \in [0,1]$, such that for each s, $H(s)$ has a non-vanishing gap above its ground state and with the two initial states being the ground states of $H(0)$ and $H(1)$, respectively. In this work, we give precise conditions under which any two gapped ground states of a given quantum spin system that 'belong to the same phase' are automorphically equivalent and show that this equivalence can be implemented as a flow generated by an s-dependent interaction which decays faster than any power law (in fact, almost exponentially). The flow is constructed using Hastings' 'quasi-adiabatic evolution' technique, of which we give a proof extended to infinite-dimensional Hilbert spaces. In addition, we derive a general result about the locality properties of the effect of perturbations of the dynamics for quantum systems with a quasi-local structure and prove that the flow, which we call the *spectral flow*, connecting the gapped ground states in the same phase, satisfies a Lieb-Robinson bound. As a result, we obtain that, in the thermodynamic limit, the spectral flow converges to a co-cycle of automorphisms of the algebra of quasi-local observables of the infinite spin system. This proves that the ground state phase structure is preserved along the curve of models $H(s)$, $0 \leq s \leq 1$.

1. Introduction

Since the discovery of the fractional quantum Hall effect [50] and its description in terms of model wave functions with special 'topological' properties [31], there has been great interest in quantum phase transition [48]. Experimental and theoretical discoveries of exotic states in strongly correlated systems [13] and, more recently, the possibility of using topologically ordered quantum phases for quantum information computation [30], have further increased our need to understand the nature of quantum phase transitions, and especially of gapped ground states. It is natural to ask whether gapped quantum phases and the transitions between them can be classified. The first and simplest question is to define precisely what it means for two gapped ground states to belong to the same phase. A pragmatic definition that has recently been considered in the literature declares two gapped ground states of a quantum spin system to belong to the same phase if there exists a family of Hamiltonians $H(s)$, with finite range interactions depending continuously on $s \in [0,1]$, such that for each s, $H(s)$ has a non-vanishing gap above its ground state, and the two given states are the ground states of $H(0)$ and $H(1)$. In other words there is a family of Hamiltonians with gapped ground states that interpolate between the given two [10, 11]. In this paper we prove a result that supports this definition. We show that any two gapped ground states in the same phase according to this definition are unitarily equivalent, with a unitary that can be obtained as the flow of an s-dependent quasi-local interaction which decays almost exponentially fast. When applied to models on a finite-dimentsional lattice, this quasi-local structure is sufficient to prove that the unitary equivalence of finite volume leads to automorphic equivalence at the level of the C^*-algebra of quasi-local observables in the thermodynamic limit.

In statistical mechanics, lattice models with short-range interactions play a central role. Many examples of Hamiltonians that can be considered as a perturbation of a model with a known ground

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state that is sufficiently simple (typically given by finite number of classical spin configurations), have been studied by series expansion methods [27, 1, 2, 34, 28, 29, 6, 14, 15, 51]. Perturbation expansions provide a detailed understanding of the ground state under quite general conditions. Often, one can prove the existence of a finite correlation length and a non-vanishing spectral gap above the ground state, and short-range correlations can in principle be calculated to arbitrary precision. The perturbation series one employs in such situations has the structure of a cluster expansion in which the cluster geometry is based on the underlying lattice structure and the short-range nature of the interactions. The effect of the perturbations can then be understood as approximately local modifications of the ground state of the unperturbed model.

In this work, we start from a different perspective. Suppose we have a family of models defined in terms of an interaction $\Phi(s)$ which depends on a parameter $s \in [0, 1]$:

$$H_\Lambda(s) = \sum_{X \subset \Lambda} \Phi(X, s) \tag{1.1}$$

Here, Λ is a finite subset of the lattice Γ (e.g., $\Gamma = \mathbb{Z}^\nu$), and $\Phi(s)$ is a short-range interaction depending smoothly on a parameter s (see Section 5 for the precise conditions on the decay of the interactions that we assume).

Suppose that for all s the ground state(s) of this family of models are isolated from the rest of the spectrum by a gap. We prove that the ground state(s) of $H_\Lambda(s)$ can be obtained from the ground state(s) of $H_\Lambda(0)$ by a unitary transformation $U_\Lambda(s)$ which has a quasi-local structure in the sense that $U_\Lambda(s)$ can be regarded as the flow generated by a quasi-local parameter-dependent interaction $\Psi(s)$ which we construct. In the works cited above the goal was to develop a suitable perturbation theory which would allow one to prove the existence of a non-vanishing spectral gap, among other things. To do that one has to start from a sufficiently simple model at $s = 0$ and also assume that the perturbation potential is sufficiently small. Note that no such smallness condition is required on $\Phi(s)$ here. We now make some comments on the methods used in this paper.

In his 2004 paper [19] Hastings introduced a new technique, which he called 'quasi-adiabatic continuation' (see also [24]). He used it in combination with the propagation bounds for quantum lattice dynamics originally due to Lieb and Robinson [32] to construct and analyze the variational states needed for the proof of a multi-dimensional version of the Lieb-Schultz-Mattis theorem [33]. The quasi-adiabatic continuation technique was subsequently elaborated upon and used in new applications by Hastings and collaborators [9, 20, 23, 8, 7] as well as other authors [40, 44]. In this paper we give a general account of this technique and show how it allows one to exploit locality properties of the dynamics of extended quantum systems with short-range interactions without resorting to cluster expansions.

The starting point of the analysis in all the works mentioned above is a version of our Proposition 2.4. This result shows that the spectral projection associated with an isolated part of the spectrum of a family of self-adjoint operators $H(s)$ depending smoothly on a parameter s, can be obtained as a unitary evolution. Concretely, let $I(s) \subset \mathbb{R}$ denote an interval such that for all s the spectrum of $H(s)$ contained in $I(s)$ is separated by a uniform gap $\gamma > 0$ from the rest of the spectrum of $H(s)$, then there exists a curve of unitary operators $U(s)$ such that

$$P(s) = U(s)P(0)U(s)^*$$

When we apply this result to families of Hamiltonians $H_\Lambda(s)$ of the form (1.1), i.e., with a quasi-local structure, we find that the unitaries $U_\Lambda(s)$ then have the structure of a quasi-local dynamics itself. Explicitly,

$$\frac{d}{ds} U_\Lambda(s) = i D_\Lambda(s) U_\Lambda(s)$$

where $D_\Lambda(s)$ is a self-adjoint operator with the structure of a time-dependent Hamiltonian,*i.e.* , there is an interaction $\Psi(s)$ such that

$$D_\Lambda(s) = \sum_{X \subset \Lambda} \Psi(X, s) .$$

Because of this quasi-local structure, the flow on the algebra of observables defined by conjugation with the unitaries $U_\Lambda(s)$, *i.e.* :

$$\alpha_s^\Lambda(A) = U_\Lambda^*(s) A U_\Lambda(s) \tag{1.2}$$

satisfies a propagation bound of Lieb-Robinson type (see Section 4). These propagation bounds—as a second application of Lieb-Robinson bounds in this paper— can be used to prove the existence of the thermodynamic limit (Section 5). The main result of this paper is Theorem 5.5. Stated in words, it says that if for a differentiable curve of Hamiltonians of the form (1.1) the gap above the ground states does not close along the curve, then, for each s there is an automorphism α_s of the algebra of quasi-local observables which maps the ground states at $s = 0$ to the ground states at s. In particular the simplex of infinite-volume ground states for all values of s is isomorphic to the one for $s = 0$.

We find the designation 'quasi-adiabatic' of the flow α_s^Λ somewhat misleading since there is nothing adiabatic about it. The flow does, however, follow the spectral subspace belonging to the isolated interval $I(s)$. We will therefore call it the *spectral flow*.

This paper is organized as follows. In Section 2, we give a rigorous and self-contained presentation of the construction of the spectral flow in a form that allows for applications with an infinite-dimensional Hilbert space. A number of applications where the infinite-dimensional context has proven useful have already been considered in the literature, see e.g. [12, 37, 38, 3, 45, 46]. We expect that more applications will be found.

In Section 3 we use Lieb-Robinson bounds to obtain a locality property of the spectral flow and prove that *local perturbations perturb locally* in the sense that the dependence of gapped ground states (or any other isolated eigenstates) on any given local term in the Hamiltonian is significant only in a neighborhood of the support of that term. In Lemma 3.2 we generalize the notion of normalized partial trace to infinite-dimensional Hilbert spaces.

In the final two sections we consider quantum lattice models, or more generally, models defined on a metric graph (satisfying suitable conditions) with sufficiently fast decaying interactions. Section 4 is devoted to showing that the spectral flow can be generated by time-dependent Hamiltonians defined in terms of local interactions. As a consequence, this flow then also satisfies a Lieb-Robinson bound. In Section 5, we restrict our attention to quantum spin systems, and use the results of Section 4 to obtain the existence of the thermodynamic limit of the spectral flow as automorphisms on the algebra of quasi-local observables. We conclude the paper with a brief discussion of the notion of 'gapped ground state phase', which has been a topic of particular interest in the recent literature.

2. The curve of spectral projections for an isolated part of the spectrum of a Hamiltonian with a parameter

We consider a smooth family of self-adjoint Hamiltonians $H(s) = H(s)^*$ parametrized by $s \in [0, 1]$, acting on a Hilbert space $\mathcal{H}$. We do not assume that $H(s)$ itself is bounded but the $s-$dependent portion should be. We are interested in the spectral projection $P(s)$ associated with an isolated part of the spectrum of $H(s)$. Explicitly, our main assumption on $H(s)$ is the following.

Assumption 2.1. $H(s)$ is a densely defined self-adjoint operator with bounded derivative $H'(s)$, such that $\|H'(s)\|$ is uniformly bounded for $s \in [0, 1]$. Furthermore, we assume that the spectrum, $\Sigma(s)$ of $H(s)$ can be decomposed in two parts: $\Sigma(s) = \Sigma_1(s) \cup \Sigma_2(s)$, such that $\inf\{|\lambda_1 - \lambda_2| \mid \lambda_1 \in \Sigma_1, \lambda_2 \in \Sigma_2\} = \gamma$ for a constant $\gamma > 0$, uniformly in s. We also assume there are compact intervals

$I(s)$, with end points depending smoothly on s and such that $\Sigma_1(s) \subset I(s) \subset (\mathbb{R} \setminus \Sigma_2(s))$, in such a way that the distance between $I(s)$ and $\Sigma_2(s)$ has a strictly positive lower bound uniformly in s.

Typically, we have in mind a family of Hamiltonians of the form $H(s) = H(0) + \Phi(s)$, with $H'(s) = \Phi'(s)$ bounded. Specifically, if $H(s)$ is unbounded, this is due to $H(0)$, which is obviously independent of s. Let $E_\lambda(s)$ be the spectral family associated with $H(s)$ and let $P(s) := \int_{I(s)} dE_\lambda(s)$ be the spectral projection on the isolated part of the spectrum $\Sigma_1(s)$.

The formulation of the main result of this section uses a function $w_\gamma \in L^1(\mathbb{R})$, depending on a parameter $\gamma > 0$, with the following properties.

Assumption 2.2. $w_\gamma \in L^1(\mathbb{R})$ satisfies

i. w_γ is real-valued and $\int dt\, w_\gamma(t) = 1$,
ii. The Fourier transform $\widehat{w}_\gamma$ is supported in the interval $[-\gamma, \gamma]$, i.e., $\widehat{w}_\gamma(\omega) = 0$, if $|\omega| \geq \gamma$.

Such functions exist and were already considered in [21]. In the following lemma, we present a family of such functions derived from [25, 16] and give explicit bounds on their decay that we will need in this work and which may also prove useful in future applications.

Lemma 2.3. *Let $\gamma > 0$ and define a positive sequence $(a_n)_{n\geq 1}$ by setting $a_n = a_1(n \ln^2 n)^{-1}$ for $n \geq 2$, and choosing a_1 so that $\sum_{n=1}^\infty a_n = \gamma/2$. Then, the infinite product*

$$w_\gamma(t) = c_\gamma \prod_{n=1}^{\infty} \Big(\frac{\sin a_n t}{a_n t}\Big)^2, \tag{2.1}$$

defines an even, non-negative function $w_\gamma \in L^1(\mathbb{R})$, and we can choose c_γ such that $\int w_\gamma(t)dt = 1$. With this choice, the following estimate holds. For all $t \geq \mathrm{e}^{1/\sqrt{2}}/\gamma$,

$$0 \leq w_\gamma(t) \leq 2(\mathrm{e}\gamma)^2 t \cdot \exp\left(-\frac{2}{7}\frac{\gamma t}{\ln^2(\gamma t)}\right). \tag{2.2}$$

Proof. Without loss of generality, we shall assume $t \geq 0$. Since each term of the product lies between 0 and 1, and by Stirling's formula,

$$w_\gamma(t) \leq c_\gamma \prod_{n=1}^{N} \Big(\frac{\sin a_n t}{a_n t}\Big)^2 \leq c_\gamma (N!)^2 \ln^{4N}(N)(a_1 t)^{-2N} \leq 2\pi c_\gamma N N^{2N} \ln^{4N}(N)(a_1 t)^{-2N} \mathrm{e}^{-2N}.$$

The desired bound is obtained by choosing $N = \lfloor a_1 t / \ln^2(\gamma t) \rfloor$ and noting that $\gamma/7 < a_1 < \gamma/2$ and $\gamma/(2\pi) < c_\gamma < \gamma/\pi$. The bounds on a_1 follow directly, while the latter estimates are proven e.g. in [4]. For $t > \mathrm{e}^{1/\sqrt{2}}/\gamma$, $N \leq \gamma t$ so that

$$w_\gamma(t) \leq 2(\mathrm{e}\gamma)^2 t \cdot \exp\left(-\frac{2}{7}\frac{\gamma t}{\ln^2(\gamma t)}\right).$$

Finally, this decay estimate and the a priori bound $w_\gamma(t) \leq 1$ for all t imply that $w_\gamma \in L^1(\mathbb{R})$. □

Since the Fourier transform of $\sin(ax)/(ax)$ is the indicator function of the interval $[-a, a]$, the support of $\widehat{w}_\gamma$ corresponds to $[-2S, 2S]$, where $S = \sum_{n=1}^\infty a_n$, and thus (ii) of Assumption 2.2 also holds. Moreover, this lemma shows that the function w_γ can be chosen to decay faster than any power as $t \to \infty$. This will be important for some of our applications. We can now state and prove the main result of this section.

Proposition 2.4. *Let $H(s)$ be a family of self-adjoint operators satisfying Assumption 2.1. Then, there is a norm-continuous family of unitaries $U(s)$ such that the spectral projections $P(s)$ associated with the isolated portion of the spectrum $\Sigma_1(s)$, are given by*

$$P(s) = U(s)P(0)U(s)^*. \tag{2.3}$$

The unitaries are the unique solution of the linear differential equation

$$-\,\mathrm{i}\frac{d}{ds}U(s) = D(s)U(s)\,, \qquad U(0) = \mathbb{1}\,, \tag{2.4}$$

where

$$D(s) = \int_{-\infty}^{\infty} dt\, w_\gamma(t) \int_0^t du\, \mathrm{e}^{\mathrm{i}uH(s)} H'(s) \mathrm{e}^{-\mathrm{i}uH(s)}\,. \tag{2.5}$$

for any function w_γ satisfying Assumption 2.2.

It is obvious from (2.5) and the assumption that $w_\gamma(t) \in \mathbb{R}$, that $D(s)$ is bounded, self adjoint, and the equations (2.3) and (2.4) can be combined into

$$\frac{d}{ds}P(s) = \mathrm{i}[D(s), P(s)]\,. \tag{2.6}$$

Moreover, boundedness of $D(s)$ implies that the unitaries $U(s)$ are norm continuous.

The existence of a (bounded holomorphic) transformation function $V(s)$ such that $P(s) = V(s)P(0)V(s)^{-1}$ is a direct consequence of the smoothness of $P(s)$, see e.g. [26]. The interest of the proposition stems from having an explicit formula of a unitary family $U(s)$, from which interesting properties can be derived. This constructive aspect is essential for the applications we have in mind (see Sections 3 and 4).

Proof. On the one hand,

$$P(s) = -\frac{1}{2\pi\mathrm{i}} \int_{\Gamma(s)} dz\, R(z,s)\,, \tag{2.7}$$

where $R(z,s) = (H(s) - z)^{-1}$ is the resolvent of $H(s)$ at z, and the contour $\Gamma(s)$ encircles the real interval $I(s)$ in the complex plane. Therefore,

$$P'(s) = -\frac{1}{2\pi\mathrm{i}} \int_{\Gamma(s)} dz\, R'(z,s) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma(s)} dz\, R(z,s)H'(s)R(z,s)\,, \tag{2.8}$$

where the first equality follows by noting that the smooth dependence of $s \mapsto I(s)$ and the uniform lower bound on the gap imply that the contour $\Gamma(s)$ can be kept fixed while differentiating; Namely for ε small enough, $\Gamma(s)$ can be chosen so that it encircles all intervals $I(\sigma)$, $\sigma \in [s, s+\varepsilon]$. The s dependence of Γ can therefore be taken as purely parametric. Since $P(s)$ is an orthogonal projection, $P(s)P'(s)P(s) = (1-P(s))P'(s)(1-P(s)) = 0$ and therefore,

$$\begin{aligned} P'(s) &= \frac{1}{2\pi\mathrm{i}} \int_{\Gamma(s)} dz\, \big(P(s)R(z,s)H'(s)R(z,s)(1-P(s)) + (1-P(s))R(z,s)H'(s)R(z,s)P(s)\big) \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\Gamma(s)} dz \int_{I(s)} d\mu \int_{\mathbb{R}/I(s)} d\lambda\, \frac{1}{\mu - z}\frac{1}{\lambda - z}\big(dE_\mu(s)H'(s)dE_\lambda(s) + dE_\lambda(s)H'(s)dE_\mu(s)\big) \qquad (2.9) \\ &= -\int_{I(s)} d\mu \int_{\mathbb{R}/I(s)} d\lambda\, \frac{1}{\lambda - \mu}\big(dE_\mu(s)H'(s)dE_\lambda(s) + dE_\lambda(s)H'(s)dE_\mu(s)\big)\,. \qquad (2.10) \end{aligned}$$

In order to justify the last equality, we interpret the double spectral integral as a double operator integral, see e.g. [5], Theorem 4.1*(iii)*. Eq. (2.9) corresponds to the factorization of the symbol $\phi(\lambda,\mu) = (\lambda-\mu)^{-1}$ of (2.10), the auxiliary measure space being $(S^1, d\gamma(t))$ where $S^1 \ni t \mapsto \gamma(t) \in \mathbb{C}$ is a parametrization of $\Gamma(s)$. The uniform integrability conditions are met because of the finite size of the gap. On the other hand,

$$\begin{aligned} \mathrm{i}[D(s), P(s)] &= \mathrm{i}\big((1-P(s))D(s)P(s) - P(s)D(s)(1-P(s))\big) \qquad (2.11) \\ &= \mathrm{i}\int_{I(s)} d\mu \int_{\mathbb{R}/I(s)} d\lambda \int dt\, w_\gamma(t) \int_0^t du \big(\mathrm{e}^{\mathrm{i}u(\lambda-\mu)} dE_\lambda(s)H'(s)dE_\mu(s) - \mathrm{e}^{-\mathrm{i}u(\lambda-\mu)} dE_\mu(s)H'(s)dE_\lambda(s)\big) \end{aligned}$$

which yields (2.10) after the time integrations are performed, namely

$$\begin{aligned} \mathrm{i}\int dt\, w_\gamma(t)\int_0^t du\, \mathrm{e}^{\pm \mathrm{i}u(\lambda-\mu)} &= \pm\int dt\, w_\gamma(t)\frac{1}{\lambda-\mu}\big(\mathrm{e}^{\pm \mathrm{i}t(\lambda-\mu)}-1\big) \\ &= \pm\frac{1}{\lambda-\mu}\big(\widehat{w_\gamma}(\pm(\mu-\lambda))-1\big) = \mp\frac{1}{\lambda-\mu}\,, \end{aligned}$$

where we used first that $\int w_\gamma(t) = 1$ and then the compact support property of $\widehat{w_\gamma}$ together with the fact that $|\lambda-\mu|>\gamma$ by Assumption (2.1). □

We now introduce the weight function

$$W_\gamma(t) := \begin{cases} \int_t^\infty d\xi\, w_\gamma(\xi) & t\geq 0 \\ -\int_{-\infty}^t d\xi\, w_\gamma(\xi) & t<0 \end{cases} \tag{2.12}$$

which will play a central role in the following applications. As $w_\gamma\in L^1(\mathbb{R})$, W_γ is well-defined.

Lemma 2.5. *For $a>0$ define*

$$u_a(\eta) = \mathrm{e}^{-a\frac{\eta}{\ln^2\eta}}\,,$$

on the domain $\eta>1$. For all integers $k\geq 0$ and for all $t\geq \mathrm{e}^4$ such that also

$$a\frac{t}{\ln^2 t}\geq 2k+2\,,$$

we have the bound

$$\int_t^\infty \eta^k u_a(\eta)\,d\eta \leq \frac{(2k+3)}{a}t^{2k+2}u_a(t)\,.$$

Proof. For $\eta\geq \mathrm{e}^2$, the function

$$\tau(\eta) = a\frac{\eta}{\ln^2\eta}$$

is positive, differentiable, and monotone increasing, and

$$\frac{d\eta}{d\tau} = \frac{1}{a}\left(\frac{\ln^2\eta}{1-\frac{2}{\ln(\eta)}}\right)\leq\frac{\eta}{a}$$

If we further require $\eta\geq\mathrm{e}^4$, we can also use the bound $1\leq \eta/(\log\eta)^4$, and therefore

$$\eta\leq\left(\frac{\eta}{\ln^2\eta}\right)^2 = \frac{\tau^2}{a^2}$$

By making the substitution to the integration variable τ in the integral, we find

$$\int_t^\infty \eta^k u_a(\eta)\,d\eta \leq \frac{1}{a^{2k+3}}\,\Gamma(2k+3,\tau(t))\,,$$

where the incomplete Gamma function $\Gamma(n+1,x)$ can be computed for any integer $n\geq 0$ by repeated integration by parts:

$$\Gamma(n+1,x) = \int_x^\infty \tau^n\mathrm{e}^{-\tau}\,d\tau = n!\,\mathrm{e}^{-x}\sum_{k=0}^n\frac{x^k}{k!}\,.$$

For $x\geq n$, this yields the bound

$$\Gamma(n+1,x)\leq (n+1)x^n\mathrm{e}^{-x}\,,$$

which can be applied with $n=2k+2$ and $x=\tau(t)\leq at$ to conclude the proof. □

Lemma 2.6. *Let $\gamma>0$ and w_γ the function defined in (2.1). Then eq. (2.12) defines a bounded, odd function $W_\gamma\in L^1(\mathbb{R})$ with the following properties:*

i. $|W_\gamma(t)|$ *is continuous and monotone decreasing for* $t \geq 0$. *In particular*

$$\|W_\gamma\|_\infty = W_\gamma(0) = 1/2\,; \tag{2.13}$$

ii. $|W_\gamma(t)| \leq G^{(W)}(\gamma|t|)$, *with* $G^{(W)}(\eta)$ *defined for* $\eta \geq 0$ *by*

$$G^{(W)}(\eta) = \begin{cases} \frac{1}{2} & 0 \leq \eta \leq \eta^* \\ 35\mathrm{e}^2\eta^4 u_{2/7}(\eta) & \eta > \eta^* \end{cases} \tag{2.14}$$

where η^* *is the largest real solution of*

$$35\mathrm{e}^2\eta^4 u_{2/7}(\eta) = 1/2\,.$$

iii. *There is a constant* K *such that*

$$\|W_\gamma\|_1 \leq \frac{K}{\gamma}\,. \tag{2.15}$$

iv. *For* $t > 0$, *let*

$$I_\gamma(t) = \int_t^\infty d\xi\, W_\gamma(\xi)\,. \tag{2.16}$$

Then, $|I_\gamma(t)| \leq G^{(I)}(\gamma|t|)$, *where* $G^{(I)}(\zeta)$ *is defined for* $\zeta \geq 0$ *by*

$$G^{(I)}(\zeta) = \frac{1}{\gamma}\cdot\begin{cases} \frac{K}{2} & 0 \leq \zeta \leq \zeta^* \\ 130\mathrm{e}^2\zeta^{10}u_{2/7}(\zeta) & \zeta > \zeta^* \end{cases}\,.$$

with K *as in (iii) and a* $\zeta^* > 0$.

Remark 2.7. It is straightforward to estimate the values of the constants η^*, ζ^*, and K, by numerical integration. One finds $14250 < \eta^* < 14251$, $36057 < \zeta^* < 36058$, and $K \sim 14708$.

Proof. i. $w_\gamma \geq 0$, even, and $\int w_\gamma = 1$. With the definition (2.12) of W_γ, this implies

$$|W_\gamma(t)| \leq \int_{|t|}^\infty w_\gamma(\xi)\,d\xi \leq \int_0^\infty w_\gamma(\xi)\,d\xi = W_\gamma(0) = \frac{1}{2}. \tag{2.17}$$

ii. The bound (2.2) for w_γ gives

$$|W_\gamma(t)| = \int_{|t|}^\infty d\xi\, w_\gamma(\xi) \leq 2\mathrm{e}^2\gamma^2\int_{|t|}^\infty d\xi\, \xi u_{2/7}(\gamma\xi) = 2\mathrm{e}^2\int_{\gamma|t|}^\infty d\eta\, \eta u_{2/7}(\eta)\,.$$

With $k = 1$ and $a = 2/7$, the conditions of Lemma 2.5 are satisfied for $\gamma|t| \geq 561$, so that

$$|W_\gamma(t)| \leq 35\mathrm{e}^2(\gamma|t|)^4 u_{2/7}(\gamma|t|)\,, \qquad \text{if } \gamma|t| \geq 561. \tag{2.18}$$

Using the decay of $u_a(\eta)$ for $\eta \geq \mathrm{e}^2$ and the fact that the RHS of (2.18) exceeds the a priori bound (2.13) for $\gamma|t| = 561$, the result follows.

iii. By (ii) $W_1 \in L^1(\mathbb{R})$ and $|W_\gamma(t)| \leq |W_1(\gamma t)|$, which implies the existence of a constant K as claimed. Using the oddness of W_γ and the explicit function $G^{(W)}(\eta)$, we choose

$$K = \eta^* + 70\mathrm{e}^2\int_{\eta^*}^\infty \eta^4 u_{2/7}(\eta)\,d\eta.$$

iv. Follows by (iii) and another application of Lemma 2.5.

□

A straightforward corollary of the decay conditions of the weight function is the following equivalent form of the generator $D(s)$, eq. (2.5).

Corollary 2.8. *The conclusions of Proposition (2.4) hold with*

$$D(s) = \int_{-\infty}^{\infty} dt\, W_\gamma(t) \cdot \mathrm{e}^{\mathrm{i}tH(s)} H'(s) \mathrm{e}^{-\mathrm{i}tH(s)}\,. \tag{2.19}$$

with W_γ as in lemma 2.6.

Proof. This follows by a simple integration by parts from (2.5). By definition of the function W_γ, we have, for any $t \in \mathbb{R} \setminus \{0\}$,

$$\frac{d}{dt} W_\gamma(t) = -w_\gamma(t)\,,$$

which can be extended by continuity at $t = 0$. Proposition (2.4) then yields

$$D(s) = -\left. W_\gamma(t) \int_0^t du\, \mathrm{e}^{\mathrm{i}uH(s)} H'(s) \mathrm{e}^{-\mathrm{i}uH(s)} \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dt\, W_\gamma(t) \cdot \mathrm{e}^{\mathrm{i}tH(s)} H'(s) \mathrm{e}^{-\mathrm{i}tH(s)}\,.$$

The boundary term vanishes by Assumption (2.1) and the decay of W_γ. □

3. Local perturbations

The aim of this section is to combine the evolution formula of Section 2 with Lieb-Robinson bounds to show that the effect of perturbations with a finite support X can be, to arbitrarily good approximation, expressed by the action of a local operator with a support that is a moderate enlargement of X. In principle, the following lemma suffices to turn Lieb-Robinson bounds into an estimate for the support of a time-evolved observable.

Lemma 3.1 ([43])**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and suppose $\epsilon \geq 0$ and $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ are such that*

$$\|[A, \mathbb{1} \otimes B]\| \leq \epsilon \|B\| \ \textit{ for all } B \in \mathcal{B}(\mathcal{H}_2).$$

Then, there exists $A' \in \mathcal{B}(\mathcal{H}_1)$, such that

$$\|A' \otimes \mathbb{1} - A\| \leq \epsilon. \tag{3.1}$$

If $\dim \mathcal{H}_2 < \infty$, one can simply take

$$A' = \frac{1}{\dim \mathcal{H}_2} \mathrm{Tr}_{\mathcal{H}_2} A,$$

as is done in [9, 36] (or see (i) in the proof of Lemma 3.2 below).

For the applications we have in mind, we want the map $A \mapsto A'$ to be continuous in the weak operator topology. In finite dimensions the partial trace is of course continuous. In infinite dimensions we cannot use the partial trace and the continuity is not obvious. Moreover, it will be convenient for us to have a map $A' = \Pi(A)$ that is compatible with the tensor product structure of the algebra of local observables of a lattice system (see Section 4.1). For this purpose, we fix a normal state ρ on $\mathcal{B}(\mathcal{H}_2)$ and define the map $\Pi : \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1) \cong \mathcal{B}(\mathcal{H}_1) \otimes \mathbb{1} \subset \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ by $\Pi = \mathrm{id} \otimes \rho$. Although the map Π depends on ρ, we have the following estimate independent of ρ.

Lemma 3.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and suppose $\epsilon \geq 0$ and $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ are such that*

$$\|[A, \mathbb{1} \otimes B]\| \leq \epsilon \|B\| \ \textit{ for all } B \in \mathcal{B}(\mathcal{H}_2).$$

Then,

$$\|\Pi(A) - A\| \leq 2\epsilon. \tag{3.2}$$

Proof. (i) First, assume $\dim \mathcal{H}_2 < \infty$. Then it suffices to take for A' the normalized partial trace of A:

$$A' = \frac{1}{\dim \mathcal{H}_2} \mathrm{Tr}_{\mathcal{H}_2} A$$

Note that

$$A' \otimes \mathbb{1} = \int_{\mathcal{U}(\mathcal{H}_2)} dU\, (\mathbb{1} \otimes U^*) A (\mathbb{1} \otimes U)$$

where dU is the Haar measure on the unitary group, $\mathcal{U}(\mathcal{H}_2)$, of $\mathcal{H}_2$. Then, by the assumptions of the Lemma, one has

$$\|A' \otimes \mathbb{1} - A\| \leq \int_{\mathcal{U}(\mathcal{H}_2)} dU\, \|(\mathbb{1} \otimes U^*)[A, (\mathbb{1} \otimes U)]\| \leq \epsilon\,.$$

(ii) In the case of infinite-dimensional $\mathcal{H}_2$, we start by defining, for $\eta \in \mathcal{H}_2$, $\|\eta\| = 1$, $A_\eta \in \mathcal{B}(\mathcal{H}_1)$ by the formula

$$\langle \phi, A_\eta \psi \rangle = \langle \phi \otimes \eta, A \psi \otimes \eta \rangle, \quad \phi, \psi \in \mathcal{H}_1\,.$$

For $\eta, \xi \in \mathcal{H}_2$, let $|\xi\rangle\langle\eta|$ denote the rank-1 operator defined by $|\xi\rangle\langle\eta|\,\phi = \langle \eta, \phi \rangle \xi$, for all $\phi \in \mathcal{H}_2$. For any three $\eta, \xi, \chi \in \mathcal{H}_2$, $\|\eta\| = \|\xi\| = \|\chi\| = 1$, note that

$$A_\xi \otimes |\eta\rangle\langle\chi| = (\mathbb{1} \otimes |\eta\rangle\langle\xi|) A (\mathbb{1} \otimes |\xi\rangle\langle\chi|)\,. \tag{3.3}$$

This equation is easily verified by equating matrix elements with arbitrary tensor product vectors $\phi \otimes \alpha$ and $\psi \otimes \beta$. By the assumptions we then have

$$\|(\mathbb{1} \otimes |\eta\rangle\langle\xi|)\, [A, \mathbb{1} \otimes |\xi\rangle\langle\eta|]\, (\mathbb{1} \otimes |\eta\rangle\langle\xi|)\| \leq \|\, [A, \mathbb{1} \otimes |\xi\rangle\langle\eta|]\, \| \leq \epsilon\,.$$

By expanding the commutator and simplifying the products in the left hand side of this inequality and using (3.3) we obtain

$$\|A_\xi - A_\eta\| = \|A_\xi \otimes |\eta\rangle\langle\xi| - A_\eta \otimes |\eta\rangle\langle\xi|\,\| \leq \epsilon\,. \tag{3.4}$$

Next, consider finite-dimensional orthogonal projections P on $\mathcal{H}_2$. Since, for each such P,

$$\|[(\mathbb{1} \otimes P) A (\mathbb{1} \otimes P), \mathbb{1} \otimes (PBP)]\| = \|[(\mathbb{1} \otimes P)[A, PBP](\mathbb{1} \otimes P)\| \leq \|[A, PBP]\| \leq \epsilon \|B\|,$$

by (i), there exists $A_P \in \mathcal{B}(\mathcal{H}_1)$ such that

$$\|A_P \otimes P - (\mathbb{1} \otimes P) A (\mathbb{1} \otimes P)\| \leq \epsilon\,. \tag{3.5}$$

Explicitly, if $\chi_1, \ldots, \chi_n$ is an o.n. basis of ran P, the construction in part (i) provides

$$A_P = \frac{1}{n} \sum_{k=1}^n A_{\chi_k}, \quad \text{and } \|A_P\| \leq \|A\|\,.$$

The diameter of the convex hull of $\{A_\chi \mid \chi \in \mathcal{H}_2, \|\chi\| = 1\}$ is bounded by ϵ due to (3.4). It follows that for any two finite-dimensonial projections P, Q on $\mathcal{H}_2$

$$\|A_P - A_Q\| \leq \epsilon\,.$$

Now, we prove the bound:

$$\|A_P \otimes \mathbb{1} - A\| \leq 2\epsilon$$

by contradiction. Suppose that for some P, $\|A_P \otimes \mathbb{1} - A\| > 2\epsilon$. Then, there exists $\delta > 0$ such that $\|A_P \otimes \mathbb{1} - A\| > 2\epsilon + \delta$. Therefore, there exist $\phi, \psi \in \mathcal{H}_\Lambda$, $\|\phi\| = \|\psi\| = 1$, such that

$$|\langle \phi, (A_P \otimes \mathbb{1} - A)\psi \rangle| > 2\epsilon + \frac{\delta}{2}\,.$$

Let Q be a finite-dimensional projection on $\mathcal{H}_2$ such that

$$\|(\mathbb{1} - \mathbb{1} \otimes Q)\phi\| \leq \frac{\delta}{8\|A\|}, \quad \text{and } \|(\mathbb{1} - \mathbb{1} \otimes Q)\psi\| \leq \frac{\delta}{8\|A\|}\,.$$

Then,

$$|\langle\phi,(\mathbb{1}\otimes Q)(A_P\otimes\mathbb{1})(\mathbb{1}\otimes Q)\psi\rangle-\langle\phi,(\mathbb{1}\otimes Q)A(\mathbb{1}\otimes Q)\psi\rangle|>2\epsilon+\frac{\delta}{2}-4\frac{\delta}{8\|A\|}\|A\|\,.$$

Since $\|A_P-A_Q\|\leq\epsilon$, this implies

$$|\langle\phi,(A_Q\otimes Q-(\mathbb{1}\otimes Q)A(\mathbb{1}\otimes Q))\psi\rangle|>\epsilon\,.$$

which contradicts (3.5).

To conclude the proof, note that for a density matrix in diagonal form, $\rho=\sum_k\rho_k\,|\xi_k\rangle\langle\xi_k|\,\xi_k$, we have that $\mathrm{id}\otimes\rho(A)=\sum_k\rho_k A_{\xi_k}$. Therefore we have

$$\|\Pi(A)-A\|=\|\sum_k\rho_k A_{\xi_k}\otimes\mathbb{1}-A\|\leq\sum_k\rho_k\|A_{\xi_k}\otimes\mathbb{1}-A\|\leq\sum_k\rho_k 2\epsilon=2\epsilon.$$

□

We now explain a *local perturbations perturb locally* principle that applies in general to any states corresponding to an isolated part of the spectrum of a system of which the dynamics has a quasi-locality property expressed by an estimate of Lieb-Robinson type. The basic argument can be applied for finite systems or for infinite systems in a suitable representation. For the sake of presentation, we consider a systems defined on a metric graph (Γ,d). To each site $x\in\Gamma$, we associate a Hilbert space $\mathcal{H}_x$. For finite $\Lambda\subset\Gamma$, we define

$$\mathcal{H}_\Lambda=\bigotimes_{x\in\Lambda}\mathcal{H}_x\quad\text{and}\quad\mathcal{A}_\Lambda=\bigotimes\mathcal{B}(\mathcal{H}_x)\tag{3.6}$$

where $\mathcal{B}(\mathcal{H}_x)$ denotes the bounded linear operators over $\mathcal{H}_x$. There is a natural way to identify $\mathcal{A}_{\Lambda_0}\subset\mathcal{A}_\Lambda$; namely identify each $A\in\mathcal{A}_{\Lambda_0}$ with $A\otimes\mathbb{1}_{\Lambda\setminus\Lambda_0}\in\mathcal{A}_\Lambda$. We can then inductively define

$$\mathcal{A}_{\mathrm{loc}}=\bigcup_{\Lambda\subset\Gamma}\mathcal{A}_\Lambda\tag{3.7}$$

where the union is taken over all finite subsets of Γ. The completion of $\mathcal{A}_{\mathrm{loc}}$ with respect to the operator norm is a C^*-algebra, which we will assume to be represented on a Hilbert space and assume that a family of Hamiltonians of the form $H(s)=H(0)+\Phi(s)$ on this space satisfies Assumption 2.1. Additionally, we assume that the Heisenberg dynamics $\tau_t^{H(s)}$, generated by $H(s)$, satisfies a Lieb-Robinson bound uniform in s.

Assumption 3.3. There are constants $C(A,B)$, $a>0$ and a Lieb-Robinson velocity $v\geq0$ such that for all $s\in[0,1]$

$$\|[\tau_t^{H(s)}(A),B]\|\leq C(A,B)\mathrm{e}^{-a\left(d(\mathrm{supp}\,A,\mathrm{supp}\,B)-v|t|\right)}$$

Here, $C(A,B)$ is of a suitable form such as $C\|A\|\,\|B\|\min(|\operatorname{supp}A|,|\operatorname{supp}B|)$.

Furthermore, we assume that there is a fixed finite subset $X\subset\Gamma$ such that $\Phi'(s)\in\mathcal{A}_X$ and

$$\|\Phi'\|=\sup_{0\leq s\leq1}\|\Phi'(s)\|<\infty.\tag{3.8}$$

The generator $D(s)$ defined in (2.5) and (2.19) for the local perturbation $\Phi(s)$ is not strictly local. However, the fast decay of the weight function $W_\gamma(t)$ in combination with Assumption 3.3 imply that the effect of $D(s)$ is small far away from X. To make this precise, let $R>0$, and denote by X_R the following 'fattening' of X:

$$X_R=\{x\,:\,\exists y\in X\text{ s.t. }d(x,y)\leq R\}\,.\tag{3.9}$$

The following result shows that in the situation described above the unitary $U(s)$ of (2.3) in Proposition 2.4 can be well-approximated by a unitary $V_R(s)\in\mathcal{A}_{X_R}$, i.e., with support in X_R.

Theorem 3.4 (Local Perturbations Perturb Locally). *For any $R > 0$, there exist unitary operators $V_R(s)$ with $\mathrm{supp}(V_R(s)) \subset X_R$ and a constant C, independent of R, such that*

$$\|U(s) - V_R(s)\| \leq CG^{(I)}(\frac{\gamma R}{2v})$$

with $G^{(I)}$ the subexponential function defined in Lemma 2.6. Consequently, we also have

$$\|P(1) - V_R(1)P(0)V_R(1)^*\| \leq 2CG^{(I)}(\frac{\gamma R}{2v}). \tag{3.10}$$

Proof. We begin by defining a local approximation of the self-adjoint generator $D(s)$ starting from (2.19). Consider the decomposition $\mathcal{A}_{\mathrm{loc}} = \mathcal{A}_{X_R} \otimes \mathcal{A}_{\Gamma\setminus X_R}$ and let $\Pi_R = \mathrm{id} \otimes \rho$ for some state ρ on $\mathcal{A}_{\Gamma\setminus X_R}$, and define

$$D_R(s) = \int_{-\infty}^{\infty} dt\, W_\gamma(t)\Pi_R(\mathrm{e}^{\mathrm{i}tH(s)}\Phi'(s)\mathrm{e}^{-\mathrm{i}tH(s)})\,.$$

Then, for any $T > 0$ we have the following estimate:

$$\|D(s) - D_R(s)\| \leq \|\Phi'\| \int_{|t|>T} dt\, |W_\gamma(t)| + \|W_\gamma\|_\infty \int_{-T}^{T} dt\, \|(\mathrm{id} - \Pi_R)(\mathrm{e}^{\mathrm{i}tH(s)}\Phi'(s)\mathrm{e}^{-\mathrm{i}tH(s)})\|\,.$$

For the first term, we apply the bound of Lemma 2.6 part (iv) and for the second term we use (2.13) and Lemma 3.2 and Assumption 3.3 to get

$$\|D(s) - D_R(s)\| \leq \|\Phi'\| 2CG^{(I)}(\gamma T) + \frac{1}{2}C\|\Phi'\||X|e^{-a(R-vT)}.$$

For the simple choice $T = R/(2v)$, for not too small R, the second term is negligible compared to the first , and we obtain

$$\|D(s) - D_R(s)\| \leq C'\|\Phi'\|G^{(I)}(\frac{\gamma R}{2v}). \tag{3.11}$$

Now, let $V_R(s)$ be solution of

$$-i\frac{\partial}{\partial s}V_R(s) = D_R(s)V_R(s)\,, \qquad V_R(0) = \mathbb{1}.$$

The claim follows by integrating the estimate (3.11). $\square$

To illustrate this result, we consider the case where the isolated part of the spectrum, $\Sigma_1(s)$ in Assumption 2.1, consists of a non-degenerate ground state energy. Let $\psi_0(s)$ denote the corresponding normalized eigenvector and let $A \in \mathcal{A}_{\Lambda\setminus X_R}$ be an observable supported away from the perturbation, whence $[A, V_R] = 0$. By applying Theorem 3.4 we immediately obtain

$$\begin{aligned}
|\langle\psi(s), A\psi(s)\rangle - \langle\psi(0), A\psi(0)\rangle| &= |\langle\psi(0), U(s)^*[A, U(s)]\psi(0)\rangle| \\
&= |\langle\psi(0), U(s)^*[A, U(s) - V_R(s)]\psi(0)\rangle| \\
&\leq 2\|A\|\|U(s) - V_R(s)\| \leq 2C\|A\|G^{(I)}(\frac{\gamma R}{2v})
\end{aligned}$$

This estimate clearly expresses the locality of the effect of the perturbation on the state $\psi(s)$.

4. The spectral flow and quasi-locality

The main goal of this section is to prove that the spectral flow defined in terms of the unitary operators $U(s)$, as in Proposition 2.4, satisfies a Lieb-Robinson bound. This is the content of Theorem 4.5 below. In Section 4.1, we introduce the basic models to which our result applies and state Theorem 4.5. Our proof of Theorem 4.5 demonstrates that the claimed estimate follows from a Lieb-Robinson bound for time-dependent interactions. We state and prove a general result of this type, see Theorem 4.6, in Section 4.2. The remainder of Section 4 is used prove that Theorem 4.6

is applicable in the context of the spectral flow. Section 4.3 contains a technical lemma, and Section 4.4 finishes the proof.

4.1. **The set-up and a statement of the main result.** The arguments we provide in Section 4 apply to a large class of models. In this subsection, we describe in detail the assumptions necessary to prove a Lieb-Robinson bound for the spectral flow.

We will consider models defined on a countable set Γ equipped with a metric d. Typically, Γ will be infinite, e.g., $\Gamma = \mathbb{Z}^\nu$. In the case that Γ is infinite, we require some assumptions on the structure of Γ as a set. First, we will assume a uniform bound on the rate at which balls grow, i.e., we assume there exist numbers $\kappa > 0$ and $\nu > 0$ for which

$$\sup_{x\in\Gamma} |B_r(x)| \leq \kappa r^\nu \, , \tag{4.1}$$

where $|B_r(x)|$ is the cardinality of the ball centered at x of radius r. In addition, we will assume that Γ has some 'integrable' underlying structure. We express this property in terms of a non-increasing, real-valued function $F : [0,\infty) \to (0,\infty)$ that satisfies
i) *uniform integrablility*: i.e.

$$\|F\| = \sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x,y)) \; < \; \infty \tag{4.2}$$

and
ii) *a convolution condition*: i.e., there exists a number $C_F > 0$ such that given any pair $x, y \in \Gamma$,

$$\sum_{z\in\Gamma} F(d(x,z))F(d(z,y) \leq C_F F(d(x,y)) \, . \tag{4.3}$$

For the case of $\Gamma = \mathbb{Z}^\nu$, one possible choice of F is given by $F(r) = (1+r)^{-(\nu+1)}$. The corresponding convolution constant may be taken as $C_F = 2^{\nu+1} \sum_{x\in\Gamma} F(|x|)$.

Lastly, we need an assumption on the rate at which F goes to zero. It is convenient to express this in terms of the sub-exponential function u_a introduced in Lemma 2.5. We suppose that there exists a number $0 < \delta < 2/7$ such that

$$\sup_{r\geq 1} \frac{u_\delta(r)}{F(r)} < \infty \, . \tag{4.4}$$

Clearly, if $\Gamma = \mathbb{Z}^\nu$ and $F(r) = (1+r)^{-(\nu+1)}$, then (4.4) holds for every $0 < \delta < 2/7$.

The following observations will be useful. Let $F : [0,\infty) \to (0,\infty)$ be a non-increasing function satisfying (4.2) and (4.3). For each $a \geq 0$, the function $F_a(r) = e^{-ar}F(r)$ also satisfies the properties (4.2) and (4.3) with $\|F_a\| \leq \|F\|$ and $C_{F_a} \leq C_F$. In fact, more generally, if g is positive, non-increasing, and logarithmically super-additive, i.e., $g(x+y) \geq g(x)g(y)$, then $F_g(r) = g(r)F(r)$ satisfies (4.2) and (4.3) with $\|F_g\| \leq g(0)\|F\|$ and $C_{F_g} \leq C_F$. For brevity we will write F_a to denote the case $g(r) = e^{-ar}$. Other functions g will be used later.

Recall the general quantum systems corresponding to Γ on which our models will be defined. As in Section 3, we associate a Hilbert space $\mathcal{H}_\Lambda$ and an algebra of observables $\mathcal{A}_\Lambda$ to each finite set $\Lambda \subset \Gamma$, see (3.6), and similarly define $\mathcal{A}_{\rm loc}$ as in (3.7). In this case, the models we consider are comprised of two types of terms. First, we fix a collection of Hamiltonians, which we label by $(H_\Lambda(0))_\Lambda$, with the property that for each finite $\Lambda \subset \Gamma$, $H_\Lambda(0)$ is a densely defined, self-adjoint operator on $\mathcal{H}_\Lambda$. Next, we consider a family of interactions $\Phi(s)$ parametrized by a real number s. For each s, the interaction $\Phi(s)$ on Γ is a mapping from the set of finite subsets of Γ into $\mathcal{A}_{\rm loc}$ with the property that $\Phi(X,s)^* = \Phi(X,s) \in \mathcal{A}_X$ for all finite $X \subset \Gamma$. It is convenient to write $\Phi(X,s) = \Phi_X(s)$. A model then consists of a choice of $(H_\Lambda(0))_\Lambda$ and a family of interactions $\Phi(s)$

over Γ. Given a model, we associate local Hamiltonians to each finite set $\Lambda \subset \Gamma$ by setting

$$H_\Lambda(s) = H_\Lambda(0) + \sum_{X \subset \Lambda} \Phi_X(s) \tag{4.5}$$

where the sum is taken over all subsets $X \subset \Lambda$. For notational consistency, we will assume that $\Phi_X(0) = 0$ for all X. With s fixed, the sum in (4.5) above is finite for each such $\Lambda \subset \Gamma$, and thus self-adjointness guarantees the existence of the Heisenberg dynamics, i.e.,

$$\tau_t^{H_\Lambda(s)}(A) = e^{itH_\Lambda(s)} A e^{-itH_\Lambda(s)} \quad \text{for all } A \in \mathcal{A}_\Lambda \text{ and } t \in \mathbb{R}, \tag{4.6}$$

which, again for fixed s, is a one-parameter group of automorphisms on $\mathcal{A}_\Lambda$.

To prove the results in this section, we need a boundedness assumption on the family of interactions. We make this precise by introducing a norm on the interactions $\Phi(s)$ over Γ, with respect to any non-increasing, positive function F satisfying (4.2) and (4.3), as follows:

$$\|\Phi\|_F = \sup_{x,y \in \Gamma} \frac{1}{F(d(x,y))} \sum_{\substack{Z \subset \Gamma: \\ x,y \in Z}} \sup_s \|\Phi_Z(s)\| < \infty \,. \tag{4.7}$$

The sum above is over all finite sets $Z \subset \Gamma$ containing x and y, and we will often abbreviate $\|\cdot\|_{F_a}$ by $\|\cdot\|_a$. On occasion, we will use $\|\Phi(s)\|_F$ for the norm of $\Phi(s)$ at fixed s, i.e., the norm defined by dropping the supremum over s in (4.7). The following lemma states some simple bounds in terms of $\|\Phi\|_F$ that we will frequently use.

Lemma 4.1. *Let $\Phi(s)$ be a family of interactions over Γ for which $\|\Phi\|_F < \infty$ for some non-increasing, positive function F satisfying (4.2) and (4.3) above. Then, for any finite $\Lambda \subset \Gamma$, we have*

$$\sum_{\substack{X \subset \Lambda: \\ x \in X}} \|\Phi_X(s)\| \leq F(0)\|\Phi\|_F \tag{4.8}$$

$$\sum_{X \subset \Lambda} \|\Phi_X(s)\| \leq F(0)\|\Phi\|_F |\Lambda| \,. \tag{4.9}$$

Proof. For $x \in \Gamma$ we have

$$\sum_{\substack{X \subset \Lambda: \\ x \in X}} \|\Phi_X(s)\| \leq \sup_{y \in \Gamma} F(d(x,y)) \sum_{\substack{X \subset \Gamma: \\ x,y \in X}} \frac{\|\Phi_X(s)\|}{F(d(x,y))} \leq F(0)\|\Phi\|_F$$

where we have used the definition of the norm (4.7) and the monotonicity of F. Using this estimate, for any finite subset $\Lambda \subset \Gamma$, we obtain the bound

$$\sum_{X \subset \Lambda} \|\Phi_X(s)\| \leq \sum_{x \in \Lambda} \sum_{\substack{X \subset \Lambda: \\ x \in X}} \|\Phi_X(s)\| \leq F(0)\|\Phi\|_F |\Lambda| \,.$$

□

We will also require the interactions to be smooth with bounded derivatives. More concretely, let $\Phi(s)$ be a family of interactions over Γ for which, given any finite $X \subset \Gamma$, $\Phi_X(s)$ is differentiable with respect to s. In this case, we define a corresponding family of interaction $\partial\Phi(s)$ over Γ by the the formula

$$\partial\Phi_X(s) = |X| \Phi'_X(s) \quad \text{for each finite } X \subset \Gamma.$$

We now state the main assumptions of this section.

Assumption 4.2. We will assume that the interactions $\Phi(s)$ are differentiable with respect to s. More specifically, we assume that for each finite $X \subset \Gamma$, $\Phi'_X(s) \in \mathcal{A}_X$ for all s. In addition, we suppose a uniform estimate on the norms of these derivatives as s varies in compact sets. For concreteness, we will assume that the domain of s-values is $[0,1]$, and suppose that there exists a number $a > 0$ for which

$$\|\partial\Phi\|_a < \infty.$$

Assumption 4.3. We will assume that for every finite $\Lambda \subset \Gamma$, the local Hamiltonian $H_\Lambda(s)$ has a spectrum that is uniformly gapped. More precisely, the spectrum of $H_\Lambda(s)$, which we will denote by $\Sigma^{(\Lambda)}(s)$, can be decomposed into two non-empty sets: $\Sigma^{(\Lambda)}(s) = \Sigma_1^{(\Lambda)}(s) \cup \Sigma_2^{(\Lambda)}(s)$ with $d(\Sigma_1^{(\Lambda)}(s), \Sigma_2^{(\Lambda)}(s)) \geq \gamma > 0$. In particular, the positive number γ is independent of $s \in [0,1]$ and finite $\Lambda \subset \Gamma$. We also suppose that there exist intervals $I(s)$, with endpoints depending smoothly on s, for which $\Sigma_1^{(\Lambda)}(s) \subset I(s)$.

In typical applications, the set $\Sigma_1^{(\Lambda)}(s)$ will consist of the ground state and (possibly) other low-lying energies, but this is not necessary.

Given Assumptions 4.2 and 4.3, the results of Section 2 apply to the local Hamiltonians $H_\Lambda(s)$. We need a further assumption in order to state the main result of this section.

Assumption 4.4. We will assume a uniform, exponential Lieb-Robinson bound. In fact, we assume that there exists an $a > 0$ and numbers K_a and v_a such that

$$\left\| \left[\tau_t^{H_\Lambda(s)}(A), B \right] \right\| \leq K_a \|A\| \|B\| e^{a v_a |t|} \sum_{x \in X, y \in Y} F_a(d(x,y)) \tag{4.10}$$

holds for all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $t \in \mathbb{R}$. Here, as above, $F_a(r) = e^{-ar} F(r)$, and we stress that the numbers K_a and v_a are each independent of both Λ and s.

Estimates of the form (4.10) have been demonstrated for a number of models, see e.g. [42], and references therein, for a recent review. Here we assume it holds for a class of models, and as a consequence, we get Theorem 4.5 below.

As indicated above, given Assumptions 4.2 and 4.3, the results of Proposition 2.4 apply to $H_\Lambda(s)$ for each finite $\Lambda \subset \Gamma$ and $s \in [0,1]$. In this case, there are unitaries $U_\Lambda(s)$ in terms of which we define the following *spectral flow*:

$$\alpha_s^\Lambda(A) = U_\Lambda(s)^* A U_\Lambda(s) \quad \text{for all } A \in \mathcal{A}_\Lambda \quad \text{and } 0 \leq s \leq 1\,. \tag{4.11}$$

The main result of this section is a Lieb-Robinson bound for the spectral flow, which is formulated with the aid of a function F_Ψ defined as follows:

$$F_\Psi(r) = \tilde{u}_\mu\left(\frac{\gamma}{8v_a} r\right) F\left(\frac{\gamma}{8v_a} r\right), \tag{4.12}$$

where

$$\tilde{u}_\mu(x) = \begin{cases} u_\mu(e^2) & \text{for } 0 \leq r \leq e^2, \\ u_\mu(x) & \text{otherwise.} \end{cases} \tag{4.13}$$

Since F is uniformly integrable over Γ and $\tilde{u}_\mu(r) \leq 1$, F_Ψ satisfies (4.2). Moreover, F_Ψ also satisfies (4.3). In fact, it is easy to check that $\tilde{u}_\mu$ is positive, non-increasing, and logarithmically superadditive. The Lieb-Robinson velocity in the following theorem also involves the norm $\|\Psi\|_{F_\Psi}$ of an interaction Ψ defined later in this section (see (4.46)).

Theorem 4.5. *Let Assumptions 4.2, 4.3, and 4.4 hold. Then,*

$$\left\| \left[\alpha_s^\Lambda(A), B \right] \right\| \leq 2\|A\|\|B\| \min\left[1, g(s) \sum_{x \in X, y \in Y} F_\Psi(d(x,y))\right], \tag{4.14}$$

for any $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, *and* $0 \le s \le 1$ *and* g *is given by*

$$C_{F_\Psi} \cdot g(t) = \begin{cases} e^{2\|\Psi\|_{F_\Psi} C_{F_\Psi}|t|} - 1 & \text{if } d(X,Y) > 0, \\ e^{2\|\Psi\|_{F_\Psi} C_{F_\Psi}|t|} & \text{otherwise.} \end{cases} \tag{4.15}$$

The number C_{F_Ψ} *is as in (4.3) and our estimate on* $\|\Psi\|_{F_\Psi}$ *is discussed in the next subsections.*

4.2. **Lieb-Robinson bounds for time-dependent interactions.** The estimate (4.14) in the statement of Thereom 4.5 can be understood as a Lieb-Robinson bound for the spectral flow. In this section, we demonstrate that Lieb-Robinson bounds hold for a large class of time-dependent interactions. As in the previous section, we assume that our models are defined on a countable set Γ equipped with a metric. Let Φ_t denote a family of interactions over Γ, and, for convenience, we will assume that $t \in [0,1]$. Thus, for every finite $X \subset \Gamma$ and each $t \in [0,1]$, $\Phi_t(X)^* = \Phi_t(X) \in \mathcal{A}_X$, and we will often write $\Phi_t(X) = \Phi_X(t)$.

In this case, corresponding to each finite $\Lambda \subset \Gamma$, there is a time-dependent local Hamiltonian which we denote by

$$H_\Lambda(t) = \sum_{X \subset \Lambda} \Phi_X(t)\,. \tag{4.16}$$

We will assume that, for each finite $\Lambda \subset \Gamma$, $H_\Lambda(t)$ is a strongly continuous map from $[0,1]$ into $\mathcal{A}_\Lambda$. In this case, see e.g. Theorem X.69 [47], it is well-known that there exists a two-parameter family of unitary propagators $U_\Lambda(t,s)$ with

$$\frac{d}{dt} U_\Lambda(t,s) = -iH_\Lambda(t) U_\Lambda(t,s) \quad \text{and} \quad U_\Lambda(s,s) = \mathbb{1}\,, \tag{4.17}$$

the above equation holding in the strong sense. The Heisenberg dynamics corresponding to $H_\Lambda(t)$ is then defined by setting

$$\tau_t^\Lambda(A) = U_\Lambda(t,0)^* A U_\Lambda(t,0) \quad \text{for all } A \in \mathcal{A}_\Lambda\,. \tag{4.18}$$

The following Lieb-Robinson bound holds.

Theorem 4.6. *Let* F *be a non-increasing, positive function satisfying (4.2) and (4.3) and suppose that the interactions* Φ_t *satisfy*

$$\|\Phi\|_F = \sup_{x,y \in \Gamma} \frac{1}{F(d(x,y))} \sum_{\substack{Z \subset \Gamma: \\ x,y \in Z}} \sup_{0 \le t \le 1} \|\Phi_Z(t)\| \, < \, \infty\,. \tag{4.19}$$

Then, for any subsets $X, Y \subset \Gamma$, $A \in \mathcal{A}_X$ *and* $B \in \mathcal{A}_Y$ *the estimate*

$$\left\| \left[\tau_t^\Lambda(A), B \right] \right\| \le 2\|A\|\|B\| \min \left[1, g(t) \sum_{x \in X, y \in Y} F(d(x,y)) \right], \tag{4.20}$$

where the function g *may be taken as*

$$C_F \cdot g(t) = \begin{cases} e^{2\|\Phi\|_F C_F |t|} - 1 & \text{if } d(X,Y) > 0, \\ e^{2\|\Phi\|_F C_F |t|} & \text{otherwise,} \end{cases} \tag{4.21}$$

and the number C_F *is as in (4.3).*

Proof. Let $X, Y \subset \Gamma$ be finite sets. Take $\Lambda \subset \Gamma$ finite with $X \cup Y \subset \Lambda$. Define the function $f : [0,1] \to \mathcal{A}_\Lambda$ by setting

$$f(t) = [U_\Lambda(t,0)^* U_X(t,0) A U_X(t,0)^* U_\Lambda(t,0), B] = \left[\tau_t^\Lambda \left(\tilde{\tau}_t^X(A) \right), B \right]\,, \tag{4.22}$$

where we have introduced the notation $\tilde{\tau}_t^X(A) = U_X(t,0) A U_X(t,0)^*$. Denoting by

$$S_X^\Lambda = \{ Z \subset \Lambda : Z \cap X \neq \emptyset, Z \cap X^c \neq \emptyset \}, \tag{4.23}$$

the surface of X, a short calculation shows that

$$\begin{aligned} f'(t) &= i\left[\tau_t^\Lambda\left([H_\Lambda(t)-H_X(t),\tilde{\tau}_t^X(A)]\right)B\right] \\ &= i\sum_{\substack{Z\subset\Lambda:\\ Z\in S_X^\Lambda}}\left[\tau_t^\Lambda(\Phi_Z(t)),f(t)\right]+i\sum_{\substack{Z\in\Lambda:\\ Z\in S_X^\Lambda}}\left[\tau_t^\Lambda(\tilde{\tau}_t^X(A)),[B,\tau_t^\Lambda(\Phi_Z(t))]\right]. \end{aligned}$$

As the first term above is norm-preserving, see e.g. [36], the inequality

$$\left\|\left[\tau_t^\Lambda\left(\tilde{\tau}_t^X(A)\right),B\right]\right\|\leq\|[A,B]\|+2\|A\|\sum_{\substack{Z\subset\Lambda:\\ Z\in S_X^\Lambda}}\int_0^{|t|}\left\|\left[\tau_s^\Lambda\left(\Phi_Z(s)\right),B\right]\right\|\,ds \tag{4.24}$$

follows. Consider now the quantity

$$C_B^\Lambda(X,t)=\sup_{\substack{A\in\mathcal{A}_X:\\ A\neq0}}\frac{\|[\tau_t^\Lambda(A),B]\|}{\|A\|} \tag{4.25}$$

It is easy to see that

$$C_B^\Lambda(X,t)\leq C_B^\Lambda(X,0)+2\sum_{\substack{Z\subset\Lambda:\\ Z\in S_X^\Lambda}}\sup_{0\leq r\leq1}\|\Phi_Z(r)\|\int_0^{|t|}C_B^\Lambda(Z,s)\,ds\,. \tag{4.26}$$

From here, the argument proceeds as in the proof of Theorem 2.1 in [36]. □

4.3. **Some notation and a lemma.** In this subsection, we prove a technical estimate needed in our proof of Theorem 4.5. The objective is to show that the s-dependent generator of the unitary flow $U_\Lambda(s)$ has the structure of a bonafide short-range interaction. In Theorem 3.4 we showed that each term of the perturbation, i.e., $\Phi_X(s)$ for a given X, leads to a term in the generator that can be well approximated by *local* self-adjoint operator supported in X_R with almost exponentially fast decay of the error as a function of R. A projection $\Pi_{X_R}:\mathcal{A}_\Lambda\to\mathcal{A}_{X_R}$ was used to accomplish this. In this subsection and the next we apply the same procedure to show that the differences between successive approximations can be summed leading to a decomposition of each term in the generator as a telescopic sum of finitely supported terms. To define the terms in this decomposition we need a family of projection mappings $(\Pi_X)_{X\subset\Lambda}$, and the decomposition we obtain will depend on the choice of this family. It will be convenient to choose a family which is compatible with the embeddings $\mathcal{A}_{\Lambda_0}\subset\mathcal{A}_\Lambda$, for $\Lambda_0\subset\Lambda$, and such that each of the Π_X are continuous in the norm and weak topologies on $\mathcal{A}_\Lambda$. We will therefore choose a family of normal states on $\mathcal{B}(\mathcal{H}_x)$, or equivalently, a family of density matrices, $(\rho_x)_{x\in\Gamma}$ so that we can define a product state on $\mathcal{A}_{X^c}$ by setting $\rho_{X^c}=\bigotimes_{x\in\Gamma\setminus X}\rho_x$. Then, for any finite $X\subset\Lambda$, we define

$$\Pi_X=\mathrm{id}_{\mathcal{A}_X}\otimes\rho_{X^c}|_{\mathcal{A}_\Lambda}\,. \tag{4.27}$$

Here, $\mathrm{id}_{\mathcal{A}_X}$ is the identity map on $\mathcal{A}_X$. Π_X can be considered as a map $\mathcal{A}_\Lambda\to\mathcal{A}_\Lambda$ with $\mathrm{ran}\Pi_X\subset\mathcal{A}_X$.

We let the dependence of Π_X on the ρ_x be implicit. All our estimates will be uniform in the ρ_x. Similarly, the interaction $\Psi_\Lambda(s)$ we define in the next subsection depends on the choice of ρ_x, but the estimates on its decay will not, and the unitary flow generated by these interactions also does *not* depend on the ρ_x.

Fix a finite set $\Lambda\subset\Gamma$. For any $X\subset\Gamma$ and $n\geq0$, denote by

$$X_n=\{z\in\Gamma:d(z,X)\leq n\}\,, \tag{4.28}$$

where $d(z,X)=\min_{x\in X} d(z,x)$. Keeping with the notation from the previous subsection, for any $A\in\mathcal{A}_X$ we set

$$\Delta^0_\Lambda(A,s)=\int_{-\infty}^{\infty}\Pi_X\left(\tau_t^{H_\Lambda(s)}(A)\right)W_\gamma(t)\,dt \tag{4.29}$$

and

$$\Delta^n_\Lambda(A,s)=\int_{-\infty}^{\infty}\Pi_{X_n}\left(\tau_t^{H_\Lambda(s)}(A)\right)W_\gamma(t)\,dt-\int_{-\infty}^{\infty}\Pi_{X_{n-1}}\left(\tau_t^{H_\Lambda(s)}(A)\right)W_\gamma(t)\,dt \tag{4.30}$$

for any $n\geq 1$. Since Λ is finite, $\Delta^n_\Lambda(A,s)=0$ for large n. Moreover, it is clear that $\mathrm{supp}\left(\Delta^n_\Lambda(A,s)\right)\subset X_n\cap\Lambda$. In our proof of Theorem 4.5, we will use that

$$\int_{-\infty}^{\infty}\tau_t^{H_\Lambda(s)}(A)W_\gamma(t)\,dt=\sum_{n=0}^{\infty}\Delta^n_\Lambda(A,s) \tag{4.31}$$

where the series is actually a finite sum. In fact, the following estimate is also important.

Lemma 4.7. *Under Assumptions 4.3 and 4.4, let $\Lambda\subset\Gamma$ be a finite set. For any $X\subset\Lambda$, $A\in\mathcal{A}_X$, and integer $n\geq 0$,*

$$\|\Delta^n_\Lambda(A,s)\|\leq 2\|A\|\min\left[\|W_\gamma\|_1,|X|G(n-1)\right] \tag{4.32}$$

where

$$G(n)=4I_\gamma\left(\frac{n}{2v_a}\right)+\frac{K_a\|F\|}{av_a}e^{-an/2} \tag{4.33}$$

and I_γ is as in Lemma 2.6.

Proof. It is easy to see that

$$\left\|\Delta^0_\Lambda(A,s)\right\|\leq\|A\|\,\|W_\gamma\|_1\quad\text{and}\quad\|\Delta^n_\Lambda(A,s)\|\leq 2\|A\|\,\|W_\gamma\|_1\,. \tag{4.34}$$

A better estimate in n is achieved by inserting and removing an identity. In fact, we need only estimate the norm of

$$\int_{-\infty}^{\infty}(\Pi_{X_n}-\mathrm{id})\left(\tau_t^{H_\Lambda(s)}(A)\right)W_\gamma(t)\,dt\,. \tag{4.35}$$

To do so, we follow the same strategy as in the proof of Theorem 3.4. By Assumption 4.4, we know that

$$\left\|\left[\tau_t^{H_\Lambda(s)}(A),B\right]\right\|\leq K_a\|F\||X|\|A\|e^{av_a|t|}e^{-an}\|B\| \tag{4.36}$$

for all $B\in\mathcal{A}_{X_n^c}$. Hence, for any $T>0$, we have that

$$\begin{aligned}\left\|\int_{|t|\leq T}(\Pi_{X_n}-\mathrm{id})\left(\tau_t^{H_\Lambda(s)}(A)\right)W_\gamma(t)\,dt\right\| &\leq \frac{1}{2}\int_{|t|\leq T}\left\|(\Pi_{X_n}-\mathrm{id})\left(\tau_t^{H_\Lambda(s)}(A)\right)\right\|\,dt\\ &\leq K_a\|F\||X|\|A\|\int_0^T e^{av_at}\,dt\,e^{-an}\,,\end{aligned} \tag{4.37}$$

using Lemma 3.2, whereas

$$\left\|\int_{|t|>T}(\Pi_{X_n}-\mathrm{id})\left(\tau_t^{H_\Lambda(s)}(A)\right)W_\gamma(t)\,dt\right\|\leq 4\|A\|I_\gamma(T)\,. \tag{4.38}$$

The choice of $T=n/2v_a$ yields an estimate of the form

$$\left\|\int_{-\infty}^{\infty}(\Pi_{X_n}-\mathrm{id})\left(\tau_t^{H_\Lambda(s)}(A)\right)W_\gamma(t)\,dt\right\|\leq 4\|A\|I_\gamma\left(\frac{n}{2v_a}\right)+\frac{K_a\|F\|}{av_a}|X|\|A\|e^{-an/2}\,. \tag{4.39}$$

The bound (4.32) readily follows. ∎

As indicated by the proof above, a stronger inequality is true. We have actually shown that for every $n \geq 1$,

$$\|\Delta_\Lambda^n(A,s)\| \leq 2\|A\| \min\left[\|W_\gamma\|_1, G_A(n-1) + G_A(n)\right] \tag{4.40}$$

where

$$G_A(n) = 2I_\gamma\left(\frac{n}{2v_a}\right) + \frac{K_a\|F\|}{2av_a}|X|e^{-an/2}\,. \tag{4.41}$$

For the arguments we use below, it is convenient to extract a decaying quantity that is independent of the given observable A and use the monotonicity of G. This explains the form of the bound (4.32) appearing in Lemma 4.7.

4.4. **The proof of Theorem 4.5.** In this subsection, we prove Theorem 4.5. The basic idea is that Theorem 4.5 follows from a Lieb-Robinson bound for time-dependent interactions, see e.g. Theorem 4.6 in Section 4.2. To see that such a result is applicable, we demonstrate that the generator of the spectral flow can be written as a sum of local interaction terms which satisfy an appropriate decay assumption. This is the content of Theorem 4.8 below.

Under Assumptions 4.2 and 4.3, we have defined (for each finite $\Lambda \subset \Gamma$) a spectral flow by setting

$$\alpha_s^\Lambda(A) = U_\Lambda(s)^* A U_\Lambda(s) \quad \text{for all } A \in \mathcal{A}_\Lambda. \tag{4.42}$$

In fact, the unitary $U_\Lambda(s)$ is the one constructed in Proposition 2.4, and as a consequence of Corollary 2.8, we know that $U_\Lambda(s)$ is generated by

$$\begin{aligned} D_\Lambda(s) &= \int_{-\infty}^{\infty} \tau_t^{H_\Lambda(s)}\left(H'_\Lambda(s)\right) W_\gamma(t)\,dt \\ &= \sum_{Z\subset\Lambda} \int_{-\infty}^{\infty} \tau_t^{H_\Lambda(s)}\left(\Phi'_Z(s)\right) W_\gamma(t)\,dt\,. \end{aligned} \tag{4.43}$$

Here γ is as in Assumption 4.3, and W_γ appears in Lemma 2.6 . The previous subsection demonstrated that each term

$$\int_{-\infty}^{\infty} \tau_t^{H_\Lambda(s)}\left(\Phi'_Z(s)\right) W_\gamma(t)\,dt = \sum_{n=0}^{\infty} \Delta_\Lambda^n(\Phi'_Z(s),s) \tag{4.44}$$

where the series is actually a finite sum. Combining (4.43) and (4.44) above, we write

$$D_\Lambda(s) = \sum_{Z\subset\Lambda}\sum_{n=0}^{\infty} \Delta_\Lambda^n(\Phi'_Z(s),s) = \sum_{Z\subset\Lambda} \Psi_\Lambda(Z,s)\,, \tag{4.45}$$

where

$$\Psi_\Lambda(Z,s) = \sum_{n\geq 0}\sum_{\substack{Y\subset\Lambda:\\ Y_n=Z}} \Delta_\Lambda^n(\Phi'_Y(s),s)\,. \tag{4.46}$$

It is important here to note that $\mathrm{supp}(\Psi_\Lambda(Z,s)) \subset Z$, i.e., the s-dependent, interaction terms $\Psi_\Lambda(Z,s)$ are strictly local. The following estimate holds.

Theorem 4.8. *Let Assumptions 4.2, 4.3, and 4.4 hold. Then, there exists a function F_Ψ satisfying (4.2) and (4.3) such that*

$$\|\Psi_\Lambda\|_{F_\Psi} = \sup_{x,y\in\Lambda} \frac{1}{F_\Psi(d(x,y))} \sum_{\substack{Z\subset\Lambda:\\ x,y\in Z}} \sup_{0\leq s\leq 1} \|\Psi_\Lambda(Z,s)\| < \infty\,. \tag{4.47}$$

Here we note that the function F_Ψ is independent of Λ.

It is now clear that Theorem 4.5 follows from Theorem 4.8 via an application of Theorem 4.6.

Proof. In the argument below, it is convenient to set $a > 0$ to be the minimum of the a's whose existences are guaranteed by Assumptions 4.2 and 4.4.

We begin by re-writing the quantity of interest. Clearly,

$$\sup_{0\leq s\leq 1} \|\Psi_\Lambda(Z,s)\| \leq \sum_{\substack{Y,n\geq 0:\\ Y_n=Z}} \sup_{0\leq s\leq 1} \|\Delta^n_\Lambda(\Phi'_Y(s),s)\| \,, \tag{4.48}$$

and so

$$\begin{aligned}
\sum_{\substack{Z\subset\Lambda:\\ x,y\in Z}} \sup_{0\leq s\leq 1} \|\Psi_\Lambda(Z,s)\| &\leq \sum_{\substack{Z\subset\Lambda:\\ x,y\in Z}} \sum_{\substack{Y,n\geq 0:\\ Y_n=Z}} \sup_{0\leq s\leq 1} \|\Delta^n_\Lambda(\Phi'_Y(s),s)\| \\
&= \sum_{Y\subset\Lambda} \sum_{n\geq 0} \mathrm{Ind}\left[x,y\in Y_n\right] \sup_{0\leq s\leq 1} \|\Delta^n_\Lambda(\Phi'_Y(s),s)\| \\
&= \sum_{\substack{Y\subset\Lambda:\\ x,y\in Y}} \sum_{n\geq 0} \sup_{0\leq s\leq 1} \|\Delta^n_\Lambda(\Phi'_Y(s),s)\| \qquad (4.49)\\
&\quad + \sum_{m=1}^{\infty} \sum_{\substack{Y\subset\Lambda:\\ x,y\in Y_m}} \mathrm{Ind}\left[\{x,y\}\cap Y^c_{m-1}\neq\emptyset\right] \sum_{n\geq m} \sup_{0\leq s\leq 1} \|\Delta^n_\Lambda(\Phi'_Y(s),s)\| \,.
\end{aligned}$$

The first equality above follows from the observation that

$$\begin{aligned}
\sum_{\substack{Z\subset\Lambda:\\ x,y\in Z}} \sum_{\substack{Y,n\geq 0:\\ Y_n=Z}} &= \sum_{Z\subset\Lambda} \sum_{Y\subset\Lambda} \sum_{n\geq 0} \mathrm{Ind}\left[Y_n=Z\right] \mathrm{Ind}\left[x,y\in Z\right] \\
&= \sum_{Y\subset\Lambda} \sum_{n\geq 0} \left[\sum_{Z\subset\Lambda} \mathrm{Ind}\left[Y_n=Z\right] \right] \mathrm{Ind}\left[x,y\in Y_n\right] \\
&= \sum_{Y\subset\Lambda} \sum_{n\geq 0} \mathrm{Ind}\left[x,y\in Y_n\right] \,, \qquad (4.50)
\end{aligned}$$

while the second is a consequence of the fact that for any pair x, y

$$\sum_{Y\subset\Lambda} = \sum_{\substack{Y\subset\Lambda:\\ x,y\in Y}} + \sum_{m\geq 1} \sum_{\substack{Y\subset\Lambda:\\ x,y\in Y_m}} \mathrm{Ind}\left[\{x,y\}\cap Y^c_{m-1}\neq\emptyset\right] \,. \tag{4.51}$$

The first sum on the right-hand-side of (4.49) is easy to bound. In fact, using Lemma 4.7, it is clear that

$$\sup_{0\leq s\leq 1} \|\Delta^n_\Lambda(\Phi'_Y(s),s)\| \leq 2|Y| \sup_{0\leq s\leq 1} \|\Phi'_Y(s)\| G(n-1) \,, \tag{4.52}$$

where G is as in (4.33) with $G(-1)$ set to be $\|W_\gamma\|_1$. Thus,

$$\begin{aligned}
\sum_{\substack{Y\subset\Lambda:\\ x,y\in Y}} \sum_{n\geq 0} \sup_{0\leq s\leq 1} \|\Delta^n_\Lambda(\Phi'_Y(s),s)\| &\leq 2\sum_{n\geq 0} G(n-1) \sum_{\substack{Y\subset\Lambda:\\ x,y\in Y}} |Y| \sup_{0\leq s\leq 1} \|\Phi'_Y(s)\| \\
&\leq 2\|\partial\Phi\|_a F_a(d(x,y)) \sum_{n\geq 0} G(n-1) \,. \qquad (4.53)
\end{aligned}$$

From the estimates in Lemma 2.6, it is clear that G is summable.

For the remaining terms in (4.49), we use the following over-counting estimate:

$$\sum_{\substack{Y\subset\Lambda:\\ x,y\in Y_m}} \mathrm{Ind}\left[\{x,y\}\cap Y^c_{m-1}\neq\emptyset\right] \leq \sum_{y_1\in B_m(x)} \sum_{y_2\in B_m(y)} \sum_{\substack{Y\subset\Lambda:\\ y_1,y_2\in Y}} \tag{4.54}$$

Combining (4.32) with (4.54), we find that

$$\begin{aligned}
&\sum_{m\geq 1}\sum_{\substack{Y\subset\Lambda:\\ x,y\in Y_m}} \mathrm{Ind}\left[\{x,y\}\cap Y_{m-1}^c\neq\emptyset\right]\sum_{n\geq m}\sup_{0\leq s\leq 1}\|\Delta_\Lambda^n(\Phi'_Y(s),s)\| \\
&\leq 2\sum_{m\geq 1}\sum_{y_1\in B_m(x)}\sum_{y_2\in B_m(y)}\sum_{\substack{Y\subset\Lambda:\\ y_1,y_2\in Y}}|Y|\sup_{0\leq s\leq 1}\|\Phi'_Y(s)\|\sum_{n\geq m}G(n-1) \\
&\leq 2\|\partial\Phi\|_a\sum_{m\geq 1}\hat{G}(m)\sum_{y_1\in B_m(x)}\sum_{y_2\in B_m(y)}F_a(d(y_1,y_2))\,, \qquad (4.55)
\end{aligned}$$

where we have set

$$\hat{G}(m)=\sum_{n\geq m}G(n-1)\,. \qquad (4.56)$$

We now perform a rough optimization over $m\geq 1$. Take $0<\epsilon<1$ and declare $m_0=m_0(\epsilon)\geq 0$ to be the largest integer less than $(1-\epsilon)d(x,y)/2$. We claim that, for $m\leq m_0$ and y_1 and y_2 as in (4.55) above, $\epsilon d(x,y)\leq d(y_1,y_2)$. This follows from

$$d(x,y)\leq d(x,y_1)+d(y_1,y_2)+d(y_2,y)\leq d(y_1,y_2)+2m\leq d(y_1,y_2)+2m_0\,, \qquad (4.57)$$

and the choice of m_0. In this case we have

$$\begin{aligned}
\sum_{m=1}^{m_0+1}\hat{G}(m)\sum_{y_1\in B_m(x)}\sum_{y_2\in B_m(y)}F_a(d(y_1,y_2)) &\leq \hat{G}(1)F_a(\epsilon d(x,y))\sum_{m=1}^{m_0+1}|B_m(x)||B_m(y)| \\
&\leq \kappa^2\hat{G}(1)F_a(\epsilon d(x,y))\sum_{m=1}^{m_0+1}m^{2\nu}\,, \qquad (4.58)
\end{aligned}$$

where we have used (4.1).

The remaining terms we bound as follows.

$$\begin{aligned}
\sum_{m>m_0+1}\hat{G}(m)\sum_{y_1\in B_m(x)}\sum_{y_2\in B_m(y)}F_a(d(y_1,y_2)) &\leq \|F_a\|\sum_{m>m_0+1}|B_m(x)|\hat{G}(m) \\
&\leq \kappa\|F_a\|\sum_{m>m_0+1}m^\nu\hat{G}(m)\,. \qquad (4.59)
\end{aligned}$$

Now, from the definition of $\hat{G}$,

$$\sum_{m>m_0+1}m^\nu\hat{G}(m)=\sum_{m=m_0+2}^{\infty}m^\nu\sum_{n=m-1}^{\infty}\left(4I_\gamma\left(\frac{n}{2v_a}\right)+\frac{K_a\|F\|}{av_a}e^{-an/2}\right)\,, \qquad (4.60)$$

and the sum

$$\sum_{m=m_0+2}^{\infty}m^\nu\sum_{n=m-1}^{\infty}e^{-an/2}=e^{a/2}\sum_{y\geq 0}e^{-ay/2}\cdot\sum_{m=m_0+2}m^\nu e^{-am/2} \qquad (4.61)$$

decays exponentially in m_0. Using the results in Lemma 2.5 and 2.6, we find that

$$\begin{aligned}
\sum_{m=m_0+2}^{\infty} m^{\nu} \sum_{n=m-1}^{\infty} I_{\gamma}\left(\frac{n}{2v_a}\right) &\leq \frac{C}{\gamma} \sum_{m=m_0+2}^{\infty} m^{\nu} \sum_{n=m-1}^{\infty} \left(\frac{\gamma n}{2v_a}\right)^{10} u_{2/7}\left(\frac{\gamma n}{2v_a}\right) \\
&\leq \frac{2v_a C}{\gamma^2} \sum_{m=m_0+2}^{\infty} m^{\nu} \int_{\frac{\gamma(m-1)}{2v_a}}^{\infty} y^{10} u_{2/7}(y)\, dy \\
&\leq \frac{161 v_a C}{\gamma^2} \sum_{m=m_0+2}^{\infty} m^{\nu} \left(\frac{\gamma(m-1)}{2v_a}\right)^{22} u_{2/7}\left(\frac{\gamma(m-1)}{2v_a}\right) \\
&\leq \frac{322 v_a^2 C}{\gamma^3} \int_{\frac{\gamma(m_0+1)}{2v_a}}^{\infty} \left(\frac{2v_a y}{\gamma}+1\right)^{\nu} y^{22} u_{2/7}(y)\, dy \\
&\leq \frac{2254 \cdot 2^{2\nu}}{\gamma} \left(\frac{v_a}{\gamma}\right)^{\nu+2} (47+2\nu) \left(\frac{\gamma(m_0+1)}{2v_a}\right)^{46+2\nu} u_{2/7}\left(\frac{\gamma(m_0+1)}{2v_a}\right). \qquad (4.62)
\end{aligned}$$

This proves that

$$\begin{aligned}
\sum_{\substack{Z\subset\Lambda:\\ x,y\in Z}} \sup_{0\leq s\leq 1} \|\Psi_{\Lambda}(Z,s)\| &\leq C_1 F_a(\epsilon d(x,y))(m_0+1)^{2\nu+1} + C_2 \sum_{m=m_0+2} m^{\nu} e^{-am/2} \\
&\quad + C_3 \left(\frac{\gamma(m_0+1)}{2v_a}\right)^{p} u_{2/7}\left(\frac{\gamma(m_0+1)}{2v_a}\right) \qquad (4.63)
\end{aligned}$$

for some number p depending only on ν. Since $2m_0 \leq (1-\epsilon)d(x,y)$, it is clear that the final term above decays the slowest in $d(x,y)$. Thus we have shown that

$$\sum_{\substack{Z\subset\Lambda:\\ x,y\in Z}} \sup_{0\leq s\leq 1} \|\Psi_{\Lambda}(Z,s)\| \leq C \left(\frac{\gamma}{2v_a}\left(\frac{1-\epsilon}{2}d(x,y)+1\right)\right)^{p} u_{2/7}\left(\frac{\gamma(1-\epsilon)}{4v_a}d(x,y)\right), \qquad (4.64)$$

for each $0<\epsilon<1$. For concreteness, take $\epsilon = 1/2$. With $\delta > 0$ as in (4.4) and any $0 < \delta' < 2/7 - \delta$, we will set $\mu = 2/7 - \delta - \delta' > 0$ and see that

$$\begin{aligned}
\sum_{\substack{Z\subset\Lambda:\\ x,y\in Z}} \sup_{0\leq s\leq 1} \|\Psi_{\Lambda}(Z,s)\| &\leq C' \left(\frac{5\gamma}{8v_a}d(x,y)\right)^{p} u_{2/7}\left(\frac{\gamma}{8v_a}d(x,y)\right) \\
&\leq C'' \left(\frac{5\gamma}{8v_a}d(x,y)\right)^{p} u_{2/7-\delta}\left(\frac{\gamma}{8v_a}d(x,y)\right) F\left(\frac{\gamma}{8v_a}d(x,y)\right) \\
&\leq C''' u_{\mu}\left(\frac{\gamma}{8v_a}d(x,y)\right) F\left(\frac{\gamma}{8v_a}d(x,y)\right). \qquad (4.65)
\end{aligned}$$

With the definition of F_{Ψ} given in (4.12), this completes the proof of (4.47). □

5. Existence of the thermodynamic limit and gapped quantum phases

The Lieb-Robinson bound for the flow α_s^{Λ} given in Theorem 4.5 of the previous section, can be used to obtain the thermodynamic limit of this flow defined as a strongly continuous cocycle of automorphisms of the C^*-algebra of quasi-local observables. The standard setting is the same as in the previous section, but we now assume that the Hilbert spaces $\mathcal{H}_x$ associated to each $x \in \Gamma$, are all finite-dimensional. The C^*-algebra of quasi-local observables $\mathcal{A}_{\Gamma}$ is then obtained as the completion with respect to the operator norm of $\mathcal{A}_{\rm loc}$:

$$\mathcal{A}_{\Gamma} = \overline{\mathcal{A}_{\rm loc}} = \overline{\bigcup_{\Lambda\subset\Gamma} \mathcal{A}_{\Lambda}}. \qquad (5.1)$$

If $\mathcal{H}_x$ is allowed to be infinite-dimensional it is typically necessary to work in the GNS representation of a reference state in order to have a well-defined thermodynamic limit. Such an approach was used in [38] to define the dynamics of an infinite lattice of anharmonic oscillators. In order to avoid the need for additional technical assumptions, for the remainder of this section we restrict ourselves to quantum spin systems, i.e., the case of finite-dimensional $\mathcal{H}_x$. It is not necessary, however, that $\dim \mathcal{H}_x$ is independent of x or even uniformly bounded.

This section has two subsections. In the first, we prove that the finite volume spectral flows, defined as in (4.11), have a well-defined thermodynamic limit. With these results in hand we can then, in the second subsection, complete the proof that gapped ground states connected by a curve of quasi-local interactions satisfying a suitable norm condition are equivalent under a quasi-local automorphism, in finite volume as well as in the thermodynamic limit. But first we describe in detail the class of systems to which our main result applies.

The systems under consideration here have finite dimensional local Hilbert spaces. In this case, we can make a convenient choice of the projection map introduced in Section 4.3 and needed for the application of Lemma 3.2, namely the natural extension of the partial trace. For any finite subset $\Lambda \subset \Gamma$, we define the conditional expectation $\Pi_\Lambda : \mathcal{A}_\Gamma \to \mathcal{A}_\Lambda$ as

$$\Pi_\Lambda = \mathrm{id}_{\mathcal{A}_\Lambda} \otimes \tau_{\mathcal{A}_{\Lambda^c}} ,$$

where for $\Lambda' \subset \Gamma$,

$$\tau_{\mathcal{A}_{\Lambda'}} = \bigotimes_{x \in \Lambda'} \tau_{\mathcal{A}_x} , \qquad \tau_{\mathcal{A}_x} = \frac{1}{\dim \mathcal{H}_x} \mathrm{Tr}_{\mathcal{H}_x}$$

is the normalized trace over $\mathcal{A}_{\Lambda'}$. In particular, for any $Z \subset \Lambda \subset \Gamma$, the subprojections

$$\Pi_{\Lambda,Z} = \Pi_Z|_{\mathcal{A}_\Lambda}$$

form a consistent family, namely for any $A \in \mathcal{A}_X$, with $Z, X \subset \Lambda_m \subset \Lambda_n \subset \Gamma$, they satisfy

$$\Pi_{\Lambda_n,Z}(A) = \Pi_{\Lambda_m,Z}(A) \tag{5.2}$$

and the first index may be dropped.

Let Γ be a countable set equipped with a metric and a function F satisfying (4.2) and (4.3). For $s \in [0,1]$, let $\Phi(s)$ be a family of interactions, differentiable in s, for which there exists a number $a > 0$ so that

$$\|\Phi\|_a + \|\partial\Phi\|_a < \infty . \tag{5.3}$$

where the norm is defined in the paragraph containing (4.7).

Our proof of the existence of the thermodynamic limit requires some assumptions on the sequence of finite volumes $(\Lambda_n)_n$ on which the spectral flows are defined. Let $(\Lambda_n)_n$ be an increasing sequence of finite sets which exhaust Γ as $n \to \infty$. For convenience, we will regard the parameter n as continuous with the understanding that, for any $n \geq 0$, $\Lambda_n = \Lambda_{[n]}$, where $[n]$ denotes the integer part of n. We will assume that there exist positive numbers b_1, b_2, and p such that

$$d(\Lambda_m, \Lambda_n^c) \geq b_1(n-m), \quad \text{and } |\Lambda_n| \leq b_2 n^p . \tag{5.4}$$

We assume that there are finite intervals $I(s)$, smoothly depending on $s \in [0,1]$ such that, for all n, the finite-volume Hamiltonians $H_{\Lambda_n}(s) = \sum_{Z \subset \Lambda_n} \Phi(Z,s)$ have one or more eigenvalues in $I(s)$, and no eigenvalues outside $I(s)$ within a distance $\gamma > 0$ of it.

Let us summarize the results of the previous sections, given these assumptions. If $P_{\Lambda_n}(s)$ denotes the spectral projections of $H_{\Lambda_n}(s)$ on $I(s)$, then there is a cocycle $\alpha_s^{\Lambda_n}$, the dual of which maps $P_{\Lambda_n}(0)$ to $P_{\Lambda_n}(s)$ for all $s \in [0,1]$. Its generator has a local structure given by

$D_{\Lambda_n}(s) = \sum_{Z\subset\Lambda_n} \Psi_{\Lambda_n}(Z,s)$ where the interactions $\Psi_{\Lambda_n}(s)$ decay almost exponentially in the following sense,

$$\|\Psi_{\Lambda_n}\|_{F_\Psi} = \sup_{x,y\in\Lambda_n} \frac{1}{F_\Psi(d(x,y))} \sum_{\substack{Z\subset\Lambda_n \\ x,y\in Z}} \sup_{0\le s\le 1} \|\Psi_{\Lambda_n}(Z,s)\| < \infty\,, \tag{5.5}$$

uniformly in n, where F_Ψ satisfies again the uniform integrability and convolution property for a constant C_Ψ. Our estimates in Section 4 demonstrate that a possible choice of F_Ψ is given by (4.12) which decays sub-exponentially.

5.1. **Thermodynamic limit for the spectral flow.** In order to prove the existence of the thermodynamic limit of the spectral flow α_s^Λ, it is convenient to recall an estimate from the proof of the existence of the thermodynamic limit of Heisenberg evolutions $\tau_t^{H_\Lambda(s)}$, as proven e.g. in [36]. In fact, assuming that $\|\Phi\|_a < \infty$, the following bound is valid.

Take finite sets $X\subset\Lambda_m\subset\Lambda_n$. Note that for any $A\in\mathcal{A}_X$, each $s\in[0,1]$, and any $t\in\mathbb{R}$,

$$\begin{aligned}
\left\|\tau_t^{H_{\Lambda_n}(s)}(A) - \tau_t^{H_{\Lambda_m}(s)}(A)\right\| &\le \sum_{\substack{Z\subset\Lambda_n: \\ Z\cap\Lambda_n\setminus\Lambda_m\neq\emptyset}} \int_0^{|t|} \left\|\left[\Phi_Z(s), \tau_{|t|-r}^{H_{\Lambda_m}(s)}(A)\right]\right\| dr \\
&\le \frac{K_a\|A\|}{av_a}(e^{av_a|t|}-1) \sum_{\substack{Z\subset\Lambda_n: \\ Z\cap\Lambda_n\setminus\Lambda_m\neq\emptyset}} \|\Phi_Z(s)\| \sum_{z\in Z, x\in X} F_a(d(x,z)) \\
&\le \frac{K_a\|A\|}{av_a} C_{F_a}\|\Phi\|_a (e^{av_a|t|}-1) \sum_{\substack{x\in X \\ y\in\Lambda_n\setminus\Lambda_m}} F_a(d(x,y))\,. \qquad (5.6)
\end{aligned}$$

Since F_a is uniformly integrable, this proves that the sequence $\left(\tau_t^{H_{\Lambda_n}(s)}(A)\right)_n$ is Cauchy. We will denote the limit by $\tau_t^{\Gamma,s}(A)$, and observe that it satisfies

$$\left\|\tau_t^{\Gamma,s}(A) - \tau_t^{H_{\Lambda_m}(s)}(A)\right\| \le \frac{K_a\|A\|}{av_a} C_{F_a}\|\Phi\|_a (e^{av_a|t|}-1) \sum_{\substack{x\in X \\ y\in\Gamma\setminus\Lambda_m}} F_a(d(x,y))\,, \tag{5.7}$$

uniformly for $s\in[0,1]$.

The following analogue of Lemma 4.7 will be useful. Recall the definitions of $\Delta_\Lambda^n(A,s)$ from (4.29) and (4.30). Define similarly $\Delta_\Gamma(A,s)$ with $\tau_t^{\Gamma,s}(A)$ replacing $\tau_t^{H_\Lambda(s)}(A)$ as appropriate.

Lemma 5.1. *Let $\Lambda\subset\Gamma$ be a finite set. For any $X\subset\Lambda$ and $A\in\mathcal{A}_X$,*

$$\|\Delta_\Lambda^n(A,s) - \Delta_\Gamma^n(A,s)\| \le 4\|A\| \min\left[\|W_\gamma\|_1, |X|\sqrt{G(n-1)K(d(X,\Lambda^c))}\right] \tag{5.8}$$

where G is as in (4.33) of Lemma 4.7 and

$$K(x) = 4I_\gamma\left(\frac{x}{2v_a}\right) + \frac{K_a C_{F_a}\|\Phi\|_a\|F\|}{a^2v_a^2} e^{-ax/2}. \tag{5.9}$$

Proof. A uniform estimate, as shown in Lemma 4.7, clearly holds for $n=0$. We need only consider $n\ge 1$. Using the consistency of the mappings Π_{X_n}, the difference $\Delta_\Lambda^n(A,s) - \Delta_\Gamma^n(A,s)$ can be written as a difference of two terms. As such, we need only bound the norm of

$$\int_{-\infty}^{\infty} (\Pi_{X_n} - \mathrm{id})\left(\tau_t^{\Gamma,s}(A) - \tau_t^{H_\Lambda(s)}(A)\right) W_\gamma(t)\,dt\,. \tag{5.10}$$

By Assumption 4.4, $\tau_t^{H_\Lambda(s)}$ satisfies a Lieb-Robinson bound uniform in Λ and s. In this case, the limit $\tau_t^{\Gamma,s}$ does as well. Arguing then as in Lemma 4.7, it is clear that

$$\left\| \int_{-\infty}^{\infty} (\Pi_{X_n} - \mathrm{id}) \left(\tau_t^{\Gamma,s}(A) - \tau_t^{H_\Lambda(s)}(A) \right) W_\gamma(t) \, dt \right\| \leq 2|X| \|A\| \left(4 I_\gamma \left(\frac{n}{2v_a} \right) + \frac{K_a \|F\|}{a v_a} e^{-an/2} \right) . \tag{5.11}$$

Since the projections Π_{X_n} are norm one maps, we may also argue using the thermodynamic estimate (5.7). In fact,

$$\left\| \int_{-\infty}^{\infty} (\Pi_{X_n} - \mathrm{id}) \left(\tau_t^{\Gamma,s}(A) - \tau_t^{H_\Lambda(s)}(A) \right) W_\gamma(t) \, dt \right\| \leq 2 \int_{-\infty}^{\infty} \left\| \tau_t^{\Gamma,s}(A) - \tau_t^{H_\Lambda(s)}(A) \right\| |W_\gamma(t)| \, dt \, .$$

Now for $|t| \leq T$, we have that

$$\begin{aligned} 2 \int_{|t| \leq T} \left\| \tau_t^{\Gamma,s}(A) - \tau_t^{H_\Lambda(s)}(A) \right\| |W_\gamma(t)| \, dt &\leq \frac{K_a C_{F_a} \|\Phi\|_a}{a v_a} \|A\| \sum_{\substack{x \in X \\ y \in \Gamma \setminus \Lambda}} F_a(d(x,y)) \int_{|t| \leq T} e^{a v_a |t|} \, dt \\ &\leq \frac{2 K_a C_{F_a} \|\Phi\|_a}{a^2 v_a^2} \|A\| |X| \|F\| e^{-a d(X, \Lambda^c)} e^{a v_a T} \, , \end{aligned} \tag{5.12}$$

whereas for $|t| > T$, the bound

$$2 \int_{|t| > T} \left\| \tau_t^{\Gamma,s}(A) - \tau_t^{H_\Lambda(s)}(A) \right\| |W_\gamma(t)| \, dt \leq 8 \|A\| I_\gamma(T) \, , \tag{5.13}$$

is clearly true. In this case, the choice $T = d(X, \Lambda^c)/(2v_a)$ yields the estimate

$$\left\| \int_{-\infty}^{\infty} (\Pi_{X_n} - \mathrm{id}) \left(\tau_t^{\Gamma,s}(A) - \tau_t^{H_\Lambda(s)}(A) \right) W_\gamma(t) \, dt \right\| \leq 2|X| \|A\| \left(4 I_\gamma \left(\frac{d(X, \Lambda^c)}{2v_a} \right) + \frac{K_a C_{F_a} \|\Phi\|_a \|F\|}{a^2 v_a^2} e^{-a d(X, \Lambda^c)/2} \right) . \tag{5.14}$$

Combining the results from (5.11) and (5.14), as well as the bound corresponding to $\Pi_{X_{n-1}}$, the estimate (4.32) follows. □

We can now state and prove the existence of the thermodynamic limit for the spectral flow $\alpha_s^{\Lambda_n}$. Recall that for any finite sets $Z \subset \Lambda \subset \Gamma$, we have defined

$$\Psi_\Lambda(Z, s) = \sum_{\substack{Y, n \geq 0: \\ Y_n = Z}} \Delta_\Lambda^n(\Phi'_Y(s), s) \, . \tag{5.15}$$

By analogy, set

$$\Psi_\Gamma(Z, s) = \sum_{\substack{Y, n \geq 0: \\ Y_n = Z}} \Delta_\Gamma^n(\Phi'_Y(s), s) \, . \tag{5.16}$$

We will show later in this subsection that the s-dependent interaction $\Psi_\Gamma(s)$ is the limit as $\Lambda \to \Gamma$ of $\Psi_\Lambda(s)$. First, we show the existence of the limiting spectral flow α_s^Γ in Theorem 5.2. Then, we argue that it is also the limit of the automorphisms generated by finite volume restrictions of the limiting interaction $\Psi_\Gamma(s)$.

Theorem 5.2. *Let $\left(\alpha_s^{\Lambda_n}\right)_n$ denote the sequence of flows associated with the sets $\Lambda_n \subset \Gamma$. Then there exists a flow α_s^Γ defined on the quasi-local algebra $\mathcal{A}_\Gamma$ such that for all $A \in \mathcal{A}_{\mathrm{loc}}$,*

$$\lim_{n \to \infty} \|\alpha_s^{\Lambda_n}(A) - \alpha_s^\Gamma(A)\| = 0 \, ,$$

uniformly for all $s \in [0,1]$.

Proof. We begin by noting that the strong limit of an automorphism is automatically an automorphism and that convergence of a sequence of automorphisms $\sigma_n \to \sigma$, is equivalent to the convergence of the inverses to the inverse automorphism, i.e., $\sigma_n^{-1} \to \sigma^{-1}$. Using these observations and by standard completeness arguments it is therefore sufficient to establish that for all $A \in \mathcal{A}_{\rm loc}$, the sequence $(\alpha_s^{\Lambda_n})^{-1}(A)$ is Cauchy. Without loss of generality, we assume that $A \in \mathcal{A}_{\Lambda_0}$ and we use the notation $\tilde{\alpha}_s^{\Lambda_n} = (\alpha_s^{\Lambda_n})^{-1}$. Then, for $n > m$, define

$$f(s) = \tilde{\alpha}_s^{\Lambda_n}(A) - \tilde{\alpha}_s^{\Lambda_m}(A).$$

and observe that

$$\begin{aligned} f'(s) &= i[D_{\Lambda_n}(s), \tilde{\alpha}_s^{\Lambda_n}(A)] - i[D_{\Lambda_m}(s), \tilde{\alpha}_s^{\Lambda_m}(A)] \\ &= i[D_{\Lambda_n}(s), f(s)] + i[D_{\Lambda_n}(s) - D_{\Lambda_m}(s), \tilde{\alpha}_s^{\Lambda_m}(A)]\,. \end{aligned}$$

Hence,

$$\left\|\tilde{\alpha}_s^{\Lambda_n}(A) - \tilde{\alpha}_s^{\Lambda_m}(A)\right\| = \|f(s)\| \le \int_0^s \|[D_{\Lambda_n}(r) - D_{\Lambda_m}(r), \tilde{\alpha}_r^{\Lambda_m}(A)]\|\, dr\,. \tag{5.17}$$

We will show that the right-hand-side goes to zero as $n, m \to \infty$.

We begin by writing the difference as

$$D_{\Lambda_n}(r) - D_{\Lambda_m}(r) = \sum_{\substack{Z\subset\Lambda_n:\\ Z\cap(\Lambda_n\setminus\Lambda_m)\neq\emptyset}} \Psi_{\Lambda_n}(Z,r) + \sum_{Z\subset\Lambda_m} \left(\Psi_{\Lambda_n}(Z,r) - \Psi_{\Lambda_m}(Z,r)\right)\,.$$

For the first term, the Lieb-Robinson bound of Theorem 4.5, which clearly applies to $\tilde{\alpha}_r^{\Lambda_m}$ as well, yields

$$\|[\Psi_{\Lambda_n}(Z,r), \tilde{\alpha}_r^{\Lambda_m}(A)]\| \le 2\|A\|\|\Psi_{\Lambda_n}(Z,r)\|g(r) \sum_{x\in\Lambda_0, y\in Z} F_\Psi(d(x,y))\,.$$

After summing over Z and integrating, we find that

$$\begin{aligned} &\int_0^s \sum_{\substack{Z\subset\Lambda_n:\\ Z\cap(\Lambda_n\setminus\Lambda_m)\neq\emptyset}} \|[\Psi_{\Lambda_n}(Z,r), \tilde{\alpha}_r^{\Lambda_m}(A)]\|\, dr \\ &\qquad \le 2\|A\| \int_0^s \sum_{\substack{Z\subset\Lambda_n:\\ Z\cap(\Lambda_n\setminus\Lambda_m)\neq\emptyset}} \|\Psi_{\Lambda_n}(Z,r)\|g(r)\, dr \sum_{x\in\Lambda_0, y\in Z} F_\Psi(d(x,y)) \\ &\qquad \le 2\|A\| \int_0^s g(r)\, dr \sum_{y\in\Lambda_n, z\in\Lambda_n\setminus\Lambda_m} \sum_{\substack{Z\subset\Lambda_n:\\ z,y\in Z}} \sup_{0\le r\le 1} \|\Psi_{\Lambda_n}(Z,r)\| \sum_{x\in\Lambda_0} F_\Psi(d(x,y)) \\ &\qquad \le 2\|A\|\|\Psi\|C_\Psi \int_0^s g(r)\, dr \sum_{z\in\Lambda_n\setminus\Lambda_m} \sum_{x\in\Lambda_0} F_\Psi(d(x,z)) \end{aligned}$$

which vanishes as $m < n \to \infty$ by the uniform integrability of F_Ψ.

To control the second term, we arrange the set of subsets of Λ_m, which we denote by $\mathcal{P}(\Lambda_m)$, as a union of three sets: $\mathcal{P}(\Lambda_m) = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ where

$$\mathcal{P}_1 = \{Z \in \mathcal{P}(\Lambda_m) : Z \subset \Lambda_{m/3}^c\}, \quad \mathcal{P}_2 = \{Z \in \mathcal{P}(\Lambda_m) : Z \subset \Lambda_{2m/3}\}, \tag{5.18}$$

and

$$\mathcal{P}_3 = \{Z \in \mathcal{P}(\Lambda_m) : Z \cap \Lambda_{m/3} \neq \emptyset \quad \text{and} \quad Z \cap \Lambda_{2m/3}^c \neq \emptyset\}. \tag{5.19}$$

We first sum over $\mathcal{P}_1$. Repeating the argument we used above, in particular using the uniform Lieb-Robinson estimate for both $\Psi_{\Lambda_n}(Z,r)$ and $\Psi_{\Lambda_m}(Z,r)$, we find that

$$\int_0^s \sum_{Z\in\mathcal{P}_1} \| \left[\Psi_{\Lambda_n}(Z,r) - \Psi_{\Lambda_m}(Z,r), \tilde{\alpha}_r^{\Lambda_m}(A)\right] \| \, dr \; \leq 4\|A\|\|\Psi\|C_\Psi \int_0^s g(r)\, dr \sum_{z\in\Lambda_{m/3}^c} \sum_{x\in\Lambda_0} F_\Psi(d(x,z)),$$

and this bound decays to zero as $m\to\infty$.

We next estimate the sum over $\mathcal{P}_2$. We begin by trivially bounding

$$\begin{aligned} \left\| \left[\sum_{Z\in\mathcal{P}_2} \left(\Psi_{\Lambda_n}(Z,r) - \Psi_{\Lambda_m}(Z,r)\right), \tilde{\alpha}_r^{\Lambda_m}(A)\right]\right\| \;&\leq\; 2\|A\| \left\| \sum_{Z\in\mathcal{P}_2} \left(\Psi_{\Lambda_n}(Z,r) - \Psi_\Gamma(Z,r)\right)\right\| \\ &\quad + 2\|A\| \left\| \sum_{Z\in\mathcal{P}_2} \left(\Psi_\Gamma(Z,r) - \Psi_{\Lambda_m}(Z,r)\right)\right\| \end{aligned} \tag{5.20}$$

where we are using the notation from (5.16). Each of the terms on the right-hand-side above will be estimated similarly. In fact, note that

$$\sum_{Z\in\mathcal{P}_2} \left(\Psi_\Gamma(Z,r) - \Psi_{\Lambda_m}(Z,r)\right) = \sum_{Z\subset\Lambda_{2m/3}} \sum_{n\geq 0} \sum_{\substack{Y\subset\Gamma:\\ Y_n = Z}} \left(\Delta_\Gamma^n(\Phi'_Y(r), r) - \Delta_{\Lambda_m}(\Phi'_Y(r), r)\right) \tag{5.21}$$

implies a bound of the form

$$\begin{aligned} \left\| \sum_{Z\in\mathcal{P}_2} \left(\Psi_\Gamma(Z,r) - \Psi_{\Lambda_m}(Z,r)\right)\right\| \;&\leq\; \sum_{n\geq 0} \sum_{\substack{Y\subset\Gamma:\\ Y_n\subset\Lambda_{2m/3}}} \left\|\Delta_\Gamma^n(\Phi'_Y(r), r) - \Delta_{\Lambda_m}(\Phi'_Y(r), r)\right\| \\ &\leq\; 4\sum_{n\geq 0}\sqrt{G(n-1)} \sum_{y\in\Lambda_{2m/3}} \sum_{\substack{Y\subset\Gamma:\\ y\in Y}} |Y| \sup_{0\leq r\leq 1} \|\Phi'_Y(r)\| \sqrt{K(d(\Lambda_{2m/3}, \Lambda_m^c))} \\ &\leq\; 4\|\partial\Phi\|_a F_a(0) \sum_{n\geq 0} \sqrt{G(n-1)} \cdot |\Lambda_{2m/3}| \sqrt{K(b_1 m/3)}\,. \end{aligned} \tag{5.22}$$

Since $|\Lambda_{2m/3}| \leq b_2(2m/3)^p$, it is clear that the above goes to zero as $m\to\infty$; uniformly for $0\leq r\leq 1$. The bound corresponding to (5.22) with Λ_m replaced with Λ_n goes to zero at least as fast.

Finally, we sum over $\mathcal{P}_3$. These sets extend over a large fraction of Λ_m, and therefore, they must correspond to terms with small norms. Indeed,

$$\begin{aligned} &\int_0^s \sum_{Z\in\mathcal{P}_3} \| \left[\Psi_{\Lambda_n}(Z,r) - \Psi_{\Lambda_m}(Z,r), \tilde{\alpha}_r^{\Lambda_m}(A)\right] \| dr \\ &\qquad \leq 2s\|A\| \sum_{x\in\Lambda_{m/3}} \sum_{y\in\Lambda_{2m/3}^c} \sum_{\substack{Z\subset\Gamma:\\ x,y\in Z}} \left(\sup_{0\leq r\leq 1} \|\Psi_{\Lambda_n}(Z,r)\| + \sup_{0\leq r\leq 1} \|\Psi_{\Lambda_m}(Z,r)\|\right) \\ &\qquad\qquad \leq 4s\|A\|\|\Psi\| \sum_{x\in\Lambda_{m/3}} \sum_{y\in\Lambda_{2m/3}^c} F_\Psi(d(x,y))\,. \end{aligned}$$

As is proven in Theorem 4.8, the function $F_\Psi(r) = u_\mu(r)F(r)$ for some $\mu>0$ and r large enough. Thus the sum

$$\sum_{x\in\Lambda_{m/3}} \sum_{y\in\Lambda_{2m/3}^c} F_\Psi(d(x,y)) \leq \|F\| |\Lambda_{m/3}| u_\mu(b_1 m/3) \tag{5.23}$$

which goes to zero as $m \to \infty$. We have shown that all terms vanish in the limit, and therefore, the sequence $(\tilde{\alpha}_s^{\Lambda_n}(A))_n$ is Cauchy as claimed. $\square$

The above result establishes the existence of the spectral flow in the thermodynamic limit, and we have denoted that limiting flow by α_s^Γ. Arguments similar to those used in the proof of Theorem 5.2 show that α_s^Γ is also the thermodynamic limit of the flows generated by the interaction $\Psi_\Gamma(s)$, defined in (5.16), restricted to the sequence of finite volumes Λ_m. This is not a surprise since, as the next proposition shows, $\Psi_\Gamma(s)$ is the limit of $\Psi_\Lambda(s)$ as $\Lambda \to \Gamma$. In this proposition, we consider the interactions $\Psi_\Lambda(s)$ as functions defined on the power set of Λ, $\mathcal{P}(\Lambda)$, with values in the algebra of observables. As such, we can consider the interactions obtained by restriction to a subset of $\mathcal{P}(\Lambda)$, such as $\Psi_\Lambda(s)|_{\mathcal{P}(\Lambda_0)}$, for $\Lambda_0 \subset \Lambda$.

Proposition 5.3. *For any finite $\Lambda \subset \Gamma$ and $Z \subset \Lambda$, the following estimate holds*

$$\|\Psi_\Lambda(Z,s) - \Psi_\Gamma(Z,s)\| \leq C\|\partial\Phi\|_a |Z| \sqrt{K(d(Z,\Lambda^c))} \tag{5.24}$$

where

$$C = 4F(0)\left(\sqrt{\|W_\gamma\|_1} + \sum_{n\geq 0}\sqrt{G(n)}\right).$$

Let $(\Lambda_m)_m$ be a sequence of finite volumes satisfying the properties (5.4). Then, for any $\beta \in (0,1)$, one has

$$\lim_{m\to\infty} \|\Psi_{\Lambda_m}|_{\mathcal{P}(\Lambda_{m-m^\beta})} - \Psi_\Gamma|_{\mathcal{P}(\Lambda_{m-m^\beta})}\|_{F_\Psi} = 0 \tag{5.25}$$

Proof. To prove the estimate (5.24) for fixed Z, we apply Lemma 5.1 with $A = \Phi'_Z(s)$ and then Lemma 4.1 as follows:

$$\begin{aligned}
\|\Psi_\Lambda(Z,s) - \Psi_\Gamma(Z,s)\| &\leq \sum_{\substack{Y,n\geq 0:\\ Y_n = Z}} \left\|\Delta_\Lambda^n(\Phi'_Y(s),s) - \Delta_\Gamma^n(\Phi'_Y(s),s)\right\| \\
&\leq 4\sum_{\substack{Y,n\geq 0:\\ Y_n = Z}} |Y|\|\Phi'_Y(s)\|\sqrt{G(n-1)K(d(Y,\Lambda^c))} \\
&\leq 4\left(\sqrt{\|W_\gamma\|_1} + \sum_{n\geq 0}\sqrt{G(n)}\right)\sqrt{K(d(Z,\Lambda^c))}\sum_{Y\subset Z}|Y|\|\Phi'_Y(s)\| \\
&\leq 4\|\partial\Phi\|_a F(0)\left(\sqrt{\|W_\gamma\|_1} + \sum_{n\geq 0}\sqrt{G(n)}\right)|Z|\sqrt{K(d(Z,\Lambda^c))}, \qquad (5.26)
\end{aligned}$$

which is the claimed result. To prove (5.25) is now a straightforward application of (5.24) and the properties of the function K defined in Lemma 5.1. $\square$

Proposition 5.4. *The spectral flow α_s^Γ for the infinite system has the following properties:*

i. $\left(\alpha_s^\Gamma\right)_{s\in[0,1]}$ *is a strongly continuous cocycle of automorphisms of the C^*-algebra of quasi-local observables, and it is the thermodynamic limit of the finite-volume cocycles generated by the interaction $\Psi_\Gamma(s)$.*
ii. α_s^Γ *satisfies the Lieb-Robinson bound*

$$\left\|\left[\alpha_s^\Gamma(A), B\right]\right\| \leq 2\|A\|\|B\| \min\left[1, g(s)\sum_{x\in X, y\in Y} F_\Psi(d(x,y))\right], \tag{5.27}$$

for any $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $0 \le s \le 1$, with g given by

$$C_{F_\Psi} \cdot g(t) = \begin{cases} e^{2\|\Psi\| C_{F_\Psi}|t|} - 1 & \text{if } d(X,Y) > 0, \\ e^{2\|\Psi\| C_{F_\Psi}|t|} & \text{otherwise.} \end{cases} \tag{5.28}$$

and the quantities F_Ψ, C_{F_Ψ}, and $\|\Psi\|_{F_\Psi}$ as given in Theorem 4.5.

iii. *If β is a local symmetry of Φ, i.e. , an automorphism such that $\beta(\Phi(X,s)) = \Phi(X,s)$, for all $X \subset \Gamma$ and $s \in [0,1]$, then β is also a symmetry of α_s^Γ, i.e. , $\alpha_s^\Gamma \circ \beta = \alpha_s^\Gamma$ for all $s \in [0,1]$.*

iv. *Suppose Γ is a lattice with a group of translations $(T_x)_x$ and $(\pi_{T_x})_x$ is the representation of the translations as automorphisms of the quasi-local algebra $\mathcal{A}_\Gamma$. Then, if Φ is translation invariant, i.e. , $\Phi(T_x(X), s) = \pi_{T_x}(\Phi(X,s))$, for all $X \subset \Gamma$, and $s \in [0,1]$, then α_s^Γ commutes with π_{T_x}, for all x and s.*

Proof. All these properties follow from the preceding results. □

5.2. **Automorphic equivalence of gapped ground states.** We can now describe more precisely the problem of equivalence of quantum phases discussed in the introduction. Let $\mathcal{S}_\Lambda(s)$ denote the set of states of the system in volume Λ that are mixtures of eigenstates with energy in $I(s)$ and let $\mathcal{S}(s)$ be the set of weak-$*$ limit points as $n \to \infty$ of $\mathcal{S}_{\Lambda_n}(s)$. Note that these sets are non-empty. The result of Section 2 immediately implies

$$\mathcal{S}_{\Lambda_n}(s) = \mathcal{S}_{\Lambda_n}(0) \circ \alpha_s^{\Lambda_n} \,, \tag{5.29}$$

where $\alpha_s^{\Lambda_n}$ is the automorphism defined in (4.11). In Section 4 we proved that $\alpha_s^{\Lambda_n}$ satisfy a Lieb-Robinson bound with a uniformly bounded Lieb-Robinson velocity and decay rate outside the 'light cone'. In the previous subsection we obtained the thermodynamic limit of these automorphisms leading to the cocycle α_s^Γ which automatically satisfies a Lieb-Robinson bound with the same estimates for the velocity and the decay. The following theorem states that (5.29) carries over to the thermodynamic limit.

Theorem 5.5. *The states $\omega(s) \in \mathcal{S}(s)$ in the thermodynamic limit are automorphically equivalent to the states $\omega(0) \in \mathcal{S}(0)$ for all $s \in [0,1]$. Indeed,*

$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s^\Gamma \tag{5.30}$$

Moreover, the connecting automorphisms α_s^Γ can be generated by a s-dependent quasi-local interaction $\Psi(s)$ with $\|\Psi\|_{F_\Psi} < \infty$, where the norm is defined in (5.5). α_s^Γ then satisfies the same Lieb-Robinson bound as α_s^Λ in Theorem 4.5.

Proof. This is a direct consequence of (5.29), theorem 5.2 and the lemma below. □

Lemma 5.6. *Let $(\sigma_n)_n$ be a strongly convergent sequence of automorphisms of a C^*-algebra $\mathcal{A}$, converging to σ and let $(\omega_n)_n$ be a sequence of states on $\mathcal{A}$. Then the following are equivalent:*

i. *ω_n converges to ω in the weak-$*$ topology;*
ii. *$\omega_n \circ \sigma$ converges to $\omega \circ \sigma$ in the weak-$*$ topology;*
iii. *$\omega_n \circ \sigma_n$ converges to $\omega \circ \sigma$ in the weak-$*$ topology.*

Proof. (i)⇔(ii) follows immediately from the fact that σ and σ^{-1} are automorphisms. Now if (ii) holds, the second term of

$$|(\omega_n \circ \sigma_n)(A) - (\omega \circ \sigma)(A)| \le |\omega_n(\sigma_n(A) - \sigma(A))| + |\omega_n(\sigma(A)) - \omega(\sigma(A))| \,,$$

vanishes. So does the first one

$$|\omega_n(\sigma_n(A) - \sigma(A))| \le \|\omega_n\| \|\sigma_n(A) - \sigma(A)\| \longrightarrow 0$$

since ω_n are states, and therefore (iii) holds. A similar argument yields (iii)⇒(ii). □

In the recent literature [10, 11], a 'ground state phase' has been defined as an equivalence class of ground states with the equivalence defined as follows: the states ω_0 and ω_1 are equivalent (*i.e.* , belong to the same phase) if there exists a continuous family of Hamiltonians $H(s)$, $0 \leq s \leq 1$, such that for each s, $H(s)$ has a gap above the ground state and ω_0 and ω_1 are ground states of $H(0)$ and $H(1)$, respectively. As an alternative definition the authors of [10] state that ω_0 and ω_1 should be related by a 'local unitary transformation'. With Theorem 5.5 we provide precise conditions under which the first property implies the second. At the same time we have clarified the role of the thermodynamic limit left implicit in the cited works.

Based on Theorem 5.5 it seems reasonable to define the ground states of two interactions $\Phi(0)$ and $\Phi(1)$ to be in the same phase if there exists a differentiable interpolating family of interactions $\Phi(s)$, $0 \leq s \leq 1$, such that there exists $a > 0$ for which $\|\Phi\|_a + \|\partial\Phi\|_a < \infty$, and if the spectral gap above the ground states of the corresponding finite-volume Hamiltonians $H_{\Lambda_m}(s)$ have a uniform lower bound $\gamma > 0$. The increasing sequence of finite volumes Λ_m should satisfy a condition of the type (5.4). One should allow for a space of nearly degenerate eigenstates of H_{Λ_m} which, in the thermodynamic limit, converge to a set of ground states $\mathcal{S}(s)$. We have proved that under these conditions the sets of thermodynamic limits of ground states are connected by a flow of automorphisms generated by a quasi-local interaction with almost exponential decay and satisfying a Lieb-Robinson bound. We believe that these are *sufficient conditions* for belonging to the same gapped ground state phase. More work is needed to identify *necessary conditions*.

We remark that a 'ground state phase' should be defined as an equivalence relation on simplices of states of a quantum lattice system. This is an equivalence of sets of states rather than of models because it is possible that different quantum phases coexist as ground states of one model, while the same states also appear as unique ground states of other models. Examples of this situation can easily be constructed using frustration free models in one dimension with finitely correlated ground states, also known as matrix product states [17, 35]. In particular, if $\mathcal{S}(s)$ denotes the set of infinite-volume ground states of a model with parameter s, the relation $\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s$, does not imply that the states in the sets $\mathcal{S}(s)$ are automorphically equivalent among themselves. E.g., if for a model with a discrete symmetry we find that symmetry broken states coexists with symmetric states, α_s cannot map these two classes into each other. In general, as emphasized in Proposition 5.4, the α_s we constructed posses all symmetries of the Hamiltonians.

There are plenty of examples of models to which our results apply. Clearly, the various perturbation results mentioned in the introduction provide many interesting examples of sets of models with ground states in a variety of types of gapped phases. Another class of examples is provided by the rich class of gapped quantum spin chains with matrix product ground states. In Yarotsky's work [51] it is shown how perturbation theory around a matrix product ground states can be applied to connect these two classes of examples. Exactly solvable models with gapped ground states depending on a parameter, such as the anisotropic XY chain [33], is another set of examples. More recently, stability under small perturbations of the interaction was proved for a class of models with topologically ordered ground states [7]; these include e.g. Kitaev's toric code model [30]. Our results are also applicable to this class of models. It seems likely that other applications will be found. As an example of an application left to be explored, we mention that the existence of a connecting automorphism of the type α_s can provide a means to distinguish true quantum phase transitions from isolated critical (i.e., gapless) points around which it is possible to circumnavigate with suitably chosen perturbations.

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