Performance Limits for FDMA Cellular Systems Described by Hypergraphs

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1. Introduction and Summary.

In this paper we shall obtain performance limits for frequency-assignment algorithms for certain models of FDMA cellular telephone systems. Our models can be described as follows. There is an underlying set $V = \{v_1, v_2, \ldots, v_N\}$ of N cells, and n available user frequencies. Call requests arrive and depart randomly throughout the system, according to a probabilistic model we shall leave largely unspecified, except to say that it is measured in ***Erlangs per available frequency***, a parameter we shall call the ***intensity of the offered traffic*** and denote by r. Thus if the offered traffic intensity is r, the expected number of offered calls at any time is rn. We allow the traffic to be nonuniform, i.e., it may differ in intensity from cell to cell. We describe this nonuniformity by a probability vector $p = (p_1, p_2, \ldots, p_N)$ which we call the ***traffic pattern***. Thus if the traffic intensity is r, the expected number of offered calls in cell i is $p_i rn$. When a call request arrives in a given cell, the system's frequency-assignment algorithm can either honor the request or block it. Honoring a call request means assigning it to one of the n user frequencies; blocking a call means ignoring it completely. The frequencies assigned to calls must satisfy certain ***frequency reuse constraints***, which can be described as follows. There is a fixed collection $E = \{E_1, E_2, \ldots, E_K\}$, of subsets of cells, called "forbidden" subsets. It is illegal for the same frequency to be in use simultaneously in each cell of a forbidden set. As we will discuss in the next section, a finite set together with a collection of subsets is called a ***hypergraph***, and so we call our cellular systems ***hypergraph systems***. In this paper we will investigate performance limits for arbitrary hypergraph cellular systems described by a pair (H, p), where $H = (V, E)$ is a hypergraph and p is a traffic pattern.

For example, consider the simple seven-cell system shown in Figure 1. For purposes of illustration, we shall assume that two cells which are adjacent are forbidden from using the same frequency, and in addition that the sets $\{1, 3, 5\}$ and $\{2, 4, 6\}$ are also forbidden; thus there are 14 forbidden sets for this system. We shall also assume that the traffic pattern is $(p_1, \ldots, p_7) = (1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/4)$. We shall use this particular system throughout the paper to illustrate our results.

A natural measure of the performance of a given frequency-assignment algorithm is its ***blocking probability***, which, as a function of the intensity r of the offered traffic, we denote by $P_{\text{block}}(r)$. We have found it easier, however, to work with an equivalent performance measure which we call the ***carried traffic function*** $T(r)$, defined as

$$T(r) = r(1 - P_{\text{block}}(r)). \tag{1.1}$$

This function measures the expected number of calls being serviced by the system as a function of the offered traffic intensity. A study of the function $T(r)$ is the central concern of this paper. For a given system (H, p), we shall define a function $T_{H,p}(r)$, which can be computed by linear programming, and which has the following significance. If $T(r)$ denotes the carried traffic function for any frequency-assignment algorithm for the S-system, then $T(r) \le T_{H,p}(r)$. On the other hand, in the limit as the number of available frequencies becomes large, there exist frequency-assignment algorithms for the (H, p)-system whose carried traffic functions are arbitrarily close to $T_{H,p}(r)$. Thus $T_{H,p}(r)$ can fairly be called ***the*** carried traffic function for the (H, p)-system.

The organization of the paper is as follows. In Section 2, we present some preliminary material about hypergraphs, including a discussion of what we call ***random hypergraph multicolorings***, a notion which is central to our analysis of frequency-assignment algorithms. In Section 3 we will show that for any frequency-assignment algorithm, the carried-traffic function must satisfy $T(r) \le T_0(r)$, where $T_0(r)$ is a simple function that can be computed by linear programming. In Section 4, on the other hand, we will give an asymptotic analysis of a class of "fixed" frequency-assignment algorithms, and show that in the limit as $n \to \infty$, these algorithms achieve carried traffic functions that are at least as large as $T_1(r)$, another simple function that can be computed by linear programming. In Section 5 we will show that $T_0(r) = T_1(r)$. This common value, denoted by $T_{H,p}(r)$, is the function referred to above. In Section 5 we will also describe some of the most important properties of the function $T_{H,p}(r)$, and identify the "most favorable" traffic patterns for a given hypergraph H.

2. Hypergraph Multicolorings and Random Hypergraph Multicolorings.

A ***hypergraph*** H is a pair (V, E), where $V = \{v_1, v_2, \ldots, v_N\}$ is a finite set of ***vertices***, and $E = \{E_1, E_2, \ldots, E_K\}$ is a finite collection of subsets of V, called the ***edges*** of H. (See Berge [1] as a general reference for hypergraphs. Note that an ordinary graph is just a hypergraph in which every edge has two elements.) We shall assume that each edge of H contains at least two vertices. An ***independent set*** for H is a set of vertices which contains no edge as a subset. A ***maximal independent set*** is an independent set which is not a proper subset of any other independent set. We assume H has M maximal independent sets $\{V_1, V_2, \ldots, V_M\}$. For future reference, we also define the ***indicator set*** I_j for the maximal independent set V_j as $I_j = \{i : v_i \in V_j\}$, and the incidence matrix $A = (a_{ij})$ as

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \in V_j \\ 0 & \text{if } v_i \notin V_j. \end{cases}$$

The hypergraph H can be reconstructed from A, and for our purposes it is the preferred representation of H.

If $w = (w_1, w_2, \ldots, w_N)$ is a list of real numbers assigned to the vertices of H, define the ***v–m transform*** (vertex-maximal independent set) of w as $W = (W_1, W_2, \ldots, W_M)$, where $W = wA$, i.e.,

$$W_j = \sum_{i:i \in I_j} w_i = \sum_{i=1}^{N} w_i a_{ij} \quad \text{for } j = 1, 2, \ldots M. \tag{2.1}$$

For example, if $w = (1, 1, \ldots, 1)$, then $W = (N_1, N_2, \ldots, N_M)$, where N_j is the size of the jth maximal independent set V_j. Similarly, if $X = (X_1, X_2, \ldots, X_M)$ is a list of real numbers assigned to the maximal independent sets of H, the ***m-v transform*** (maximal independent set-vertex transform) of X is $x = (x_1, x_2, \ldots, x_N)$, where $x^T = AX^T$, i.e.,

$$x_i = \sum_{j:i \in I_j} X_j = \sum_{j=1}^{M} X_j a_{ij} \quad \text{for } i = 1, 2, \ldots N. \tag{2.2}$$

For example, if $X = (1,1,\ldots,1)$, then $x = (M_1, M_2, \ldots, M_N)$, where M_i denotes the number of maximal independent sets containing vertex v_i.

For example, consider the hypergraph of Figure 1. There are 14 edges, viz., the 12 pairs of adjacent cells, together with $\{1,3,5\}$ and $\{2,4,6\}$. There are 10 maximal independent sets, viz., $V = \{V_1, V_2, \ldots, V_{10}\}$, where the corresponding adjacency matrix A is given by

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\!\!\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \tag{2.3}$$

Thus e.g., $V_5 = \{2,5\}$ and $V_{10} = \{7\}$. Note that the N_j's are the column sums of A, and the M_i's are the row sums of A. In the example, $(N_1, N_2, \ldots, N_{10}) = (2,2,2,2,2,2,2,2,2,1)$ and $(M_1, M_2, \ldots, M_7) = (3,3,3,3,3,3,1)$.

An *n-multicoloring* of a hypergraph $H = (V, E)$ is an assignment of a set of distinct elements ("colors") from $\{1,2,\ldots,n\}$ to each vertex in such a way that for all colors $c = 1,2,\ldots,n$, the set of vertices assigned color c cannot contain an edge as a subset. Equivalently, the set of vertices assigned color c must be an independent set. An n-multicoloring can be described by an $N \times n$ matrix (m_{ic}) of 0's and 1's such that $m_{ic} = 1$ if color c is assigned to vertex i, and 0 otherwise. Given an n-multicoloring, if the number of colors assigned to v_i is m_i, then plainly

$$m_i = \sum_{c=1}^{n} m_{ic}, \qquad i = 1,2,\ldots,N. \tag{2.4}$$

Theorem 2.1. *Suppose (m_{ic}) is an n-multicoloring of the hypergraph H. If $(w_1, w_2, \ldots, w_N)$ is any set of nonnegative weights attached to the vertices of H, we have*

$$\sum_{i=1}^{N} w_i m_i \le n W_{\max}, \tag{2.5}$$

where $W_{\max}$ is the maximum component of the v–m transform of $(w_1, w_2, \ldots, w_N)$.

Proof: From (2.3) we have

$$\begin{aligned} \sum_{i=1}^{N} w_i m_i &= \sum_{i=1}^{N} w_i \sum_{c=1}^{n} m_{ic} \\ &= \sum_{c=1}^{n} \sum_{i=1}^{N} w_i m_{ic}. \end{aligned} \tag{2.6}$$

For a fixed value of c, the inner sum in (2.6) is equal to $\sum_{i \in J_c} w_i$, where $J_c = \{i : m_{ic} = 1\}$. But by the definition of a multicoloring, the set of vertices assigned a fixed color must be an independent set, so that $J_c \subseteq I_j$ for some index j. Hence (recall that the weights w_i are nonnegative)

$$\sum_{i=1}^{N} w_i m_{ic} = \sum_{i \in J_c} w_i \le \sum_{i \in I_j} w_i = W_j \le W_{\max}. \tag{2.7}$$

Thus combining (2.5) and (2.6), we have

$$\sum_{i=1}^{N} w_i m_i \le \sum_{c=1}^{n} W_{\max} = n W_{\max},$$

as asserted. ❒

We next define a *random n-multicoloring of H* as a random $N \times n$ matrix $M = (m_{ic})$ of 0's and 1's such that for each point ω in the underlying sample space, $M(\omega)$ is an n-multicoloring of H.

Theorem 2.2. *If M is a random n-multicoloring for H, then for any set of numbers $(y_1, y_2, \ldots, y_N)$ satisfying $0 \le y_i \le 1$, we have*

$$E(\sum_{i=1}^{N} m_i) \le \sum_{i=1}^{N} E(m_i) y_i + n \max_j (N_j - Y_j),$$

where N_j is the size of the jth maximal independent set V_j, and $(Y_1, Y_2, \ldots, Y_M)$ is the v–m transform of $(y_1, y_2, \ldots, y_N)$.

Proof: For any ω we have

$$\sum_{i=1}^{N} m_i(\omega) = \sum_{i=1}^{N} m_i(\omega) y_i + \sum_{i=1}^{N} m_i(\omega)(1 - y_i).$$

The theorem now follows by applying Theorem 2.1 to the second sum on the right side, and then taking expectations of both sides. ❒

3. Upper Bounds on the Performance of Frequency-Assignment Algorithms.

We now assume that our hypergraph H serves as a model for a FDMA cellular telephone exchange. (One of the earliest appearances of graph models for cellular systems is in Pennotti [5]. The first appearance of hypergraph models is in Sivarajan [6].) There are n frequencies available and the vertices of H represent the cells of the system. The edges of H give the frequency reuse restrictions, i.e., an edge of H represents a set of cells in which it is forbidden to use the same frequency simultaneously. Conversely, a given frequency can be used simultaneously in each cell of an independent set. We assume that calls arrive randomly, and that the normalized traffic intensity is r Erlangs per available frequency, i.e., the expected number of offered calls in the system is rn. The traffic pattern $p = (p_1, p_2, \ldots, p_N)$ determines the offered traffic in each cell, i.e., the expected number of offered calls in the ith cell is $p_i rn$. We assume each arriving call is either assigned a frequency, or blocked, according to a particular frequency assignment algorithm. We shall make no formal attempt to define a frequency-assignment algorithm except to assume that any such algorithm produces a random n-multicoloring of H, viz., at any point in time the set of cells using a given frequency must be a subset of a maximal independent set of H. If we denote the number of frequencies being used in cell i by m_i, then the expected value of the carried traffic is $E(\sum_{i=1}^{N} m_i)$. As mentioned in Section 1, we measure the performance of a given frequency-assignment algorithm by its carried-traffic function $T(r)$ defined in (1.1), which is the expected number of accepted calls per cell per available frequency:

$$T(r) = \frac{1}{n} E(m_1 + \cdots + m_N). \tag{3.1}$$

The main result of this section is the following.

Theorem 3.1. *Let $(y_1, y_2, \ldots, y_N)$ be any list of N numbers satisfying $0 \le y_i \le 1$ for $i = 1, 2, \ldots, N$. Then for any frequency-assignment algorithm,*

$$T(r) \le r\sum_{i=1}^{N} p_i y_i + \max_j (N_j - Y_j), \tag{3.2}$$

where $(Y_1, Y_2, \ldots, Y_M)$ is the v–m transform of $(y_1, y_2, \ldots, y_N)$.

Proof: Since the average carried traffic cannot exceed the average offered traffic, and since the average offered traffic in the ith cell is $p_i rn$, then $E(m_i) \le p_i rn$ for $i = 1, 2, \ldots, N$. The result now follows from (3.1) and Theorem 2.2. ❒

The following result is a simple corollary to Theorem 3.1 but it it allows us to define the important function $T_0(r)$, which is an upper bound on the carried traffic function for any frequency-assignment algorithm for the (H, p)-system.

Theorem 3.2. *Suppose $T_0(r)$ is the value of the following linear program:*

$$r\sum_{i=1}^{N} p_i y_i + y_{N+1} = \text{minimum, subject to} \tag{3.3}$$

$$0 \le y_i \le 1 \qquad i = 1, 2, \ldots, N \tag{3.4}$$

$$\sum_{i=1}^{N} y_i a_{ij} + y_{N+1} \ge N_j \qquad j = 1, 2, \ldots, M. \tag{3.5}$$

Then for any frequency-assignment algorithm for the (H, p) system, $T(r) \le T_0(r)$.

Proof: If (3.4) is satisfied, then by Theorem 3.1, the bound (3.2) holds. If now y_{N+1} is a real number satisfying (3.5), then by the definition (2.1) of the v–m transform,

$$y_{N+1} \ge N_j - Y_j, \qquad j = 1, 2, \ldots, M, \tag{3.6}$$

Thus from (3.2), it follows that

$$T(r) \le r\sum_{i=1}^{N} p_i y_i + y_{N+1} \tag{3.7}$$

for any set of numbers $y_1, y_2, \ldots, y_{N+1}$ satisfying (3.4) and (3.5). This completes the proof. ❒

We can illustrate Theorem 3.1 with the (H, p) system of Figure 1, for which $(N_1, \ldots, N_{10}) = (2,2,2,2,2,2,2,2,2,1)$. If we take $y = (1,1,1,1,1,1,1)$, then $Y = (2,2,2,2,2,2,2,2,2,1)$, and $\max_j(N_j - Y_j) = 0$. Thus Theorem 3.1 implies $T(r) \le r$ for all $r \ge 0$. If $y = (0,0,0,0,0,0,0)$, then $Y = (0,0,0,0,0,0,0,0,0,0)$, and $\max_j(N_j - Y_j) = 2$, so that $T(r) \le 2$ for all $r \ge 0$. Finally if $y = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 0)$, then $Y = (1,1,1,1,1,1,1,1,1,0)$, and $\max_j(N_j - Y_j) = 1$, so that $T(r) \le \frac{3}{8}r + 1$ for all $r \ge 0$. Combining these three inequalities, we find that the carried traffic function $T(r)$ for the (H, p) system in Figure 1 must satisfy $T(r) \le \min(r, \frac{3}{8}r + 1, 2)$. Indeed it is possible to show that $T_0(r) = \min(r, \frac{3}{8}r + 1, 2)$ for this system. (See Figure 2.)

4. Asymptotic Performance of Fixed Frequency Assignment Algorithms.

In this section we will study the asymptotic performance of a class of frequency-assignment algorithms which we call *fixed frequency-assignment algorithms*. By asymptotic we mean that n, the number of available frequencies, is large. We shall not be precise about the underlying model of the offered traffic except to require that it satisfy the following "asymptotic traffic property (ATP)," originally introduced by McEliece and Sivarajan [4], which can be defined by the performance of a simple one-cell frequency-assignment algorithm.

Suppose then that there is just one cell, and that there are n available frequencies. An obvious frequency-assignment algorithm in this situation is a "greedy" algorithm, i.e., one in which when a new call request arrives, it is assigned to any unoccupied frequency, if there is one, and otherwise it is blocked. If the intensity of the offered traffic is k Erlangs, we denote by $C(k, n)$ the carried traffic, i.e., the expected number of occupied frequencies for the greedy algorithm. The ATP referred to above is

$$\lim_{n\to\infty} \frac{C(k_n, n)}{n} = \min(r, 1) \qquad \text{if } k_n/n \to r. \tag{4.1}$$

The ATP says that if the offered traffic is less than the number of available frequencies, then asymptoticaly all call requests can be honored, whereas if the offered traffic exceeds the number of available frequencies, then asymptotically all available frequencies will be occupied. It is thus a kind of law of large numbers. Most common traffic models satisfy the ATP, including the standard Poisson arrivals with exponential call durations. (The "Erlang B formula" can be used to prove this—see Bertsekas and Gallager [2], Sec. 3.4.3).

We will now define a family of frequency-assignment algorithms, which we call *fixed frequency-assignment algorithms*, and proceed to analyze their asymptotic performance, assuming the ATP.

Thus let $X = (X_1, X_2, \ldots, X_M)$ be a list of M real numbers satisfying $X_j \ge 0$ and $\sum_j X_j = 1$. We use these numbers to define an n-multicoloring of the hypergraph H, as follows. For $j = 1, 2, \ldots, M$, we define $n_j = \lfloor nX_j \rfloor$, and create M disjoint classes of frequencies $C_1, C_2, \ldots, C_M$, with class C_j containing exactly n_j frequencies. Then for $j = 1, 2, \ldots, M$, we assign each of the frequencies in C_j to each vertex in the jth maximal independent set V_j. In this way we obtain an n-multicoloring of H with $m_i = \sum_j n_j a_{ij}$ frequencies assigned to vertex v_i. But since $nX_j \le n_j < nX_j + 1$, and since $\sum_j X_j a_{ij} = x_i$, where where $(x_1, x_2, \ldots, x_N)$ is the m-v transform of X, it follows that $nx_i \le m_i < nx_i + M_i$, and so

$$\lim_{n\to\infty} \frac{m_i}{n} = x_i \qquad \text{for } i = 1, 2, \ldots, N. \tag{4.2}$$

For a given X and n, we define a frequency-assignment algorithm as follows. When a call request arrives at v_i, assign it one of the m_i frequencies available at v_i, if at least one is not in use; otherwise, block the call. We call this the *X fixed frequency-assignment algorithm*. It is, in effect, N independent greedy algorithms, one for each cell in the system.

Theorem 4.1. *Assume the ATP. If $T_X(r)$ denotes the carried traffic function of the X fixed frequency-assignment algorithm for the (H, p) system, then*

$$\lim_{n\to\infty} T_X(r) = \sum_{i=1}^{N} \min(rp_i, x_i) \qquad \text{for all } r \ge 0. \tag{4.3}$$

Proof: In the X fixed frequency-assignment algorithm, each of the N cells operates independently of the others. Within the ith cell, the algorithm is a greedy algorithm with m_i available frequencies and the offered traffic is $p_i rn$ Erlangs. Thus the carried traffic in the ith cell is $C(p_i rn, m_i)$, and the carried traffic for the entire system is $\sum_{i=1}^{N} C(p_i rn, m_i)$, so that

$$T_X(r) = \frac{1}{n}\sum_{i=1}^{N} C(p_i rn, m_i). \qquad (4.4)$$

But from the ATP property (4.1), and the known rate of growth of m_i (4.2),

$$\lim_{n\to\infty} \frac{1}{n} C(p_i rn, m_i) = \min(p_i r, x_i) \qquad (4.5)$$

Combining (4.4) and (4.5) we obtain (4.3). □

Theorem 4.2. *Suppose $T_1(r)$ is the value of the following linear program:*

$$\sum_{j=1}^{M} N_j X_j - \sum_{i=1}^{N} z_i = \text{maximum, subject to} \qquad (4.6)$$

$$X_j \ge 0 \quad j = 1, 2, \ldots, M \qquad (4.7)$$

$$z_i \ge 0 \quad i = 1, 2, \ldots, N \qquad (4.8)$$

$$\sum_{j=1}^{M} X_j = 1 \qquad (4.9)$$

$$\sum_{j=1}^{M} X_j a_{ij} - z_i \le p_i r \qquad i = 1, 2, \ldots, N \qquad (4.10)$$

Then for any fixed r, there exists a fixed frequency-assignment algorithm for the (H, p) system whose asymptotic carried traffic function is arbitrarily close to $T_1(r)$.

Proof: We begin with vectors $X = (X_1, X_2, \ldots, X_M)$ and $z = (z_1, z_2, \ldots, z_N)$ whose components satisfy (4.7), (4.8), (4.9), and (4.10). Let $(x_1, x_2, \ldots, x_N)$ be the m-v transform of X. We note that (4.10) is equivalent to $x_i - z_i \le p_i r$, and so by (4.8) and (4.10) we have $x_i - z_i \le \min(p_i r, x_i)$. Therefore

$$\sum_{i=1}^{N} (x_i - z_i) \le \sum_{i=1}^{N} \min(p_i r, x_i). \qquad (4.11)$$

Note also

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N}\sum_{j=1}^{M} X_j a_{ij} = \sum_{j=1}^{M} X_j \sum_{i=1}^{N} a_{ij} = \sum_{j=1}^{M} X_j N_j, \qquad (4.12)$$

so that

$$\sum_{i=1}^{N} (x_i - z_i) = \sum_{j=1}^{M} X_j N_j - \sum_{i=1}^{N} z_i. \qquad (4.13)$$

Therefore we have

$$\lim_{n\to\infty} T_X(r) = \sum_{i=1}^{N} \min(p_i r, x_i) \quad \text{by Theorem 4.1}$$
$$\ge \sum_{i=1}^{N} (x_i - z_i) \quad \text{by (4.11)}$$
$$= \sum_{j=1}^{S} N_j X_j - \sum_{i=1}^{N} z_i \quad \text{by (4.13)} \qquad (4.14)$$

Thus the X fixed frequency-assignment algorithm has a carried traffic function which is asymptotically at least as large as the objective function (4.6), and this proves the theorem. □

To illustrate Theorem 4.1, we return to the hypergraph of Figure 1. If we let $X_2 = X_5 = X_8 = 1/5$, $X_{10} = 2/5$, and $X_j = 0$ for all other values of j, then the m-v transform of X is $(1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 2/5)$, and Theorem 4.1 implies that the X-algorithm's asymptotic carried traffic curve is $T_X(r) = 6\min(\frac{1}{8}r, \frac{1}{5} + \min(\frac{1}{4}r, 2/5) = \min(\frac{3}{4}r, \frac{6}{5}) + \min(\frac{1}{4}r, 2/5) = \min(r, \frac{8}{5})$. Similarly, if $Y_2 = Y_5 = Y_8 = 1/3$, and $Y_j = 0$ for all other values of j, then the m-v transform of Y is $(1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 0)$, and by Theorem 4.1, the Y-algorithm's asymptotic carried traffic curve is $T_Y(r) = 6\min(\frac{1}{8}r, \frac{1}{3}) = \min(\frac{3}{4}r, 2)$. These two functions are shown in Figure 3. By taking all possible convex combinations of X and Y, i.e., vectors of the form $Z = \lambda X + (1-\lambda)Y$, we obtain a family of curves which give the convex hull of $T_X(r)$ and $T_Y(r)$. But this convex hull is the same as the curve $T_0(r)$ given in Figure 2. Since we saw in Section 3 that no point above this curve shaded region is achievable by any frequency-assignment algorithm, and we have shown that every point below the curve is asymptotically achievable, we are justified in asserting that the function $T_0(r)$ is *the* achievable region for the performance of frequency-assignment algorithms for the (H, p) system of Figure 1. In the next section we will see that this is no accident, but an instance of a general rule.

5. Equality of $T_0(r)$ and $T_1(r)$. General Properties of this Function.

In Section 3 we showed that for any frequency-assignment algorithm for the (H, p) system, the corresponding carried traffic function was bounded above by $T_0(r)$. On the other hand, in Section 4, we showed that if n is sufficiently large, then the performance of certain fixed frequency-assignment algorithms for the (H, p) system is bounded *below* by $T_1(r)$. Interestingly, however, these two functions are equal.

Theorem 5.1. *$T_0(r) = T_1(r)$, for all $r \ge 0$.*

Proof: By Theorem 3.2, $T_0(r)$ is the value of a certain linear program; and by Theorem 4.2, $T_1(r)$ is the value of another linear program. However, these programs are dual programs (see Franklin [3], Section 1.2), and so by the Duality Theorem of Linear Programming ([3], Section 1.8), the values of these two programs are equal, provided both programs are feasible. It is easy to show that both programs are feasible: A feasible solution for the $T_0(r)$ program is $y_1 = y_2 = \cdots = y_N = 0$ and $y_{N+1} = \max_j N_j$; and a feasible solution for the $T_1(r)$ program is $X_1 = 1, X_2 = \cdots = X_M = 0$ and $z_i = a_{i1}$ for $i = 1, 2, \ldots, N$. Thus $T_0(r) = T_1(r)$, as asserted. □

Let us denote the common value of the functions $T_0(r)$ and $T_1(r)$ by $T_{H,p}(r)$. The next theorem gives the most important general properties of this function.

Theorem 5.2. *The function $T_{H,p}(r)$ has the following properties.*

(a) *$T_{H,p}(r)$ is nondecreasing, continuous, piecewise linear, and convex $\cap$.*

(b) *$T_{H,p}(r) = r$ for all $r \leq r_0$, where r_0 is the value of the following linear program:*

$$r = \text{maximum, subject to} \tag{5.1}$$
$$X_j \geq 0 \qquad j = 1, 2, \ldots, M \tag{5.2}$$
$$\sum_{j=1}^{M} X_j = 1 \tag{5.3}$$
$$\sum_{j=1}^{M} X_j a_{ij} \geq p_i r \qquad i = 1, 2, \ldots, N. \tag{5.4}$$

(c) *$T_{H,p}(r) = \max_j N_j$ for all $r \geq r_1$, where r_1 is the value of the following linear program (here $V_1, V_2, \ldots, V_{M_1}$ are the maximal independent sets of largest cardinality):*

$$r = \text{minimum, subject to} \tag{5.5}$$
$$X_j \geq 0 \qquad j = 1, 2, \ldots, M_1 \tag{5.6}$$
$$\sum_{j=1}^{M_1} X_j = 1 \tag{5.7}$$
$$\sum_{j=1}^{M_1} X_j a_{ij} \leq p_i r \qquad i = 1, 2, \ldots, N. \tag{5.8}$$

Proof: (a). According to Theorem 4.2, a feasible solution for the $T_1(r)$ program will also be feasible for all $r' \geq r$, and so $T_1(r') \geq T_1(r)$, for all $r' \geq r$. Thus $T_{H,p}(r)$ is nondecreasing. Furthermore, the $T_1(r)$ program is a parametric linear program (with parameter r) in the sense of ([3] Section 1.9), and so by a result proved there (page 70), the function $T_1(r)$ is continuous, convex $\cap$, and piecewise linear.

(b). We use the $T_1(r)$ program (Theorem 4.2) as our definition of $T_{H,p}(r)$. Since $N_j = \sum_i a_{ij}$, the objective function (4.6) can be written as

$$\sum_{i=1}^{N} \left(\sum_{j=1}^{M} X_j a_{ij} - z_i \right). \tag{5.9}$$

By the constraint (4.10), it follows that $T_1(r) \leq r$, with equality if and only if $\sum_j X_j a_{ij} - z_i = p_i r$ for $i = 1, 2, \ldots, N$. Thus since $z_i \geq 0$, we have that $T_1(r) = r$ if and only if there exists a vector $(X_1, \ldots, X_M)$ satisfying (5.2)–(5.4).

(c). We again use the $T_1(r)$ program definition of $T_{H,p}(r)$, and define $N_{\max} = \max_j N_j$. Because of the constraints (4.7)–(4.9), the objective function (4.6) is at most $N_{\max}$, with equality if and only if the z_i's are all zero and $X_j = 0$ for $j > M_1$. Thus $T_1(r) = N_{\max}$ if and only if there exists a vector $(X_1, \ldots, X_M)$ satisfying (5.6)–(5.8). ❒

For a given system (H, p), Theorem 5.2 tells us that $T_{H,p}(r) \leq \min(r, N_{\max})$, independent of the traffic pattern p. If, for a particular p we have $T_{H,p}(r) = \min(r, N_{\max})$, we say that p is a ***favorable traffic pattern*** for H. The next theorem identifies these traffic patterns.

Theorem 5.3. *p is a favorable traffic pattern for H if and only if the vector $N_{\max} p$ is a convex combination of the first M_1 columns of the incidence matrix A, i.e., if there exists a vector $(X_1, \ldots, X_{M_1})$ satisfying (5.6) and (5.7) such that*

$$\sum_{j=1}^{M} X_j a_{ij} = N_{\max} p_i \qquad \text{for } i = 1, 2, \ldots, N.$$

Proof: By Theorem 5.2, p is favorable if and only if $r_0 = r_1$. If this holds, then $r_0 = T_{H,p}(r_0) = T_{H,p}(r_1) = N_{\max}$, and so p is favorable if and only if $T_1(N_{\max}) = N_{\max}$. We saw in the proof of Theorem 5.2(c), however, that $T_1(r) = N_{\max}$ if and only if $z_i = 0$ and $X_j = 0$ for $j > M_1$ in the $T_1(r)$ program. Then by (4.10) with $r = N_{\max}$, the components x_i of the m-v transform of X satisfy

$$x_i = \sum_{j=1}^{M_1} X_j a_{ij} \leq p_i N_{\max}, \qquad i = 1, \ldots, N.$$

But also

$$\sum_{i=1}^{N} x_i = \sum_{j=1}^{M_1} X_j \sum_{i=1}^{N} a_{ij} = \sum_{j=1}^{M_1} X_j N_j = N_{\max}.$$

On the other hand, $\sum_i N_{\max} p_i = N_{\max}$, so if $x_i < N_{\max} p_i$ for any value of i, we would have $\sum_i x_i < N_{\max}$, a contradiction. ❒

To illustrate Theorem 5.3, we return to the hypergraph described in Figure 1. Here $N_{\max} = 2$ and so a traffic pattern p is favorable if and only if $2p$ is a convex combination of the first 9 columns of the incidence matrix A given in (2.3). It turns out that any traffic pattern with $p_7 = 0$ and $p_i \leq p_{i+2} + p_{i+3} + p_{i+4}$ for $i = 1, 2, \ldots, 6$ (subscripts taken modulo 6) is of this form (see Figure 4).

Acknowledgements.

A preliminary version of this paper, dealing with uniform traffic on systems described by ordinary graphs, was presented in [4]. This work was supported by grant from GTE Laboratories, by AFOSR grant 88-0247, and by a grant from Pacific Bell.

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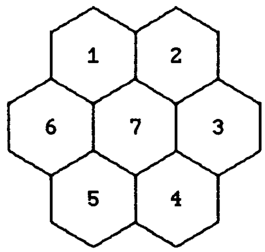

Figure 1. A seven cell system. There are 14 forbidden subsets, the 12 pairs of adjacent cells, and the sets $\{1,3,5\}$ and $\{2,4,6\}$. The traffic pattern is $(p_1, p_2, \ldots, p_7) = (1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/4)$.

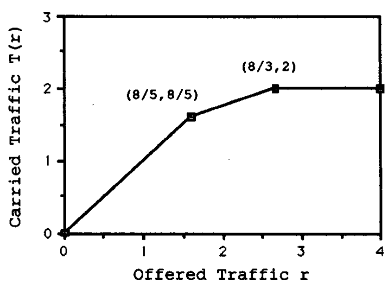

Figure 2. The function $T_0(r) = \min(r, \frac{3}{8}r + 1, 2)$ for the (H, p) system of Figure 1.

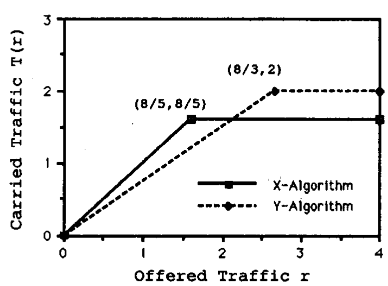

Figure 3. The functions $T_X(r) = \min(r, \frac{8}{5})$ and $T_Y(r) = \min(\frac{3}{4}r, 2)$ for the (H, p) system described in Figure 1. Here $X = (0, \frac{1}{5}, 0, 0, \frac{1}{5}, 0, 0, \frac{1}{5}, 0, \frac{2}{5})$ and $Y = (0, \frac{1}{3}, 0, 0, \frac{1}{3}, 0, 0, \frac{1}{3}, 0, 0)$.

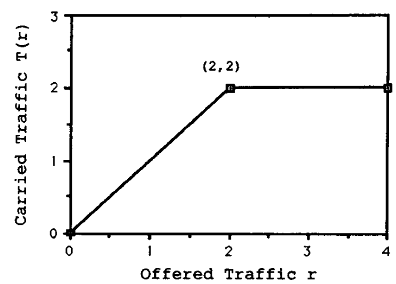

Figure 4. The carried traffic function for the (H, p) system, where H is the hypergraph in Figure 1, for a favorable traffic pattern p.