Harmonic Analysis and the Complexity of Computing with Threshold (Neural) Elements

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The main purpose of this talk is to introduce a useful tool for the analysis of discrete neural networks in which every node is a Boolean threshold gate. The difficulty in the analysis of neural networks arises from the fact that the basic processing elements (linear threshold gates) are nonlinear. The key idea in harmonic analysis of threshold functions is to represent the functions as polynomials over the field of real numbers. Answering different questions regarding neural networks becomes equivalent to answering questions related to the coefficients of these polynomials. We have applied these techniques and obtained many interesting and surprising results [1, 2, 3, 4]. The focus of this talk will be on presenting a theorem that characterizes–using spetral norms–the complexity of computing a Boolean function with threshold circuits [2, 3]. This result establishes the first known link between harmonic analysis and the complexity of computing with neural networks.

A Boolean function $f(X)$ is a *threshold function* if

$$f(X) = sgn\,(F(X)) = \begin{cases} 1 & \text{if } F(X) > 0 \\ -1 & \text{if } F(X) < 0 \end{cases}$$

where $F(X)$ is a multilinear polynomial with rational coefficients and $X = (x_1, x_2, \ldots, x_n)$.

Definition: Let PT_1, for *Polynomial Threshold* functions, be the set of all Boolean functions that can be computed by a single threshold gate where the number of monomials is bounded by a polynomial in n.

Let PL_1 be the class of Boolean functions for which the spectral norm L_1 is bounded by a polynomial in n and PL_∞ the class of Boolean functions for which L_∞^{-1} is bounded by a polynomial in n. Then our main result is

Characterization Theorem:

$$PL_1 \subset PT_1 \subset PL_\infty.$$

Applications: There are two possible applications to the characterization result. The sufficient condition can be used to obtain upper bounds [4] while the necessary condition can be used to obtain lower bounds [2, 3].

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