State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems

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Abstract—**Simple state-space formulas are derived for all controllers solving a standard $\mathcal{H}_\infty$ problem: for a given number $\gamma > 0$, find all controllers such that the $\mathcal{H}_\infty$ norm of the closed-loop transfer function is (strictly) less than γ. A controller exists if and only if the unique stabilizing solutions to two algebraic Riccati equations are positive definite and the spectral radius of their product is less than γ^2. Under these conditions, a parametrization of all controllers solving the problem is given as a linear fractional transformation (LFT) on a contractive, stable free parameter. The state dimension of the coefficient matrix for the LFT, constructed using these same two Riccati solutions, equals that of the plant, and has a separation structure reminiscent of classical LQG (i.e., $\mathcal{H}_2$) theory. This paper is also intended to be of tutorial value, so a standard $\mathcal{H}_2$ solution is developed in parallel.**

I. Introduction

A. Overview

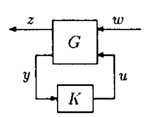
Two popular performance measures in optimal control theory are $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, defined in the frequency-domain for a stable transfer matrix $G(s)$ as

$$\|G\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\,[G(j\omega)^* G(j\omega)]\, d\omega\right)^{1/2}$$

$$\|G\|_\infty := \sup_\omega \sigma_{\max}[G(j\omega)] \quad (\sigma_{\max} := \text{maximum singular value}).$$

The former arises when the exogenous signals either are fixed or have a fixed power spectrum; the latter arises from (weighted) balls of exogenous signals. $\mathcal{H}_2$-optimal control theory was heavily studied in the 1960's as the linear quadratic Gaussian (LQG) optimal control problem; $\mathcal{H}_\infty$-optimal control theory is continuing to be developed.

The basic block diagram used in this paper is



where G is the generalized plant and K is the controller. Only finite-dimensional linear time-invariant (LTI) systems and controllers will be considered in this paper. The generalized plant G contains what is usually called the plant in a control problem plus all weighting functions. The signal w contains all external inputs, including disturbances, sensor noise, and commands; the output z is an error signal; y is the measured variables; and u is the control input. The diagram is also referred to as a linear fractional transformation (LFT) on K, and G is called the coefficient matrix for the LFT. The resulting closed-loop transfer function from w to z is denoted by T_{zw}.

The main $\mathcal{H}_\infty$ output feedback results of this paper as described in the Abstract are presented in Section III. The proofs of these results in Section V exploit the "separation" structure of the controller, which is reminiscent of the classical $\mathcal{H}_2$ controller. Of course, there are significant differences that reflect the fact that the $\mathcal{H}_\infty$ criterion corresponds to designing for the worst exogenous signal. These are also discussed in Section V. Special attention will be given to the central controller, obtained by setting to zero the free parameter in the LFT formula for the controller.

If full state feedback is available, then the central controller is simply a gain matrix F_∞, obtained through solving a single Riccati equation. Also, the optimal estimator is an observer whose gain is obtained as a solution to a Riccati equation. These special cases are described in Section IV and the proofs are given in Sections VII and VIII. In the general output feedback case the central controller can be interpreted as an optimal estimator for $F_\infty x$. Furthermore, the two Riccati equations involved in this solution can be associated with state feedback and estimation problems.

The algebraic Riccati equations in the $\mathcal{H}_\infty$ solution are those that arise in the theory of linear quadratic differential games. The game theoretic analogy is intuitively appealing for in the $\mathcal{H}_\infty$ control problem the exogenous input and the control input can be viewed as strategies employed by opposing players in a game: the exogenous input is chosen to maximize the norm of the output and the control input is chosen to minimize it. The Riccati equations have indefinite quadratic terms, however, so solutions cannot be guaranteed as simply as in the $\mathcal{H}_2$ problem. Indeed, as mentioned in the Abstract, the existence of solutions to the Riccati equations is part of the necessary and sufficient conditions for existence of $\mathcal{H}_\infty$ (sub)optimal controllers. The preliminary machinery needed to establish these conditions in terms of Riccati equations is developed in Section II.

To facilitate exposition, the problem chosen for treatment in this paper is the simplest special case which captures all the essential features of the general problem. Although the assumptions used in this special case involve some sacrifice of generality, the formulas are simple and easy to interpret. Also, these assumptions are commonly used in treatments of the $\mathcal{H}_2$ problem. The general formulas are presented in [14]. Our entire approach to the $\mathcal{H}_\infty$ problem has parallels in the conventional $\mathcal{H}_2$ theory. In fact, it is interesting to note that as γ tends to ∞, the central controller actually approaches the $\mathcal{H}_2$ controller. Because the $\mathcal{H}_2$ theory is well-understood throughout the control community, these two problems are developed side-by-side. It is hoped that this will enhance the paper's tutorial value.

This paper is organized in a top-down manner and is intended to be accessible to a variety of readers. While this organization may be a bit awkward for the experts who plan to study all the details, it is hoped that it will enhance the tutorial value of the paper. The main results are stated in Sections I-IV. The reader who is only interested in seeing the new theorems and formulas could stop

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there. The reader may find some deeper interpretation and see how the separation argument is used to prove the output feedback results by additionally reading Section V. Finally, the more technical aspects of the proofs are in Sections VI–VIII. Section IX on connections with risk sensitivity and 2×2-block mixed Hankel–Toeplitz operators is intended for experts.

The proofs are constructed out of a series of lemmas, each of which has some independent interest. A possible contribution of this paper, beyond the new formulas and theorems, may be some of the machinery developed. The result is that the proofs of both the theorems and the lemmas leading to them are quite short. Furthermore, the development is reasonably self-contained, and the only background required is a knowledge of elementary aspects of state-space theory, $\mathcal{L}_2$ spaces, and operators on $\mathcal{L}_2$, including projections and adjoints.

The main results in this paper appeared, without proofs, in [8].

B. Historical Perspective

Since the state-space methods in this paper stand in contrast to previous work in $\mathcal{H}_\infty$ theory, it is useful to provide some historical perspective. This section is not intended as a review of the literature in $\mathcal{H}_\infty$ theory, but rather an attempt to outline some of the work that most closely touches on this paper. For a more extensive bibliography and review of the literature, the interested reader might see [11] and [12].

Zames' [40] original formulation of $\mathcal{H}_\infty$ optimal control theory was in an input–output setting. Most solution techniques available at that time involved analytic functions (Nevanlinna–Pick interpolation) or operator-theoretic methods [33], [1], [4]. An earlier state-space solution was presented in [7], in which the steps were as follows: parametrize all internally-stabilizing controllers via [38]; obtain realizations of the closed-loop transfer matrix; convert the resulting model-matching problem into the equivalent 2×2-block general distance or best approximation problem involving mixed Hankel–Toeplitz operators; reduce to the Nehari problem (Hankel only); solve the Nehari problem by the procedure of Glover [13]. Both [11] and [12] give expositions of this approach, which will be referred to as the "1984" approach.

In a mathematical sense, the 1984 procedure "solved" the general rational $\mathcal{H}_\infty$ optimal control problem and much of the subsequent work in $\mathcal{H}_\infty$ control theory focused on the 2×2-block problems, either in the model-matching or general distance forms. Unfortunately, the associated complexity of computation was substantial, involving several Riccati equations of increasing dimension, and formulas for the resulting controllers tended to be very complicated and have high state dimension. Encouragement came from Limebeer, Hung, and Halikias [26], [27] who showed, for problems transformable to 2×1-block problems, that a subsequent minimal realization of the controller has state dimension no greater than that of the generalized plant G. This suggested the likely existence of similarly low dimension optimal controllers in the general 2×2 case.

Additional progress on the 2×2-block problems came from Ball and Cohen [3], who gave a state-space solution involving three Riccati equations. Jonckheere and Juang [19] showed a connection between the 2×1-block problem and previous work by Jonckheere and Silverman on linear-quadratic control. Foias and Tannenbaum [10] developed an interesting class of operators called skew Toeplitz to study the 2×2-block problem. Other approaches have been derived by Hung [18] using an interpolation theory approach, Kwakernaak [25] using a polynomial approach, and Kimura [22] using a method based on conjugation.

In addition to providing controller formulas that are simple and expressed in terms of plant data, the methods in the present paper are a fundamental departure from the earlier work described above. In particular, the Youla parametrization and the resulting 2×2-block model-matching problem of the 1984 solution are avoided entirely; replaced by a pair of 2×1 block problems and a separation argument. The entire development uses simple and familiar tools, in the style of Willems [37], relying on state feedback and observer-based control methods and more straightforward and elegant use of operator theory. Another strong influence on this paper is Redheffer's work [31] on linear fractional transformations.

Interestingly, the formulas for the controller given here were actually first obtained with the 1984 approach, but using a new 2×2-block solution, together with a cumbersome back substitution. This approach has subsequently been developed and extended in [14], where the optimal case is also treated in detail. The very simplicity of the new formulas and their similarity with the $\mathcal{H}_2$ ones suggested a more direct approach. Of course, elegance is in the eye of the beholder, and researchers who have been studying the 2×2 block problems arising in the 1984 approach may still prefer it to the new approach taken here. While the Youla parametrization and a 2×2-block problem of the 1984 solution are not needed for our new results, these techniques have played a central role in $\mathcal{H}_\infty$ theory, and so we will indicate how they fit in with this paper's development. In particular, the new results on 2×2-block mixed Hankel–Toeplitz operators are briefly described in Section IX-A, and the resulting new version of the 1984 approach is sketched.

Independent encouragement for a simpler approach to the $\mathcal{H}_\infty$ problem came from papers by Khargonekar, Petersen, Rotea, and Zhou [20], [21], [41]. They showed that for the state-feedback $\mathcal{H}_\infty$ problem one can choose a constant gain as a (sub)optimal controller. In addition, a formula for the state-feedback gain matrix was given in terms of an algebraic Riccati equation. These results are similar to those in Section IV-A below, although the proof techniques in Section VII are entirely different. Also, these papers established connections between $\mathcal{H}_\infty$-optimal control, quadratic stabilization, and linear-quadratic differential games.

We expect the results and techniques in this paper to encourage greater interest in applications of $\mathcal{H}_\infty$ methods, in alternative developments of the theory using other techniques, and in extensions to more general problems. Some of this has already taken place. A version of the Glover and Doyle [14] formulas for the controller was used by Stein in the Tutorial Workshop on $\mathcal{H}_\infty$ Control Theory which preceded the 1987 IEEE Conference on Decision and Control. The formulas also appeared in the associated workshop notes [9]. Several software packages are under development with the new results as a central feature, and these are being used in a number of applications.

On the theoretical side, Green *et al.* [17] offer an alternative development using a J-spectral factorization approach. Bernstein and Haddad [5] were able to reproduce the formulas in this paper using Lagrange multiplier techniques, which will be discussed more in Section II-C. Tadmor [34] uses the maximum principle to extend these results to the time-varying, finite horizon case. For the relationship of the results in this paper to the risk-sensitive LQG problem of Whittle [35], see [14]. There are a number of other generalizations, many of which have been completed and will appear elsewhere.

C. Notation

The notation is fairly standard. The Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ consist of square-integrable functions on the imaginary axis with analytic continuation into, respectively, the right and left half-plane. The Hardy space $\mathcal{H}_\infty$ consists of bounded functions with analytic continuation into the right half-plane. The Lebesgue spaces $\mathcal{L}_2 = \mathcal{L}_2(-\infty, \infty)$, $\mathcal{L}_{2+} = \mathcal{L}_2[0, \infty)$, and $\mathcal{L}_{2-} = \mathcal{L}_2(-\infty, 0]$ consist, respectively, of square-integrable functions on $(-\infty, \infty)$, $[0, \infty)$, and $(-\infty, 0]$, and $\mathcal{L}_\infty$ consists of bounded functions on $(-\infty, \infty)$. As interpreted in this paper, $\mathcal{L}_\infty$ will consist of functions of frequency, $\mathcal{L}_{2+}$ and $\mathcal{L}_{2-}$ functions of time, and $\mathcal{L}_2$ will be used for both.

We will make liberal use of the Hilbert space isomorphism, via the Laplace transform and the Paley–Wiener theorem, of $\mathcal{L}_2 = \mathcal{L}_{2+} \oplus \mathcal{L}_{2-}$ in the time-domain with $\mathcal{L}_2 = \mathcal{H}_2 \oplus \mathcal{H}_2^\perp$ in the frequency-domain, and of $\mathcal{L}_{2+}$ with $\mathcal{H}_2$ and $\mathcal{L}_{2-}$ with $\mathcal{H}_2^\perp$. In fact, we will normally not make any distinction between a time-domain signal and its transform. Thus, we may write $w \in \mathcal{L}_{2+}$ and then treat w as if $w \in \mathcal{H}_2$. This style streamlines the development, as well as the notation, but when any possibility of confusion could arise, we will make it clear whether we are working in the time- or frequency-domain.

A transfer matrix in terms of state-space data is denoted

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI-A)^{-1}B+D.$$

For a matrix $M \in \mathbb{C}^{p\times r}$ or $\mathbb{R}^{p\times r}$, M' denotes its transpose, M^* denotes its conjugate transpose, $\sigma_{\max}(M) = \rho(M^*M)^{1/2}$ denotes its maximum singular value, $\rho(M)$ denotes its spectral radius (if $p = r$), and $M^\dagger$ denotes the Moore–Penrose psuedoinverse of M. Im denotes image, ker denotes kernel, and $G^\sim(s) := G(-s)'$. For operators, Γ^* denotes the adjoint of Γ. The prefix $\mathcal{B}$ denotes the open unit ball and the prefix $\mathcal{R}$ denotes real-rational.

The orthogonal projections P_+ and P_- map $\mathcal{L}_2$ to, respectively, $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ (or $\mathcal{L}_{2+}$ and $\mathcal{L}_{2-}$). For $G \in \mathcal{L}_\infty$, the Laurent or multiplication operator $M_G: \mathcal{L}_2 \to \mathcal{L}_2$ for frequency-domain $w \in \mathcal{L}_2$ is defined by $M_G w = Gw$. The norms on $\mathcal{L}_\infty$ and $\mathcal{L}_2$ in the frequency-domain were defined in Section I-A. Note that both norms apply to matrix or vector-valued functions. The unsubscripted norm $\|\cdot\|$ will denote the standard Euclidean norm on vectors. We will omit all vector and matrix dimensions throughout, and assume that all quantities have compatible dimensions.

II. Preliminaries

This section reviews some mathematical preliminaries, in particular, the computation of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of a transfer matrix G. Consider the transfer matrix

$$G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \tag{1}$$

with A stable (i.e., all eigenvalues in the open left half-plane). Although $\|G\|_2$ and $\|G\|_\infty$ can, in principle, be computed from their definitions in Section I-A, it is useful in the development of the subsequent theory to have alternative characterizations.

One useful characterization is in terms of hypothetical input-output experiments. Let e_i denote the ith standard basis vector. Apply the impulsive input $\delta(t)e_i$ (δ is the unit impulse) and denote the output by $z_i(t)$. Then $z_i \in \mathcal{L}_{2+}$, because $D = 0$, and

$$\|G\|_2^2=\sum_i \|z_i\|_2^2.$$

For the $\mathcal{H}_\infty$ norm, suppose that we apply an input $w \in \mathcal{L}_2$ and consider the output $z \in \mathcal{L}_2$. Then a standard result is that $\|G\|_\infty$ is the induced norm of the multiplication operator M_G, as well as the Toeplitz operator $P_+M_G: \mathcal{H}_2 \to \mathcal{H}_2$.

$$\|G\|_\infty = \sup_{w\in\mathcal{B}\mathcal{L}_2} \|z\|_2 = \sup_{w\in\mathcal{B}\mathcal{L}_{2+}} \|P_+z\|_2 = \sup_{w\in\mathcal{B}\mathcal{H}_2} \|P_+M_Gw\|_2.$$

The rest of this section involves additional characterizations of the norms in terms of state-space descriptions. Section II-A considers the $\mathcal{H}_2$ norm and II-C considers the $\mathcal{H}_\infty$ norm. Section II-B collects some basic material on the Riccati equation and the Riccati operator which play an essential role in the development of both theories.

A. Computing $\mathcal{H}_2$ Norm

If L_c denotes the controllability Gramian of (A, B) and L_o the observability Gramian of (C, A), then

$$AL_c+L_cA'+BB'=0 \qquad A'L_o+L_oA+C'C=0$$

$$\|G\|_2^2=\text{trace}\,(CL_cC')=\text{trace}\,(B'L_oB). \tag{2}$$

Note that this computation involves the solution of a linear equation and can be done in a finite number of steps.

B. The Riccati Operator

Let A, Q, R be real $n \times n$ matrices with Q and R symmetric. Define the $2n \times 2n$ Hamiltonian matrix

$$H := \begin{bmatrix} A & R \\ Q & -A' \end{bmatrix}.$$

Assume H has no eigenvalues on the imaginary axis. Then it must have n eigenvalues in Re $s < 0$ and n in Re $s > 0$. Consider the two n-dimensional spectral subspaces $\mathcal{X}_-(H)$ and $\mathcal{X}_+(H)$: the former is the invariant subspace corresponding to eigenvalues in Re $s < 0$; the latter, to eigenvalues in Re $s > 0$. Finding a basis for $\mathcal{X}_-(H)$, stacking the basis vectors up to form a matrix, and partitioning the matrix, we get

$$\mathcal{X}_-(H)=\text{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where $X_1, X_2 \in \mathbb{R}^{n\times n}$. If X_1 is nonsingular, or equivalently, if the two subspaces

$$\mathcal{X}_-(H),\ \text{Im}\begin{bmatrix} 0 \\ I \end{bmatrix} \tag{3}$$

are complementary, we can set $X := X_2X_1^{-1}$. Then X is uniquely determined by H, i.e., $H \mapsto X$ is a function, which will be denoted Ric; thus, $X = \text{Ric}(H)$. We will take the domain of Ric, denoted dom(Ric), to consist of Hamiltonian matrices H with two properties, namely, H has no eigenvalues on the imaginary axis and the two subspaces in (3) are complementary. For ease of reference, these will be called the stability property and the complementarity property, respectively. The following well-known results give some properties of X as well as verifiable conditions under which H belongs to dom(Ric). See, for example, Section 7.2 in [11], Theorem 12.2 in [39], and [23].

Lemma 1: Suppose $H \in$ dom(Ric) and $X = \text{Ric}(H)$. Then:

a) X is symmetric;

b) X satisfies the algebraic Riccati equation

$$A'X+XA+XRX-Q=0;$$

c) $A + RX$ is stable.

Lemma 2: Suppose H has no imaginary eigenvalues, R is either positive semidefinite or negative semidefinite, and (A, R) is stabilizable. Then $H \in$ dom(Ric).

Lemma 3: Suppose H has the form

$$H=\begin{bmatrix} A & -BB' \\ -C'C & -A' \end{bmatrix}$$

with (A, B) stabilizable and (C, A) detectable (denote the unobservable subspace by $\mathcal{X}$). Then $H \in$ dom(Ric), $X = \text{Ric}(H) \geq 0$, and $\ker(X) \subset \mathcal{X}$.

Note that if (C, A) is observable, then $\text{Ric}(H) > 0$. Also, note that $\ker(X) \subset \mathcal{X} \subset \ker(C)$, so that the equation $XM = C'$ always has a solution for M, and we will denote the least-squares solution by $X^\dagger C'$.

C. Computing $\mathcal{H}_\infty$ Norm

For the transfer matrix $G(s)$ in (1), with A stable and $\gamma > 0$, define the Hamiltonian matrix

$$H := \begin{bmatrix} A & \gamma^{-2}BB' \\ -C'C & -A' \end{bmatrix}. \tag{4}$$

The following lemma is essentially from [2], [37], and [6].

Lemma 4: The following conditions are equivalent.

a) $\|G\|_\infty < \gamma$.

b) H has no eigenvalues on the imaginary axis.

c) $H \in \mathrm{dom(Ric)}$.

d) $H \in \mathrm{dom(Ric)}$ and $\mathrm{Ric}(H) \geq 0$ ($\mathrm{Ric}(H) > 0$ if (C, A) is observable).

Proof: Without loss of generality, take $\gamma = 1$. This can be done with the scaling $G \rightarrow \gamma^{-1}G$, $B \rightarrow \gamma^{-1}B$. We begin by proving the equivalence of a) and b). We have

$$(I - G^{\sim}G)^{-1}(s) = \left[\begin{array}{cc|c} A & BB' & B \\ -C'C & -A' & 0 \\ \hline 0 & B' & I \end{array}\right]$$

so H is the A-matrix of $(I - G^{\sim}G)^{-1}$. It is easy to check that this realization has no uncontrollable or unobservable modes on the imaginary axis. Thus, H has no eigenvalues on the imaginary axis iff $(I - G^{\sim}G)^{-1}$ has no poles there, i.e., $(I - G^{\sim}G)^{-1} \in \mathcal{RL}_\infty$. So it suffices to prove that

$$\|G\|_\infty < 1 \Leftrightarrow (I - G^{\sim}G)^{-1} \in \mathcal{RL}_\infty.$$

If $\|G\|_\infty < 1$, then $I - G(j\omega)^*G(j\omega) > 0$, $\forall\omega$, and hence $(I - G^{\sim}G)^{-1} \in \mathcal{RL}_\infty$. Conversely, if $\|G\|_\infty \geq 1$, then $\sigma_{\max}[G(j\omega)] = 1$ for some ω, i.e., 1 is an eigenvalue of $G(j\omega)^*G(j\omega)$, so $I - G(j\omega)^*G(j\omega)$ is singular. Thus, a) and b) are equivalent.

The equivalence of b) and c) follows from Lemma 2, and the equivalence of c) and d) follows from Lemma 1 and standard results for solutions of Lyapunov equations. ■

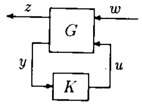
Lemma 4 suggests the following way to compute an $\mathcal{H}_\infty$ norm: select a positive number γ; test if $\|G\|_\infty < \gamma$ by calculating the eigenvalues of H; increase or decrease γ accordingly; repeat. Thus, $\mathcal{H}_\infty$ norm computation requires a search, over either γ or ω, in contrast to $\mathcal{H}_2$ norm computation, which does not. We should not be surprised by similar characteristics of the $\mathcal{H}_\infty$-optimal control problem. A somewhat analogous situation occurs for matrices with the norms $\|M\|_2^2 = \mathrm{trace}(M^*M)$ and $\|M\|_\infty = \sigma_{\max}[M]$. In principle, $\|M\|_2^2$ can be computed exactly with a finite number of operations, as can the test for whether $\sigma_{\max}(M) < \gamma$ (e.g., $\gamma^2 I - M^*M > 0$), but the value of $\sigma_{\max}(M)$ cannot. To compute $\sigma_{\max}(M)$ we must use some type of iterative algorithm.

One could use the above characterizations of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms to obtain controllers by fixing the controller order and solving the equations for controller parameters using Lagrange multiplier methods. This method was studied extensively in the 1960's and early 1970's for the $\mathcal{H}_2$ case. Bernstein and coworkers have recently explored this in a series of papers (see, for example, [5] and the references therein), and extended the approach to handle $\mathcal{H}_\infty$ specifications as well. In both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cases, one can derive coupled Riccati equations, which can, in principle, be solved to obtain a suboptimal controller of the specified dimension.

It was well-known that in the full-order $\mathcal{H}_2$ case these coupled Riccati equations obtained using Lagrange multiplier methods decouple into the two standard equations and the approach yields the optimal controller. A new result in [5] is that the simple formulas in this paper for the $\mathcal{H}_\infty$ case can be reproduced as well. While their techniques cannot currently reproduce the full theory developed in this paper, their results are still very encouraging as their techniques do extend to more general problems, such as the reduced-order controller problem, in a natural way.

III. Main Results: Output Feedback

Consider the system described by the block diagram



Both G and K are real-rational and proper. Section III-A discusses the assumptions on G that will be used in both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cases. In Section III-B we show how to pick K to minimize the $\mathcal{H}_2$ norm of T_{zw}, the transfer matrix from w to z; in Section III-C we do the same for the $\mathcal{H}_\infty$ norm. In both cases K is constrained to provide internal stability. In our application we shall have state models of G and K. Then *internal stability* will mean that the states of G and K go to zero from all initial values when $w = 0$. Since we will restrict our attention exclusively to proper, real-rational controllers which are stabilizable and detectable, these properties will be assumed throughout. Thus, the term controller will be taken to mean a controller which satisfies these properties. Controllers that have the additional property of being internally-stabilizing will be said to be *admissible*.

A. Assumptions on G

The realization of the transfer matrix G is taken to be of the form

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

The following assumptions are made.

i) (A, B_1) is stabilizable and (C_1, A) is detectable.

ii) (A, B_2) is stabilizable and (C_2, A) is detectable.

iii) $D_{12}'[C_1 D_{12}] = [0\ I]$.

iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}' = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

Assumption i) is made for a technical reason: together with ii) it guarantees that the two Hamiltonian matrices (H_2 and J_2 below) in the $\mathcal{H}_2$ problem belongs to dom(Ric). This assumption simplifies the theorem statements and proofs, but if it is relaxed, the theorems and proofs can be modified so that the given formulas are still correct. An important simplification that is a consequence of the assumptions is that internal stability is essentially equivalent to input–output stability ($T_{zw} \in \mathcal{RH}_\infty$). This is captured in the following lemma, which is proven in Section VI-B. Of course, assumption ii) is necessary and sufficient for G to be internally stabilizable, but is not needed to prove the equivalence of internal stability and $T_{zw} \in \mathcal{RH}_\infty$.

Lemma 5: Suppose that assumptions i), iii), and iv) hold. Then a controller K is admissible iff $T_{zw} \in \mathcal{RH}_\infty$.

Assumption iii) means that C_1x and $D_{12}u$ are orthogonal so that the penalty on $z = C_1x + D_{12}u$ includes a nonsingular, normalized penalty on the control u. In the conventional $\mathcal{H}_2$ setting this means that there is no cross weighting between the state and control input, and that the control weight matrix is the identity. Other nonsingular control weights can easily be converted to this problem with a change of coordinates in u. Relaxing the orthogonality condition introduces a few extra terms in the controller formulas. This is well-known in the $\mathcal{H}_2$ case and the $\mathcal{H}_\infty$ case generalizes similarly. Assumption iv) is dual to iii) and concerns how the exogenous signal w enters G: w includes both plant disturbance and sensor noise, these are orthogonal, and the sensor noise weighting is normalized and nonsingular.

Two additional assumptions that are implicit in the assumed realization for $G(s)$ are that $D_{11} = 0$ and $D_{22} = 0$. Relaxing these assumptions complicates the formulas substantially, as can be seen in [14]. There it is shown how to form an equivalent problem with $D_{22} = 0$. A transformation from D_{11} nonzero to $D_{11} = 0$ was done in [41], for the state feedback case. Finally, Safonov and Limebeer [32] have shown, in the output feedback case, how to construct an equivalent problem with both $D_{11} = 0$ and $D_{22} = 0$.

The above assumptions were chosen in part because they are reasonably standard in elementary treatments of conventional $\mathcal{H}_2$ control theory. Developing the $\mathcal{H}_\infty$ theory in parallel with the $\mathcal{H}_2$

theory in this way should facilitate the tutorial contribution of this paper. Furthermore, the entire development, including proofs, under these assumptions, contains the essential features of the general problem when these assumptions are relaxed. The only major subtlety in the general case occurs for optimal controllers, which are considered only briefly in Section V-G.

B. $\mathcal{H}_2$ Case

The first problem in this section is to find an admissible controller K which minimizes $\|T_{zw}\|_2$. By Lemma 3, the Hamiltonian matrices

$$H_2 := \begin{bmatrix} A & -B_2B_2' \\ -C_1'C_1 & -A' \end{bmatrix}, \quad J_2 := \begin{bmatrix} A' & -C_2'C_2 \\ -B_1B_1' & -A \end{bmatrix}$$

belong to dom(Ric) and, moreover, $X_2 := \mathrm{Ric}(H_2)$ and $Y_2 := \mathrm{Ric}(J_2)$ are positive semidefinite. Define $F_2 := -B_2'X_2$, $L_2 := -Y_2C_2'$, and

$$A_{F2} := A + B_2F_2, \qquad C_{1F2} := C_1 + D_{12}F_2$$

$$A_{L2} := A + L_2C_2, \qquad B_{1L2} := B_1 + L_2D_{21}$$

$$\hat{A}_2 := A + B_2F_2 + L_2C_2$$

$$G_c(s) := \left[\begin{array}{c|c} A_{F2} & I \\ \hline C_{1F2} & 0 \end{array}\right], \qquad G_f(s) := \left[\begin{array}{c|c} A_{L2} & B_{1L2} \\ \hline I & 0 \end{array}\right].$$

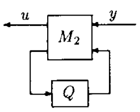
Theorem 1: The unique optimal controller is

$$K_{\mathrm{opt}}(s) := \left[\begin{array}{c|c} \hat{A}_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right].$$

Moreover, $\min \|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2 = \|G_cL_2\|_2^2 + \|C_1G_f\|_2^2$.

The controller K_{opt} has the well-known separation structure, which will be discussed in more detail in Section V. For comparison to the $\mathcal{H}_\infty$ results below, it is useful to describe all suboptimal controllers.

Theorem 2: The family of all admissible controllers such that $\|T_{zw}\|_2 < \gamma$ equals the set of all transfer matrices from y to u in



$$M_2(s) = \left[\begin{array}{c|cc} \hat{A}_2 & -L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

where $Q \in \mathcal{RH}_2$, $\|Q\|_2^2 < \gamma^2 - (\|G_cB_1\|_2^2 + \|F_2G_f\|_2^2)$.

Thus, the suboptimal controllers are parametrized by a fixed (independent of γ) linear-fractional transformation with a free parameter Q. With $Q = 0$ we recover K_{opt}. It is worth noting that the parametrization in Theorem 2 makes T_{zw} affine in Q and yields the Youla parametrization of all stabilizing controllers when the conditions on Q are replaced by $Q \in \mathcal{RH}_\infty$. This particular parametrization also has additional useful properties that will be discussed in Section IX-A.

C. $\mathcal{H}_\infty$ Case

The problem considered in this subsection is the suboptimal $\mathcal{H}_\infty$ control problem: find all admissible K such that $\|T_{zw}\|_\infty < \gamma$. Clearly, γ must be greater than the $\mathcal{H}_\infty$-optimal level. In Section V-G we will briefly discuss how to find an admissible K to minimize $\|T_{zw}\|_\infty$. Optimal $\mathcal{H}_\infty$ controllers are more difficult to characterize than suboptimal ones, and this is one major difference between the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ results. Recall that similar differences arose in the norm computation problem in Section II-C as well.

The $\mathcal{H}_\infty$ solution involves two new Hamiltonian matrices

$$H_\infty := \begin{bmatrix} A & \gamma^{-2}B_1B_1' - B_2B_2' \\ -C_1'C_1 & -A' \end{bmatrix},$$

$$J_\infty := \begin{bmatrix} A' & \gamma^{-2}C_1'C_1 - C_2'C_2 \\ -B_1B_1' & -A \end{bmatrix}.$$

The important difference here is that the (1, 2)-blocks are not sign definite, so we cannot use the lemmas in Section II to guarantee that $H_\infty \in$ dom(Ric) or $\mathrm{Ric}(H_\infty) \geq 0$. Indeed, these conditions are intimately related to the existence of $\mathcal{H}_\infty$ suboptimal controllers. Note that the (1, 2)-blocks are a suggestive combination of expressions from the $\mathcal{H}_\infty$ analysis of Section II-C and the $\mathcal{H}_2$ synthesis of Section III-B. The reasons for the form of these expressions should become clearer through the discussions and proofs for the following theorem.

Theorem 3: There exists an admissible controller such that $\|T_{zw}\|_\infty < \gamma$ iff the following three conditions hold.

i) $H_\infty \in$ dom(Ric) and $X_\infty := \mathrm{Ric}(H_\infty) \geq 0$.

ii) $J_\infty \in$ dom(Ric) and $Y_\infty := \mathrm{Ric}(J_\infty) \geq 0$.

iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

Moreover, when these conditions hold, one such controller is

$$K_{\mathrm{sub}}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

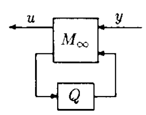
$$\hat{A}_\infty := A + \gamma^{-2}B_1B_1'X_\infty + B_2F_\infty + Z_\infty L_\infty C_2$$

$$F_\infty := -B_2'X_\infty, \quad L_\infty := -Y_\infty C_2', \quad Z_\infty := (I - \gamma^{-2}Y_\infty X_\infty)^{-1}.$$

The $\mathcal{H}_\infty$ controller displayed in Theorem 3 has certain obvious similarities to the $\mathcal{H}_2$ controller as well as some important differences. Although it is not as apparent as in the $\mathcal{H}_2$ case, the $\mathcal{H}_\infty$ controller also has an interesting separation structure. Furthermore, each of the conditions in the theorem can be given a system-theoretic interpretation in terms of this separation. These interpretations, given in Section V, require the filtering and full information (i.e., state feedback) results of the next section. The proof of Theorem 3 is constructed out of these results as well.

The following theorem parametrizes the controllers that achieve a suboptimal $\mathcal{H}_\infty$ norm less than γ.

Theorem 4: If conditions i)–iii) in Theorem 3 are satisfied, the set of all admissible controllers such that $\|T_{zw}\|_\infty < \gamma$ equals the set of all transfer matrices from y to u in



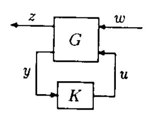
$$M_\infty(s) = \left[\begin{array}{c|cc} \hat{A}_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

where $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < \gamma$.

As in the $\mathcal{H}_2$ case, the suboptimal controllers are parametrized by a fixed linear-fractional transformation with a free parameter Q. With $Q = 0$ we recover the central controller $K_{\mathrm{sub}}(s)$.

IV. Special Problems

In this section we discuss four problems from which the output feedback solutions of the previous sections will be constructed via a separation argument. These special problems are central to the whole approach taken in this paper, and as we shall see, they are also important in their own right. All pertain to the standard block diagram



but with different structures for G. The problems are labeled as follows.

FI: Full information.

FC: Full control.

DF: Disturbance feedforward.

OE: Output estimation.

FC and OE are natural duals of FI and DF, respectively. The output feedback solutions will be constructed out of the FI and OE results.

These special problems are not, strictly speaking, special cases of the output feedback problem, as they do not satisfy all of the assumptions. Each of the four problems inherits certain of the assumptions i)–iv) from Section III as appropriate. The terminology and assumptions will be discussed in the subsections for each problem. In each of the four cases, the results are summarized as a list of five items, as follows (in all cases, K must be admissible):

1) the minimum of $\|T_{zw}\|_2$
2) the unique controller minimizing $\|T_{zw}\|_2$
3) the family of all controllers such that $\|T_{zw}\|_2 < \gamma$, where γ is greater than the minimum norm
4) necessary and sufficient conditions for the existence of a controller such that $\|T_{zw}\|_\infty < \gamma$
5) the family of all controllers such that $\|T_{zw}\|_\infty < \gamma$.

In all cases, item 3) can yield the Youla parametrization of all stabilizing controllers, as in Theorem 2. To obtain this in each case below, simply replace the given conditions on Q with $Q \in \mathcal{RH}_\infty$. Detailed proofs for the results stated for these four problems are presented in Sections VII and VIII.

A. Problem FI: Full Information

$$G(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right].$$

In this problem the controller is provided with *full information* since $y = \binom{x}{w}$. However, in the $\mathcal{H}_2$ case the optimal controller uses just x, with w providing only redundant information. And in the $\mathcal{H}_\infty$ case a suboptimal controller exists which also uses just x. This case could have been restricted to state feedback, which is more traditional, but we believe that the full information problem is more fundamental and more natural than the state feedback problem, once one gets outside the pure $\mathcal{H}_2$ setting.

One setting in which the full information case is more natural occurs when the parametrization of all suboptimal controllers is considered. It is also appropriate when studying the general case when $D_{11} \neq 0$ or when $\mathcal{H}_\infty$ optimal (not just suboptimal) controllers are desired. Even though the latter two problems are not studied in detail in this paper, we want the methods to extend to them in a natural and straightforward way.

The assumptions relevant to the FI problem which are inherited from the output feedback problem are as follows.

i) (A, B_1) is stabilizable and (C_1, A) is detectable.
ii) (A, B_2) is stabilizable.
iii) $D_{12}'[C_1\ D_{12}] = [0\ I]$.

Assumption iv) has been effectively strengthened because of the assumed structure for C_2 and D_{21}. We should note that for the FI (and FC) problem, internal stability is not equivalent to $T_{zw} \in \mathcal{RH}_\infty$, although this presents no difficulties in the proofs. We simply must remember that in the FI case K admissible means internally stabilizing, not just $T_{zw} \in \mathcal{RH}_\infty$. The results for the full information case are as follows.

FI.1: $\min \|T_{zw}\|_2 = \|G_cB_1\|_2 = (\operatorname{trace}(B_1'X_2B_1))^{1/2}$.

FI.2: $K(s) = [F_2\ 0]$.

FI.3: $K(s) = [F_2\ Q(s)]$, where $Q \in \mathcal{RH}_2$, $\|Q\|_2^2 < \gamma^2 - \|G_cB_1\|_2^2$.

FI.4: $H_\infty \in \operatorname{dom}(\operatorname{Ric})$, $\operatorname{Ric}(H_\infty) \geq 0$.

FI.5: $K(s) = [F_\infty - Q(s)\gamma^{-1}B_1'X_\infty\ \ Q(s)]$, where $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < \gamma$.

The $\mathcal{H}_2$ optimal controller in this case is a constant gain obtained from $\operatorname{Ric}(H_2)$, as is well-known, and the $\mathcal{H}_\infty$ controller is obtained similarly from $\operatorname{Ric}(H_\infty)$. A fundamental difference between the two is that the $\mathcal{H}_\infty$ controller depends on the disturbance through B_1, whereas the $\mathcal{H}_2$ controller does not. This difference is essentially captured by the necessary and sufficient conditions for the existence of a controller given in FI.4. Note that this condition is the same as condition i) in Theorem 3.

The two individual conditions in FI.4 may each be given their own interpretations. The condition that $H_\infty \in \operatorname{dom}(\operatorname{Ric})$ implies that $X_\infty := \operatorname{Ric}(H_\infty)$ exists and $K(s) = [F_\infty\ 0]$ gives T_{zw} as

$$T_{zw}=\left[\begin{array}{c|c} A_{F\infty} & B_1 \\ \hline C_{1F_\infty} & 0 \end{array}\right] \qquad \begin{array}{l} A_{F_\infty} := A+B_2F_\infty \\ C_{1F_\infty} := C_1+D_{12}F_\infty . \end{array} \tag{5}$$

Furthermore, $\|T_{zw}\|_\infty < \gamma$, which is verified as part of the proof of Lemma 18 in Section VI-D. The further condition that $X_\infty \geq 0$ is equivalent to this K stabilizing T_{zw}. Since this will be used again, it will be stated as a lemma.

Lemma 6: Suppose $H_\infty \in \operatorname{dom}(\operatorname{Ric})$. Then A_{F_∞} in (5) is stable iff $X_\infty \geq 0$.

Proof: Rearrange the Riccati equation for X_∞ and use the definition of F_∞ and C_{1F_∞} to get

$$A_{F_\infty}'X_\infty+X_\infty A_{F_\infty}+C_{1F_\infty}'C_{1F_\infty}+\gamma^{-2}X_\infty B_1B_1'X_\infty=0. \tag{6}$$

Since $H_\infty \in \operatorname{dom}(\operatorname{Ric})$, $(A_{F_\infty} + \gamma^{-2}B_1B_1'X_\infty)$ is stable by Lemma 1 c), and hence $(B_1'X_\infty, A_{F_\infty})$ is detectable. Then from standard results on Lyapunov equations (see Lemma 14 a) in Section VI-B) A_{F_∞} in (5) is stable iff $X_\infty \geq 0$. ∎

B. Problem FC: Full Control

$$G(s)=\left[\begin{array}{c|cc} A & B_1 & [I\ 0] \\ \hline C_1 & 0 & [0\ I] \\ C_2 & D_{21} & [0\ 0] \end{array}\right]$$

This problem is dual to the full information case: the FC G has the same form as the transpose of the FI G. The term *full control* is used because the controller has full access to both the state through output injection and to the output z. The only restriction on the controller is that it must work with the measurement y. The assumptions that the FC problem inherit from the output feedback problem are just the dual of those in the FI problem.

i) (A, B_1) is stabilizable and (C_1, A) is detectable.
ii) (C_2, A) is detectable.

iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}' = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

The results for the FC case are as follows.

FC.1: $\min \|T_{zw}\|_2 = \|C_1G_f\|_2 = (\operatorname{trace}(C_1Y_2C_1'))^{1/2}$.

FC.2: $K(s)=\begin{bmatrix} L_2 \\ 0 \end{bmatrix}$.

FC.3: $K(s)=\begin{bmatrix} L_2 \\ Q(s) \end{bmatrix}$, where $Q \in \mathcal{RH}_2$, $\|Q\|_2^2<\gamma^2-\|C_1G_f\|_2^2$.

FC.4: $J_\infty \in \operatorname{dom}(\operatorname{Ric})$, $\operatorname{Ric}(J_\infty) \geq 0$.

FC.5: $K(s)=\begin{bmatrix} L_\infty-\gamma^{-1}Y_\infty C_1'Q(s) \\ Q(s) \end{bmatrix}$, where $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty<\gamma$.

As expected, the condition in FC.4 is the same as that in ii) of Theorem 3.

C. Problem DF: Disturbance Feedforward

$$G(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & I & 0 \end{array}\right].$$

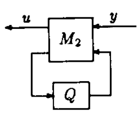
This problem inherits the same assumptions i)–iii) as in the FI problem, but for internal stability we need to strengthen i) from (A, B_1) stabilizable to $A - B_1C_2$ is stable. With this assumption, internal stability is again equivalent to $T_{zw} \in \mathcal{RH}_\infty$, as in the output feedback case. This fact will be stated in its dual form as a lemma in the next section.

To motivate the name *disturbance feedforward* consider the special case with $C_2 = 0$. Then there is no feedback and the measurement is exactly w. The feedback caused by $C_2 \neq 0$ does not affect the achievable norm as long as $A - B_1C_2$ is stable. The latter condition is equivalent to the transfer function from w to $y(G_{21})$ having neither right half-plane transmission zeros nor unstable hidden modes, so both x and w can be solved for in terms of y (since u is known as well). Thus, DF is essentially equivalent to FI, DF.1 and DF.4 are the same as FI.1 and FI.4, and DF.2, DF.3, and DF.5 can be obtained from the corresponding results in FI. This idea is formalized in the proofs in Section VIII. The DF results are as follows.

DF.1: $\min \|T_{zw}\|_2 = \|G_cB_1\|_2$.

DF.2: $$K(s) := \left[\begin{array}{c|c} A + B_2F_2 - B_1C_2 & B_1 \\ \hline F_2 & 0 \end{array}\right].$$

DF.3: The set of all transfer matrices from y to u in

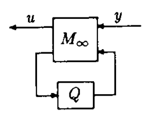


$$M_2(s) = \left[\begin{array}{c|cc} A + B_2F_2 - B_1C_2 & B_1 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

where $Q \in \mathcal{RH}_2$, $\|Q\|_2^2 < \gamma^2 - \|G_cB_1\|_2^2$.

DF.4: $H_\infty \in \text{dom(Ric)}$, $\text{Ric}(H_\infty) \geq 0$.

DF.5: The set of all transfer matrices from y to u in



$$M_\infty(s) = \left[\begin{array}{c|cc} A + B_2F_\infty - B_1C_2 & B_1 & B_2 \\ \hline F_\infty & 0 & I \\ -C_2 - \gamma^{-1}B_1'X_\infty & I & 0 \end{array}\right]$$

where $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < \gamma$.

D. Problem OE: Output Estimation

$$G(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right].$$

This problem is dual to DF, just as FC was to FI. Thus, the discussion of the DF problem is relevant here, when appropriately dualized. The OE assumptions are as follows.

i) (A, B_1) is stabilizable and $A - B_2C_1$ is stable.

ii) (C_2, A) is detectable.

iv) $$\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}' = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Assumption i), together with iv), imply that internal stability is again equivalent to $T_{zw} \in \mathcal{RH}_\infty$, as in the output feedback case. Obviously, assumption ii) is necessary for the existence of internally stabilizing controllers, but is not required to prove this equivalence. Since this is used in the proof of the output feedback problem, it will be stated as a lemma. The proof is in Section VI-C.

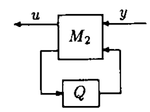
Lemma 7: Suppose that assumptions i) and iv) hold. Then for the OE problem, K is admissible iff $T_{zw} \in \mathcal{RH}_\infty$.

We are focusing on this restricted estimation problem because it is the one that arises in solving the output feedback problem. A more conventional estimation problem would be the special case where the internal stability requirement is dropped and $B_2 = 0$. Then the problem would be that of estimating the output z given the measurement y. This special case motivates the term output estimation, and can be obtained immediately from the results here. The OE results are as follows.

OE.1: $\min \|T_{zw}\|_2 = \|C_1G_f\|_2$.

OE.2: $$K(s) := \left[\begin{array}{c|c} A + L_2C_2 - B_2C_1 & L_2 \\ \hline C_1 & 0 \end{array}\right].$$

OE.3: The set of all transfer matrices from y to u in

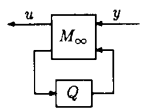


$$M_2(s) = \left[\begin{array}{c|cc} A + L_2C_2 - B_2C_1 & L_2 & -B_2 \\ \hline C_1 & 0 & I \\ C_2 & I & 0 \end{array}\right]$$

where $Q \in \mathcal{RH}_2$, $\|Q\|_2^2 < \gamma^2 - \|C_1G_f\|_2^2$.

OE.4: $J_\infty \in \text{dom(Ric)}$, $\text{Ric}(J_\infty) \geq 0$.

OE.5: The set of all transfer matrices from y to u in



$$M_\infty(s) = \left[\begin{array}{c|cc} A + L_\infty C_2 - B_2C_1 & L_\infty & -B_2 - \gamma^{-1}Y_\infty C_1' \\ \hline C_1 & 0 & I \\ C_2 & I & 0 \end{array}\right]$$

where $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < \gamma$.

It is interesting to compare $\mathcal{H}_\infty$ and $\mathcal{H}_2$ in the context of the OE problem, even though, by duality, the essence of these remarks was made before. Both optimal estimators are observers with the observer gain determined by $\text{Ric}(J_\infty)$ and $\text{Ric}(J_2)$. Optimal $\mathcal{H}_2$ output estimation consists of multiplying the optimal state estimate by the output map C_1. Thus, optimal $\mathcal{H}_2$ estimation depends only trivially on the output z that is being estimated and *state* estimation is the fundamental problem. In contrast, the $\mathcal{H}_\infty$ estimation problem depends very explicitly and importantly on the output being estimated. This will have implications for the separation properties of the $\mathcal{H}_\infty$ output feedback controller.

V. Separation Theory and Proofs of Theorems 1–4

If we assume the results of the special problems, which are proven in Sections VII and VIII, we can now prove Theorems 3 and 4 using separation arguments. This essentially involves reducing the output feedback problem to a combination of the full information and the output estimation problems. The separation properties of the $\mathcal{H}_\infty$ controller are more complicated than for the $\mathcal{H}_2$ controller, although they are no less interesting. This paper gives only a brief interpretation of these ideas, and it may be that further study will reveal deeper understanding of the structure of the $\mathcal{H}_\infty$controller. The notation and assumptions for this section are as in Section III.

A. $\mathcal{H}_2$ Controller Structure

Recall that the unique $\mathcal{H}_2$ optimal controller is

$$K_2(s) := \left[\begin{array}{c|c} \hat{A}_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right] = \left[\begin{array}{c|c} A + B_2F_2 + L_2C_2 & Y_2C_2' \\ \hline -B_2'X_2 & 0 \end{array}\right]$$

and

$$\min \|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2$$

where $X_2 := \text{Ric}(H_2)$ and $Y_2 := \text{Ric}(J_2)$ and the min is over all stabilizing controllers. Note that F_2 is the optimal state feedback in the full information problem and L_2 is the optimal output injection in the full control case. The well-known separation property of the $\mathcal{H}_2$ solution is reflected in the fact that K_2 is exactly the optimal output estimate of F_2x and can be obtained by setting $C_1 = F_2$ in OE.2. Also, the minimum cost is the sum of the FI cost (FI.1) and the OE cost for estimating F_2x (OE.1).

The controller equations can be written in standard observer form as

$$\dot{\hat{x}} = A\hat{x} + B_2u + L_2(C_2\hat{x} - y)$$

$$u = F_2\hat{x}$$

where $\hat{x}$ is the optimal estimate of x.

B. Proof of Theorem 1

The proof requires a preliminary change of variables that will also be used several times in the sequel. If we define a new control variable $\nu := u - F_2x$, the transfer function to z becomes

$$z = \left[\begin{array}{c|cc} A_{F_2} & B_1 & B_2 \\ \hline C_{1F_2} & 0 & D_{12} \end{array}\right] \begin{bmatrix} w \\ \nu \end{bmatrix} = G_cB_1w + U\nu \tag{7}$$

where

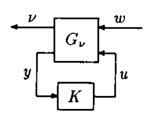
$$G_c(s) := \left[\begin{array}{c|c} A_{F_2} & I \\ \hline C_{1F_2} & 0 \end{array}\right]$$

and

$$U(s) := \left[\begin{array}{c|c} A_{F_2} & B_2 \\ \hline C_{1F_2} & D_{12} \end{array}\right].$$

This latter matrix has two useful properties given in Lemma 17 in Section VI-D and proven by simple algebra: U is inner (i.e., $U^{\sim}U = I$) and $U^{\sim}G_c$ belongs to $\mathcal{RH}_2^{\perp}$.

Proof of Theorem 1: Let K be any admissible controller and look at how ν is generated:



$$G_\nu = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F_2 & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

Note that K stabilizes G iff K stabilizes G_ν (the two closed-loop systems have identical A-matrices), and that G_ν has the form of the output estimation problem. From (7) and the properties of U we have that

$$\min \|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \min \|T_{\nu w}\|_2^2. \tag{8}$$

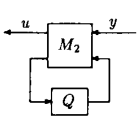
But from item OE.2, $\|T_{\nu w}\|_2$ is uniquely minimized by the controller

$$\left[\begin{array}{c|c} A + B_2F_2 + L_2C_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right]$$

and then from OE.1 $\min \|T_{\nu w}\|_2 = \|F_2G_f\|_2$. ∎

C. Proof of Theorem 2

Continuing with the development in the previous proof, we see that the set of all suboptimal controllers equal the set of all K's such that $\|T_{\nu w}\|_2^2 < \gamma^2 - \|G_cB_1\|_2^2$. Apply item OE.3 to get that such K's are parametrized by



$$M_2(s) = \left[\begin{array}{c|cc} \hat{A}_2 & -L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

with $Q \in \mathcal{RH}_2$, $\|Q\|_2^2 < \gamma^2 - \|G_cB_1\|_2^2 - \|F_2G_f\|_2^2$. ∎

D. $\mathcal{H}_\infty$ Controller Structure

The $\mathcal{H}_\infty$ controller formulas from Theorem 3 are

$$K_{\text{sub}}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

$$\hat{A}_\infty := A + \gamma^{-2}B_1B_1'X_\infty + B_2F_\infty + Z_\infty L_\infty C_2$$

$$F_\infty := -B_2'X_\infty,\; L_\infty := -Y_\infty C_2',\; Z_\infty := (I - \gamma^{-2}Y_\infty X_\infty)^{-1}$$

where $X_\infty := \text{Ric}(H_\infty)$ and $Y_\infty := \text{Ric}(J_\infty)$. The necessary and sufficient conditions for the existence of an admissible controller such that $\|T_{zw}\|_\infty < \gamma$ are as follows.

i) $H_\infty \in \text{dom(Ric)}$ and $X_\infty := \text{Ric}(H_\infty) \geq 0$.

ii) $J_\infty \in \text{dom(Ric)}$ and $Y_\infty := \text{Ric}(J_\infty) \geq 0$.

iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

We have seen that condition i) corresponds to the full information condition FI.4, and that ii) corresponds to the full control condition FC.4. It is easily shown that, given the FI and FC results, these conditions are necessary for the output feedback case as well.

Lemma 8: Suppose that there exists an admissible controller making $\|T_{zw}\|_\infty < \gamma$. Then conditions i) and ii) hold.

Proof: Let K be an admissible controller for which $\|T_{zw}\|_\infty < \gamma$. The controller $K[C_2\ D_{21}]$ solves problem FI; hence, from FI.4, $H_\infty \in \text{dom(Ric)}$ and $X_\infty := \text{Ric}(H_\infty) \geq 0$. Condition ii) follows by the dual argument. ∎

We also would expect some condition beyond these two, and that is provided by iii) which is an elegant combination of elements from the FI and FC. Note that all the conditions of Theorem 3 are symmetric in H_∞, J_∞, X_∞, and Y_∞, but the formula for the controller is not. Needless to say there is a dual form that can be obtained by inspection from the above formula. For a symmetric formula, the state equations above can be multiplied through by Z_∞ and put in descriptor form. A simple substitution from the Riccati equation for X_∞ will then yield a symmetric, although more complicated, formula. A symmetric formula of the standard form can then be obtained using $Z_\infty^{1/2}$. The details are omitted.

To emphasize its relationship to the $\mathcal{H}_2$ controller formulas, the $\mathcal{H}_\infty$ controller can be written as

$$\dot{\hat{x}} = A\hat{x} + B_1\hat{w}_{\text{worst}} + B_2u + Z_\infty L_\infty(C_2\hat{x} - y)$$

$$u = F_\infty\hat{x},\quad \hat{w}_{\text{worst}} = \gamma^{-2}B_1'X_\infty\hat{x}.$$

These equations have the structure of an observer-based compensator. The obvious questions that arise when these formulas are compared to the $\mathcal{H}_2$ formulas above are as follows.

1) Where does the term $B_1\hat{w}_{\text{worst}}$ come from?

2) Why $Z_\infty L_\infty$ instead of L_∞?

3) Is there a separation interpretation of these formulas analogous to that for $\mathcal{H}_2$?

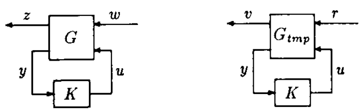
The proof of Theorem 3 reveals that there is a very well-defined separation interpretation of these formulas and that $w_{\text{worst}} := \gamma^{-2}B_1'X_\infty x$ is, in some sense, a worst-case input for the full information problem. Furthermore, $Z_\infty L_\infty$ is actually the optimal filter gain for estimating $F_\infty x$, which is the optimal full information control input, in the presence of this worst-case input. It is therefore not surprising that $Z_\infty L_\infty$ should enter in the controller equations instead of L_∞. The term $\hat{w}_{\text{worst}}$ may be thought of loosely as an estimate for w_{worst}.

E. Proof of Theorem 3

This proof also requires a preliminary change of variables that will be used repeatedly. If we define new disturbance and control variables $r := w - \gamma^{-2}B_1'X_\infty x$, $v := u + B_2'X_\infty x$, then

$$\begin{pmatrix} v \\ y \end{pmatrix} = \left[\begin{array}{c|cc} A_{tmp} & B_1 & B_2 \\ \hline -F_\infty & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right] \begin{pmatrix} r \\ u \end{pmatrix} \qquad A_{tmp} := A + \gamma^{-2}B_1B_1'X_\infty.$$

Call this transfer function G_{tmp}, and note that it has the form of the plant in problem OE. Now look at a proper controller K and its effect when applied both to G and to G_{tmp}. The two block diagrams are



The following lemma connects these two systems, T_{zw} and T_{vr}. Recall that internal and input-output stability are equivalent for admissibility of K in both the output feedback and OE problems.

Lemma 9: Assume X_∞ exists and $X_\infty \geq 0$. Then K is admissible for G and $\|T_{zw}\|_\infty < \gamma$ iff K is admissible for G_{tmp} and $\|T_{vr}\|_\infty < \gamma$.

Although the proof of this lemma is given in Section VI-D, we will give some motivation. If we assume $X_\infty := \mathrm{Ric}(H_\infty)$ exists, we can differentiate $x(t)'X_\infty x(t)$, where $x(t)$ is the solution to the plant equations for a given input w

$$\begin{aligned}\frac{d}{dt}x'X_\infty x &= \dot{x}'X_\infty x + x'X_\infty \dot{x} \\ &= x'(A'X_\infty + X_\infty A)x + 2\langle w, B_1'X_\infty x\rangle + 2\langle u, B_2'X_\infty x\rangle.\end{aligned}$$

The Riccati equation for X_∞ is

$$A'X_\infty + X_\infty A + C_1'C_1 + \gamma^{-2}X_\infty B_1B_1'X_\infty - X_\infty B_2B_2'X_\infty = 0.$$

Using this to substitute in for $A'X_\infty + X_\infty A$ gives

$$\begin{aligned}\frac{d}{dt}x'X_\infty x = &-\|C_1x\|^2 - \gamma^{-2}\|B_1'X_\infty x\|^2 \\ &+ \|B_2'X_\infty x\|^2 + 2\langle w, B_1'X_\infty x\rangle + 2\langle u, B_2'X_\infty x\rangle.\end{aligned}$$

Finally, completion of the squares along with orthogonality of C_1x and $D_{12}u$ gives the key equation:

$$\begin{aligned}\frac{d}{dt}(x'X_\infty x) = &-\|z\|^2 + \gamma^2\|w\|^2 \\ &- \gamma^2\|w - \gamma^{-2}B_1'X_\infty x\|^2 + \|u + B_2'X_\infty x\|^2. \quad (9)\end{aligned}$$

Assume $x(0) = x(\infty) = 0$, $w \in \mathcal{L}_{2+}$, and integrate (9) from $t = 0$ to $t = \infty$:

$$\begin{aligned}\|z\|_2^2 - \gamma^2\|w\|_2^2 &= \|u + B_2'X_\infty x\|_2^2 \\ &- \gamma^2\|w - \gamma^{-2}B_1'X_\infty x\|_2^2 = \|v\|_2^2 - \gamma^2\|r\|_2^2. \quad (10)\end{aligned}$$

These results motivate the change of variables to r and v, and they provide the connections between T_{zw} and T_{vr}. In particular, from (10) it is immediate that $\|T_{zw}\|_\infty \leq \gamma$ iff $\|T_{vr}\|_\infty \leq \gamma$. While this is the basic idea behind the proof of Lemma 9, the details needed for strict inequality and internal stability require a bit more work.

Note that $w_{\text{worst}} := \gamma^{-2}B_1'X_\infty x$ is the worst disturbance input in the sense that it maximizes the quantity $\|z\|_2^2 - \gamma^2\|w\|_2^2$ in (10) for the minimizing value of $u = -B_2'X_\infty x$; that is, the u making $v = 0$ and the w making $r = 0$ are values satisfying a saddle point condition. It is also interesting to note that w_{worst} is the optimal strategy for w in the corresponding LQ game problem. In terms of problem OE for G_{tmp}, the output being estimated is the optimal FI control input $F_\infty x$ and the new disturbance r is offset by the "worst case" FI disturbance input w_{worst}.

While G_{tmp}, has the form required for the OE problem, to actually use the OE results to prove Lemma 9 and the rest of Theorem 3, we will need to verify that G_{tmp} satisfies the following assumptions for the OE problem.

i) (A_{tmp}, B_1) is stabilizable and $A_{tmp} + B_2F_\infty$ is stable.

ii) (C_2, A_{tmp}) is detectable.

iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}' = \begin{bmatrix} 0 \\ I \end{bmatrix}.$

Assumptions iv) and (A_{tmp}, B_1) stabilizable from i) follow immediately from the corresponding assumptions for Theorem 3. The following lemma gives conditions for assumption ii) and the remaining part of i) to hold. Of course, the existence of an admissible controller for G_{tmp} immediately implies that assumption ii) holds. Note that the OE Hamiltonian matrix for G_{tmp} is

$$J_{tmp} := \begin{bmatrix} A_{tmp}' & \gamma^{-2}F_\infty'F_\infty - C_2'C_2 \\ -B_1B_1' & -A_{tmp} \end{bmatrix}.$$

Lemma 10: a) If $H_\infty \in \mathrm{dom(Ric)}$, then $A_{tmp} + B_2F_\infty$ is stable. b) If $J_{tmp} \in \mathrm{dom(Ric)}$ and $Y_{tmp} := \mathrm{Ric}(J_{tmp}) \geq 0$, then (C_2, A_{tmp}) is detectable.

Proof: Part a) follows immediately from Lemma 1 c) and part b) follows from the dual to Lemma 6, which gives that $(A_{tmp} - Y_{tmp}C_2'C_2)$ is stable. ■

Proof of Theorem 3 (Sufficiency): Assume conditions i)–iii) in the theorem statement hold. Using the Riccati equation for X_∞, one can easily verify that

$$T := \begin{bmatrix} I & -\gamma^{-2}X_\infty \\ 0 & I \end{bmatrix}$$

provides a similarity transformation between J_{tmp} and J_∞, i.e., $T^{-1}J_{tmp}T = J_\infty$. Thus,

$$\mathcal{X}_-(J_{tmp}) = T\mathcal{X}_-(J_\infty) = T\,\mathrm{Im}\begin{bmatrix} I \\ Y_\infty \end{bmatrix} = \mathrm{Im}\begin{bmatrix} I - \gamma^{-2}X_\infty Y_\infty \\ Y_\infty \end{bmatrix}$$

so $Y_{tmp} := \mathrm{Ric}(J_{tmp}) = Y_\infty(I - \gamma^{-2}X_\infty Y_\infty)^{-1} = Z_\infty Y_\infty$, and $\rho(X_\infty Y_\infty) < \gamma^2$ implies that $Y_{tmp} \geq 0$. Thus, by Lemma 10 the OE assumptions hold for G_{tmp}, and by item OE.4 the OE problem is solvable. For item OE.5 with $Q = 0$ one solution is

$$\left[\begin{array}{c|c} A + \gamma^{-2}B_1B_1'X_\infty - Y_{tmp}C_2'C_2 + B_2F_\infty & Y_{tmp}C_2' \\ \hline F_\infty & 0 \end{array}\right]$$

but this is precisely K_{sub} defined in Theorem 3. We conclude that K_{sub} stabilizes G_{tmp} and $\|T_{vr}\| < \gamma$. Then by Lemma 9, K_{sub} stabilizes G and $\|T_{zw}\| < \gamma$.

(Necessity): Let K be an admissible controller for which $\|T_{zw}\|_\infty < \gamma$. By Lemma 8 $H_\infty \in \mathrm{dom(Ric)}$, $X_\infty := \mathrm{Ric}(H_\infty) \geq 0$, $J_\infty \in \mathrm{dom(Ric)}$, and $Y_\infty := \mathrm{Ric}(J_\infty) \geq 0$. From Lemma 9, K is admissible for G_{tmp} and $\|T_{vr}\|_\infty < \gamma$. Together with Lemma 10 a), this implies that the OE assumptions hold for G_{tmp} and that the OE problem is solvable, so from OE.4 applied to G_{tmp}, we have that $J_{tmp} \in \mathrm{dom(Ric)}$ and $Y_{tmp} = \mathrm{Ric}(J_{tmp}) \geq 0$. Using the same argument as in the sufficiency part, we note that $Y_{tmp} = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}Y_\infty \geq 0$, which implies that $\rho(X_\infty Y_\infty) < \gamma^2$. ■

We now see exactly why the term involving $\hat{w}_{\text{worst}}$ appears and why the "observer" gain is $Z_\infty L_\infty$. Both terms are consequences of the output estimation problem of estimating the optimal full information (i.e., state feedback) control gain. While an analogous output estimation problem arises in the $\mathcal{H}_2$ output feedback problem, the resulting equations are much simpler. This is because there is no "worst-case" disturbance for the $\mathcal{H}_2$ full information problem and the problem of estimating any output, including the optimal state feedback, is equivalent to state estimation.

We may now present a separation interpretation for $\mathcal{H}_\infty$ suboptimal controllers. It will be stated in terms of the central controller, but similar interpretations could be made for the parametrization of all suboptimal controllers (see the proofs of Theorems 3 and 4).

> The $\mathcal{H}_\infty$ output feedback controller is the output estimator of the full information control law in the presence of the "worst-case" disturbance w_{worst}.

Note that the same statement holds for the $\mathcal{H}_2$ optimal controller, except that $w_{\text{worst}} = 0$.

F. Proof of Theorem 4

From Lemma 9 the set of all admissible controllers for G such that $\|T_{zw}\|_\infty < \gamma$ equals the set of all admissible controllers for G_{tmp} such that $\|T_{vr}\|_\infty < \gamma$. Apply item OE.5. ■

G. Optimality and Dependence of the Solution on γ

In this section we will discuss, without proof, the behavior of the $\mathcal{H}_\infty$ suboptimal solution as γ varies, especially as γ approaches the infimal achievable norm, denoted by γ_o. Since Theorem 3 gives necessary and sufficient conditions for existence of an admissible controller such that $\|T_{zw}\|_\infty < \gamma$, γ_o is the infimum over all γ such that conditions i)–iii) are satisfied. Theorem 3 does not give an explicit formula for γ_o, but just as for the $\mathcal{H}_\infty$ norm calculation, it can be computed as closely as desired by a search technique.

Although we have not focused on the problem of $\mathcal{H}_\infty$ *optimal* controllers, the assumptions in this paper make them relatively easy to obtain in most cases. In addition to describing the qualitative behavior of suboptimal solutions as γ varies, we will indicate why the descriptor version of the controller formulas from Section V-D can usually provide formulas for the optimal controller when $\gamma = \gamma_o$. Most of these results can be obtained relatively easily using the machinery that is developed in Sections VII and VIII. The reader interested in filling in the details is encouraged to begin by strengthening assumption i) to controllable and observable and considering the Hamiltonians for X_∞^{-1} and Y_∞^{-1}. Descriptor formulas are stated in Limebeer and Kasenally [28], and the optimal case is treated in detail in Glover *et al.* [16].

As $\gamma \to \infty$, $H_\infty \to H_2$, $X_\infty \to X_2$, etc., and $K_{\text{sub}} \to K_2$. This fact is the result of the particular choice for the central controller ($Q = 0$) that was made here. While it could be argued that K_{sub} is a natural choice, this connection with $\mathcal{H}_2$ actually hints at deeper interpretations. In fact, K_{sub} is the maximum entropy solution [29], [15] as well as the minimax controller for $\|z\|_2^2 - \gamma^2\|w\|_2^2$.

If $\gamma_2 \geq \gamma_1 > \gamma_0$, then $X_\infty(\gamma_1) \geq X_\infty(\gamma_2)$ and $Y_\infty(\gamma_2) \geq Y_\infty(\gamma_2)$. Thus, X_∞ and Y_∞ are decreasing functions of γ, as is $\rho(X_\infty Y_\infty)$. At $\gamma = \gamma_o$, any one of the three conditions in Theorem 3 can fail. If only condition iii) fails, then it is relatively straightforward to show that the descriptor formulas for $\gamma = \gamma_o$ are optimal. The formulas in Theorem 3 are not well-defined because the term $(I - \gamma_0^{-2}X_\infty Y_\infty)$ is not invertible. It is possible but far less likely that conditions i) or ii) would fail before iii). To see this, consider i) and let γ_1 be the largest γ for which H_∞ fails to be in dom(Ric), because it fails to have either the stability property or the complementarity property. The same remarks will apply to ii) by duality.

If complementarity fails at $\gamma = \gamma_1$, then $\rho(X_\infty) \to \infty$ as $\gamma \to \gamma_1$. For $\gamma < \gamma_1$, H_∞ may again be in dom(Ric), but X_∞ will be indefinite. For such γ, the controller $u = -B_2' X_\infty x$ would make $\|T_{zw}\|_\infty < \gamma$ but would not be stabilizing. If the stability property fails at $\gamma = \gamma_1$, then $H_\infty \notin$ dom(Ric) but Ric can be extended to obtain X_∞ and the controller $u = -B_2' X_\infty x$ is stabilizing and makes $\|T_{zw}\|_\infty = \gamma_1$. The stability property will also not hold for any $\gamma \leq \gamma_1$, and no controller whatsoever exists which makes $\|T_{zw}\|_\infty < \gamma_1$. In other words, if stability breaks down first then the infimum over stabilizing controllers equals the infimum over all controllers, stabilizing or otherwise. In view of this, we would expect that typically complementarity would fair first.

Complementarity failing at $\gamma = \gamma_1$ means $\rho(X_\infty) \to \infty$ as $\gamma \to \gamma_1$ so condition iii) would fail at even larger values of γ, unless the eigenvectors associated with $\rho(X_\infty)$ as $\gamma \to \gamma_1$ are in the null space of Y_∞. Thus, condition iii) is the most likely of all to fail first. If condition i) or ii) fails first because the stability property fails, the formulas in Theorem 3 as well as their descriptor versions are optimal at $\gamma = \gamma_o$. If the complementarity condition fails first, then obtaining formulas for the optimal controllers is a more subtle problem.

VI. Technical Machinery

This section collects a number of results that will be used in the proofs in the remainder of this paper. The development is in terms of a series of lemmas, each having a short and reasonably elementary proof. It is hoped that this incremental development will help reveal the essential structure of the proofs. Also, the lemmas in this section are of independent interest beyond their utility in constructing the proofs of the main theorems.

Section VI-A reviews some results on Hankel operators and introduces the 2×1-block mixed Hankel–Toeplitz operator result that will play a key role in the $\mathcal{H}_\infty$ FI problem. Section VI-B includes two lemmas on characterizing inner transfer functions and their role in certain LFT's and Section VI-C considers the equivalence of internal and input–output stability for the output feedback and OE problems. Finally, Section VI-D gives some specific state-space formulas, based on the general results in Section VI-B which will be used in the proof of Lemma 9 and the FI results. The proof of Lemma 9 is also given.

It can be assumed, without loss of generality, that $\gamma = 1$, since this is achieved by the scalings $\gamma^{-1/2}B_1$, $\gamma^{-1/2}C_1$, $\gamma^{1/2}B_2$, $\gamma^{1/2}C_2$, $\gamma^{-1}X_\infty$, $\gamma^{-1}Y_\infty$, and $\gamma^{-1}K$. This will be done for the remainder of this paper.

A. Mixed Hankel–Toeplitz Operators

It will be useful to characterize some additional induced norms of $G(s)$ in (1) and its associated differential equation

$$\dot{x} = Ax + Bw$$
$$z = Cx \tag{11}$$

with A stable. We will prove several lemmas that will be useful in the rest of the paper. It is convenient to describe all the results in the frequency domain and give all the proofs in the time domain.

Consider first the problem of using an input $w \in \mathcal{L}_{2-}$ to maximize $\|P_+ z\|_2^2$. This is exactly the standard problem of computing the Hankel norm of G (i.e., the induced norm of the Hankel operator $P_+ M_G: \mathcal{H}_2^\perp \to \mathcal{H}_2$), and can be expressed in terms of the Gramians L_c and L_o from (2). Although this result is well known, we will include a time-domain proof similar in technique to the proofs of the new results in this paper.

Lemma 11:

$$\sup_{w \in \mathcal{BL}_{2-}} \|P_+ z\|_2^2 = \sup_{w \in \mathcal{BH}_2^\perp} \|P_+ M_G w\|_2^2 = \rho(L_o L_c).$$

Proof: Assume (A, B) is controllable; otherwise, restrict attention to the controllable subspace. Then L_c is invertible and $w \in \mathcal{L}_{2-}$ can be used to produce any $x(0) = x_0$ given $x(-\infty) = 0$. The proof is in two steps. First,

$$\inf_{w \in \mathcal{L}_{2-}} \{\|w\|_2^2 \mid x(0) = x_0\} = x_0' L_c^{-1} x_0. \tag{12}$$

To show this, we can differentiate $x(t)' L_c^{-1} x(t)$ along the solutions of (11) for any given input w as follows:

$$\frac{d}{dt}(x' L_c^{-1} x) = \dot{x}' L_c^{-1} x + x' L_c^{-1} \dot{x}$$
$$= x'(A' L_c^{-1} + L_c^{-1} A)x + 2\langle w, B' L_c^{-1} x \rangle.$$

Use of (2) to substitute for $A'L_c^{-1} + L_c^{-1}A$ and completion of the squares give

$$\frac{d}{dt}(x'L_c^{-1}x) = \|w\|^2 - \|w - B'L_c^{-1}x\|^2.$$

Integration from $t = -\infty$ to $t = 0$ with $x(-\infty) = 0$ and $x(0) = x_0$ gives

$$x_0'L_c^{-1}x_0 = \|w\|_2^2 - \|w - B'L_c^{-1}x\|_2^2 \leq \|w\|_2^2.$$

If $w(t) = B'e^{-A't}L_c^{-1}x_0 = B'L_c^{-1}e^{(A+BB'L_c^{-1})t}x_0$ on $(-\infty, 0]$, then $w = B'L_c^{-1}x$ and equality is achieved, thus proving (12).

Second, given $x(0) = x_0$ and $w = 0$, the norm of $z(t) = Ce^{At}x_0$ can be found from

$$\|P_+z\|_2^2 = \int_0^\infty x_0'e^{A't}C'Ce^{At}x_0\,dt = x_0'L_ox_0.$$

These two results can be combined as in Section II of [13]

$$\sup_{w\in\mathcal{B}\mathcal{L}_{2-}} \|P_+z\|_2^2 = \sup_{0\neq w\in\mathcal{L}_{2-}} \frac{\|P_+z\|_2^2}{\|w\|_2^2} = \max_{x_0\neq 0}\frac{x_0'L_ox_0}{x_0'L_c^{-1}x_0} = \rho(L_oL_c).$$

■

If $\|G\|_\infty < 1$, then by Lemmas 1 and 4, the Hamiltonian matrix H in (4) is in dom(Ric), $X = \mathrm{Ric}(H) \geq 0$, $A + BB'X$ is stable, and

$$A'X + XA + XBB'X + C'C = 0. \tag{13}$$

The following lemma offers yet another consequence of $\|G\|_\infty < 1$. In fact, this simple time-domain characterization and its proof form the basis for the entire development to follow.

Lemma 12: Suppose $\|G\|_\infty < 1$ and $x(0) = x_0$. Then

$$\sup_{w\in\mathcal{L}_{2+}} (\|z\|_2^2 - \|w\|_2^2) = x_0'Xx_0.$$

Proof: We can differentiate $x(t)'Xx(t)$ as above, use the Riccati equation (13) to substitute for $A'X + XA$, and complete the squares to get

$$\frac{d}{dt}(x'Xx) = -\|z\|^2 + \|w\|^2 - \|w - B'Xx\|^2.$$

If $w \in \mathcal{L}_{2+}$, then $x \in \mathcal{L}_{2+}$, so integrating from $t = 0$ to $t = \infty$ gives

$$\|z\|_2^2 - \|w\|_2^2 = x_0'Xx_0 - \|w - B'Xx\|_2^2 \leq x_0'Xx_0. \tag{14}$$

If we let $w = B'Xx = B'Xe^{(A+BB'X)t}x_0$, then $w \in \mathcal{L}_{2+}$ because $A + BB'X$ is stable. Thus, the inequality in (14) can be made an equality and the proof is complete. Note that the sup is achieved for a w which is a linear function of the state. ■

Now suppose that the input is partitioned so that $B = [B_1\ B_2]$, $G(s) = [G_1(s)\ G_2(s)]$, and w is partitioned conformally. Then $\|G_2\|_\infty < 1$ iff

$$H_W := \begin{bmatrix} A & B_2B_2' \\ -C'C & -A' \end{bmatrix}$$

is in dom(Ric). For $H_W \in$ dom(Ric), define $W = \mathrm{Ric}(H_W)$. Let

$$w \in \mathcal{W} := \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \middle| w_1 \in \mathcal{H}_2^\perp,\ w_2 \in \mathcal{L}_2 \right\}. \tag{15}$$

We are interested in a test for $\sup_{w\in\mathcal{B}\mathcal{W}} \|P_+z\|_2 < 1$, or equivalently

$$\sup_{w\in\mathcal{B}\mathcal{W}} \|\Gamma w\|_2 < 1 \tag{16}$$

where $\Gamma = P_+[M_{G_1}\ M_{G_2}]: \mathcal{W} \to \mathcal{H}_2$ is a mixed Hankel–Toeplitz operator

$$\Gamma \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = P_+[G_1\ G_2]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad w_1 \in \mathcal{H}_2^\perp,\ w_2 \in \mathcal{L}_2.$$

Note that Γ is the sum of the Hankel operator $P_+M_GP_-$ with the Toeplitz operator $P_+M_{G_2}P_+$. The following lemma generalizes Lemma 4 ($B_1 = 0$) and Lemma 11 ($B_2 = 0$).

Lemma 13: Equation (16) holds iff the following two conditions hold.

i) $H_W \in$ dom(Ric).

ii) $\rho(WL_c) < 1$.

Proof: As in Lemma 11, assume (A, B) is controllable; otherwise, restrict attention to the controllable subspace. By Lemma 4, condition i) is necessary for (16), so we will prove that given condition i), (16) holds iff condition ii) holds. We will do this by showing, equivalently, that $\rho(WL_c) \geq 1$ iff $\sup_{w\in\mathcal{B}\mathcal{W}} \|\Gamma w\|_2 \geq 1$. By definition of $\mathcal{W}$, if $w \in \mathcal{W}$, then

$$\|P_+z\|_2^2 - \|w\|_2^2 = \|P_+z\|_2^2 - \|P_+w_2\|_2^2 - \|P_-w\|_2^2.$$

Note that the last term only contributes to $\|P_+z\|_2^2$ through $x(0)$. Thus, if L_c is invertible, then Lemma 12 and (12) yield

$$\sup_{w\in W} \{\|P_+z\|_2^2 - \|w\|_2^2 \mid x(0) = x_0\} = x_0'Wx_0 - x_0'L_c^{-1}x_0 \tag{17}$$

and the supremum is achieved for some $w \in \mathcal{W}$ that can be constructed from the previous lemmas. Since $\rho(WL_c) \geq 1$ iff $\exists x_0 \neq 0$ such that the right-hand side of (17) is ≥ 0, we have, by (17), that $\rho(WL_c) \geq 1$ iff $\exists w \in \mathcal{W}$, $w \neq 0$ such that $\|P_+z\|_2^2 \geq \|w\|_2^2$. But this is true iff $\sup_{w\in\mathcal{B}\mathcal{W}} \|\Gamma w\|_2 \geq 1$. ■

The FI proof of Section VII-C will make use of the adjoint $\Gamma^*: \mathcal{H}_2 \to \mathcal{W}$, which is given by

$$\Gamma^* z = \begin{bmatrix} P_-(G_1^\sim z) \\ G_2^\sim z \end{bmatrix} = \begin{bmatrix} P_-G_1^\sim \\ G_2^\sim \end{bmatrix} z \tag{18}$$

where $P_-Gz := P_-(Gz) = (P_-M_G)z$. That the expression in (18) is actually the adjoint of Γ is easily verified from the definition of the inner product on vector-valued $\mathcal{L}_2$, expressed in the frequency-domain as

$$\langle x_1, x_2 \rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} x_1(j\omega)^* x_2(j\omega)\,d\omega. \tag{19}$$

The adjoint of $\Gamma: \mathcal{W} \to \mathcal{H}_2$ is the operator $\Gamma^*: \mathcal{H}_2 \to \mathcal{W}$ such that $\langle z, \Gamma w \rangle = \langle \Gamma^* z, w \rangle$ for all $w \in \mathcal{W}$, $z \in \mathcal{H}_2$. Directly using the definition in (19), we get

$$\begin{aligned} \langle z, \Gamma w \rangle &= \langle z, P_+(G_1w_1 + G_2w_2) \rangle = \langle z, G_1w_1 \rangle + \langle z, G_2w_2 \rangle \\ &= \langle P_-(G_1^\sim z), w_1 \rangle + \langle G_2^\sim z, w_2 \rangle \\ &= \langle \Gamma^* z, w \rangle. \end{aligned}$$

B. LFT's and Inner Matrices

A transfer function G in $\mathcal{R}\mathcal{H}_\infty$ is called *inner* if $G^\sim G = I$, and hence $G(j\omega)^*G(j\omega) = I$ for all ω. Note that G inner implies that G has at least as many rows as columns. For G inner, and any $q \in \mathbb{C}^m$, $w \in \mathcal{L}_2$, then $\|G(j\omega)q\| = \|q\|$, $\forall\omega$, and $\|Gw\|_2 = \|w\|_2$. Because of these norm preserving properties inner matrices will be central to several of the proofs. In this section we give a characterization of inner functions and some properties of linear fractional transformations. First, we present a state-space characterization of inner transfer functions analogous to Lemma 4 that is well known and simple to verify (see [2], [39], and [13]).

Lemma 14: Suppose

$$G=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with (C, A) detectable and $L_o = L_o'$ satisfies

$$A'L_o + L_oA + C'C = 0.$$

Then the following holds.

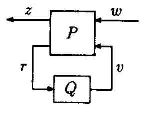
a) $L_o \geq 0$ iff A is stable.

b) $D'C + B'L_o = 0$ implies $G^{\sim}G = D'D$.

c) $L_o \geq 0$, (A, B) is controllable, and $G^{\sim}G = D'D$ implies $D'C + B'L_o = 0$.

The next lemma considers linear fractional transformations with inner matrices and is based on the work of Redheffer [31].

Lemma 15: Consider the following feedback system:



$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{RH}_\infty$$

Suppose that $P^{\sim}P = I$, $P_{21}^{-1} \in \mathcal{RH}_\infty$, and Q is a proper rational matrix. Then the following are equivalent.

a) The system is internally stable and well-posed, and $\|T_{zw}\|_\infty < 1$.

b) $Q \in \mathcal{RH}_\infty$ and $\|Q\|_\infty < 1$.

Proof: b) $\Rightarrow$ a). Internal stability and well-posedness follow from $P, Q \in \mathcal{RH}_\infty$, $\|P_{22}\|_\infty \leq 1$, $\|Q\|_\infty < 1$, and a small gain argument. To show that $\|T_{zw}\|_\infty < 1$ consider the closed-loop system at any frequency $s = j\omega$ with the signals fixed as complex constant vectors. Let $\|Q\|_\infty =: \epsilon < 1$ and note that $T_{wr} = P_{21}^{-1}(I - P_{22}Q) \in \mathcal{RH}_\infty$. Also let $\kappa := \|T_{wr}\|_\infty$. Then $\|w\| \leq \kappa\|r\|$, and P inner implies that $\|z\|^2 + \|r\|^2 = \|w\|^2 + \|v\|^2$. Therefore,

$$\|z\|^2 \leq \|w\|^2 + (\epsilon^2 - 1)\|r\|^2 \leq [1 - (1-\epsilon^2)\kappa^{-2}]\|w\|^2$$

which implies $\|T_{zw}\|_\infty < 1$.

a) $\Rightarrow$ b). To show that $\|Q\|_\infty < 1$ suppose there exists a (real or infinite) frequency ω and a constant nonzero vector r such that at $s = j\omega$, $\|Qr\| \geq \|r\|$. Then setting $w = P_{21}^{-1}(I - P_{22}Q)r$, $v = Qr$ gives $v = T_{vw}w$. But as above, P inner implies that $\|z\|^2 + \|r\|^2 = \|w\|^2 + \|v\|^2$, and hence $\|z\|^2 \geq \|w\|^2$, which is impossible since $\|T_{zw}\|_\infty < 1$. It follows that $\sigma_{\max}(Q(j\omega)) < 1$ for all ω, i.e., $\|Q\|_\infty < 1$, since Q is rational.

Finally, Q has a right-coprime factorization $Q = NM^{-1}$ with $N, M \in \mathcal{RH}_\infty$. We shall show that $M^{-1} \in \mathcal{RH}_\infty$. Since $T_{vw}P_{21}^{-1} = Q(I - P_{22}Q)^{-1}$ it has the right-coprime factorization $T_{vw}P_{21}^{-1} = N(M - P_{22}N)^{-1}$. But since $T_{vw}P_{21}^{-1} \in \mathcal{RH}_\infty$, so does $(M - P_{22}N)^{-1}$. This implies that the winding number of $\det(M - P_{22}N)$, as s traverses the Nyquist contour, equals zero. Furthermore, since $\det(M - \alpha P_{22}N) \neq 0$ for all α in [0, 1] and all $s = j\omega$ (this uses the fact that $\|P_{22}\|_\infty \leq 1$ and $\|Q\|_\infty < 1$), we have that the winding number of $\det M$ equals zero too. Therefore, $Q \in \mathcal{RH}_\infty$ and the proof is complete. ∎

C. LFT's and Stability

In this section, we consider the equivalence of internal and input-output stability for the output feedback problem and the DF problem, and in particular, Lemmas 5 and 7. The proofs in this section are very routine and use standard techniques, principally the PBH test for controllability or observability, so they will only be sketched.

Recall the realization of G from Section III-A and suppose that $A \in \mathbb{R}^{n\times n}$, and that z, y, w, and u have dimension p_1, p_2, m_1, and m_2, respectively. Thus, $C_1 \in \mathbb{R}^{p_1\times n}$, $B_2 \in \mathbb{R}^{n\times m_2}$, and so on. Define

$$n_{12}(\lambda) := \operatorname{rank}\begin{bmatrix} A-\lambda I & B_2 \\ C_1 & D_{12}\end{bmatrix},$$

$$n_{21}(\lambda) := \operatorname{rank}\begin{bmatrix} A-\lambda I & B_1 \\ C_2 & D_{21}\end{bmatrix}.$$

Now suppose we apply a controller K to G to obtain T_{zw}. Recall that K is admissible iff it internally stabilizes G. For the following lemma, we do not need the assumptions from Section III-A on G for the output feedback problem.

Lemma 16: Suppose that $n_{12}(\lambda) = n + m_2$ and $n_{21}(\lambda) = n + p_2$ for all Re $(\lambda) \geq 0$. Then K is admissible iff $T_{zw} \in \mathcal{RH}_\infty$.

Proof: Form the closed-loop state-space matrices and perform a PBH test for controllability and observability. It is easily checked that any unobservable or uncontrollable modes must occur at λ violating the above rank conditions (see Limebeer and Halikias [26] for more details). Hence, the closed-loop system is stabilizable and detectable and the result follows. ∎

It is easy to prove Lemma 5 by verifying that assumptions i), iii), and iv) for the output feedback problem in Section III-A imply that the rank conditions in Lemma 16 hold. Similarly, Lemma 7 is proven by verifying that the assumptions for the OE problem also imply that the above rank conditions hold. Note that for the OE problem, $D_{12} = I$. Further details are left to the reader.

D. Specific State-Space Formulas

Recall from Section V-B that if we define a new control variable in the $\mathcal{H}_2$ problem, $\nu := u - F_2x$, the transfer function to z becomes (7)

$$z=\left[\begin{array}{c|cc} A_{F_2} & B_1 & B_2 \\ \hline C_{1F_2} & 0 & D_{12}\end{array}\right]\begin{bmatrix} w \\ \nu \end{bmatrix} = G_cB_1w + U\nu$$

where

$$G_c(s) := \left[\begin{array}{c|c} A_{F_2} & I \\ \hline C_{1F_2} & 0\end{array}\right]$$

and

$$U(s) := \left[\begin{array}{c|c} A_{F_2} & B_2 \\ \hline C_{1F_2} & D_{12}\end{array}\right].$$

Suppose $D_\perp$ is any matrix making $[D_{12}\ D_\perp]$ an orthogonal matrix, and define

$$U_\perp = \left[\begin{array}{c|c} A_{F_2} & -X_2^\dagger C_1'D_\perp \\ \hline C_{1F_2} & D_\perp\end{array}\right]. \tag{20}$$

The following is easily proven using Lemma 14 by obtaining a state-space realization, and then eliminating uncontrollable states using a little algebra involving the Riccati equation for X_2.

Lemma 17: $[U\ U\perp]$ is square and inner and a realization for $G_c^{\sim}[U\ U_\perp]$ is

$$G_c^{\sim}[U\ U_\perp] = \left[\begin{array}{c|cc} A_{F_2} & -B_2 & X_2^\dagger C_1'D_\perp \\ \hline X_2 & 0 & 0\end{array}\right] \in \mathcal{RH}_2. \tag{21}$$

This implies that U and $U_\perp$ are each inner, and both $U_\perp^{\sim}G_c$ and $U^{\sim}G_c$ are in $\mathcal{RH}_2^\perp$.

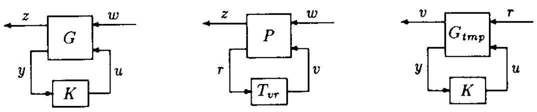
Under the assumption that X_∞ exists, the change of variables for the $\mathcal{H}_\infty$ problem introduced in Section V-E is $r := w - B_1'X_\infty x$, $v := u + B_2'X_\infty x$. If we recall the definitions of G and

G_{tmp} in Section V-E and define P as

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A_{F_\infty} & B_1 & B_2 \\ \hline C_{1F_\infty} & 0 & D_{12} \\ -B_1' X_\infty & I & 0 \end{array}\right]$$

$$\begin{aligned} A_{F_\infty} &:= A + B_2 F_\infty \\ C_{1F_\infty} &:= C_1 + D_{12} F_\infty \end{aligned} \tag{22}$$

we can show with a little algebra the equivalence of the first two of the following block diagrams, with T_{vr} given by the third one.



Equation (10) suggests that P is inner when $X_\infty \geq 0$, which is verified by the following lemma.

Lemma 18: If X_∞ exists and $X_\infty \geq 0$, then P in (22) is in $\mathcal{RH}_\infty$ and inner, and $P_{21}^{-1} \in \mathcal{RH}_\infty$.

Proof: Note that P_{11} equals T_{zw} in (5). Thus by Lemma 6, A_{F_∞} is stable and $P \in \mathcal{RH}_\infty$. That P is inner ($P^\sim P = I$) follows from Lemma 14 upon noting that the observability Gramian of P is X_∞ [see (6)] and

$$\begin{bmatrix} 0 & I \\ D_{12}' & 0 \end{bmatrix} \begin{bmatrix} C_{1F_\infty} \\ -B_1' X_\infty \end{bmatrix} + \begin{bmatrix} B_1' \\ B_2' \end{bmatrix} X_\infty = 0.$$

Finally, the state matrix for P_{21}^{-1} is $(A_{F_\infty} + B_1 B_1' X_\infty)$, which is stable from the proof of Lemma 6. Thus, $P_{21}^{-1} \in \mathcal{RH}_\infty$. ■

We are now in a position to prove Lemma 9, which was the central part of the separation argument in Section V-E.

Proof of Lemma 9: We may assume without loss of generality that the realization of K is stabilizable and detectable. Recall from Lemma 5 that for T_{zw} internal stability is equivalent to $T_{zw} \in \mathcal{RH}_\infty$, and similarly from Lemma 7 for T_{vr}, provided that OE assumptions i) and iv) hold for G_{tmp}. OE assumption iv) for G_{tmp} is the same as for the output feedback problem, and assumption i) follows from Lemma 10 a). Thus, internal stability is equivalent to input–output stability for both G and G_{tmp}. The result then follows immediately from Lemmas 15 and 18 along with the above block diagrams. ■

VII. Proofs for Problem FI: Full Information

We will prove the FI results and the FC results follow by duality.

A. Items FI.1 and FI.2

The minimum of $\|T_{zw}\|_2$ equals $\|G_c B_1\|_2$ and the unique optimal controller is $K(s) = [F_2 \; 0]$.

Proof of FI.1 and FI.2: We have verified the steps in the proof of Theorem 1 in Section V-B through (8) (see Lemma 17). Now T_{vw} can be made identically 0 by setting $u = F_2 x$ so that $v = u - F_2 x = 0$. This uniquely minimizes $\|T_{zw}\|_2$ and makes $\|T_{zw}\|_2 = \|G_c B_1\|_2$ and $K(s) = [F_2 \; 0]$. ■

We shall also prove a slightly stronger result. Let w be a fixed impulse, $w_0 \delta$, and allow u to be an arbitrary function in $\mathcal{L}_{2+}$ instead of restricting it to be generated through y. It turns out that the optimal u is actually obtained by state-feedback.

Proposition 1: For each $w(t) = w_0 \delta(t)$ there exists a unique u in $\mathcal{L}_{2+}$ minimizing $\|z\|_2$, namely, $u = F_2 x$. Moreover,

$$\min \|z\|_2 = \|G_c B_1 w_0\|_2 = w_0' B_1' X_2 B_1 w_0.$$

Proof: Consider (7) and observe that v is a free function in $\mathcal{H}_2$: from any v in $\mathcal{H}_2$ we can recover u via $u = v + F_2 x$, and then $u \in \mathcal{H}_2$. It follows from Lemma 17 that the functions $G_c B_1 w_0$ and Uv are orthogonal for every v in $\mathcal{H}_2$. Hence, with U inner

$$\|z\|_2^2 = \|G_c B_1 w_0\|_2^2 + \|Uv\|_2^2 = \|G_c B_1 w_0\|_2^2 + \|v\|_2^2.$$

This equation gives the desired conclusion immediately: the optimal v is $v = 0$ (i.e., $u = F_2 x$) and the minimum norm of z equals $\|G_c B_1 w_0\|_2$. ■

B. Item FI.3

The set of all admissible controllers such that $\|T_{zw}\|_2 < \gamma$ is described by

$$K(s) = [F_2 \;\; Q(s)], \; Q \in \mathcal{RH}_2, \; \|Q\|_2^2 < \gamma^2 - \|G_c B_1\|_2^2. \tag{23}$$

Proof: Let K be an admissible controller such that $\|T_{zw}\|_2 < \gamma$. Denote by Q the transfer matrix from w to v; it belongs to $\mathcal{RH}_2$ by internal stability. Then $u = F_2 x + v = F_2 x + Qw$ so $K = [F_2 \; Q]$, and from (8), $\|T_{zw}\|_2^2 = \|G_c B_1\|_2^2 + \|Q\|_2^2$, and hence

$$\|Q\|_2^2 = \|T_{zw}\|_2^2 - \|G_c B_1\|_2^2 < \gamma^2 - \|G_c B_1\|_2^2.$$

Likewise, one can show that every controller of the form (23) is admissible and suboptimal. ■

C. Item FI.4 and FI.5: Necessity

If there exists an admissible controller such that $\|T_{zw}\|_\infty < 1$, then

$$H_\infty \in \mathrm{dom(Ric)}, \; \mathrm{Ric}(H_\infty) \geq 0. \tag{24}$$

As in the $\mathcal{H}_2$ case, we will also prove a slightly stronger result. Before beginning the proof, however, we will show that we can, without loss of generality, strengthen the assumption on (C_1, A) from detectable to observable. Suppose there exists a controller

$$\hat{K} = \left[\begin{array}{c|c} \hat{A} & \hat{B}_1 \hat{B}_2 \\ \hline \hat{C} & \hat{D}_1 \hat{D}_2 \end{array}\right]$$

such that $\|T_{zw}\|_\infty < 1$. If (C_1, A) is detectable but not observable, then change coordinates for the state of G to $[{}^{x_1}_{x_2}]$ with x_2 unobservable, (C_{11}, A_{11}) observable and A_{22} stable, giving the following closed-loop state equations:

$$\dot{x}_1 = A_{11} x_1 + B_{11} w + B_{21} u$$

$$z = C_{11} x_1 + D_{21} u$$

$$\dot{x}_2 = A_{21} x_1 + A_{22} x_2 + B_{21} w + B_{22} u$$

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}_{11} x_1 + \hat{B}_{12} x_2 + \hat{B}_2 w$$

$$u = \hat{C}\hat{x} + \hat{D}_{11} x_1 + \hat{D}_{12} x_2 + \hat{D}_2 w.$$

If we group the first two equations as a new plant G_{obs} with state x_1 and group the last three equations as a new controller K_{obs} with state made up of x_2 and $\hat{x}$, then

$$G_{obs}(s) = \left[\begin{array}{c|cc} A_{11} & B_{11} & B_{21} \\ \hline C_{11} & 0 & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$$

still satisfies the assumptions of the FI problem and is stabilized by K_{obs} with $\|T_{zw}\|_\infty < 1$. If we now show that there exists $\hat{X}_\infty > 0$ solving the $\mathcal{H}_\infty$ Riccati equation for G_{obs}, then

$$\mathrm{Ric}(H_\infty) = X_\infty = \begin{bmatrix} \hat{X}_\infty & 0 \\ 0 & 0 \end{bmatrix} \geq 0$$

exists for G. We can therefore assume without loss of generality that (C_1, A) is observable, which by Lemma 3 implies that $X_2 > 0$.

Proposition 2: If $\sup_{w \in \mathcal{BL}_{2+}} \min_{u \in \mathcal{L}_{2+}} \|z\|_2 < 1$, then $H_\infty \in \text{dom(Ric)}$ and $\text{Ric}(H_\infty) > 0$.

Proof: Again define $\nu := u - F_2 x$ to get $z = G_c B_1 w + U\nu$, and note that by Lemma 3, (C_1, A) observable implies that $X_2 > 0$. The hypothesis implies that

$$\sup_{w \in \mathcal{BH}_2} \sup_{\nu \in \mathcal{H}_2} \|z\|_2 < 1. \tag{25}$$

With $\mathcal{W}$ from (15) define the operator $\Gamma: \mathcal{W} \to \mathcal{H}_2$ as

$$\Gamma \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = P_+(B_1' G_c^{\sim}(Uq_1 + U_\perp q_2)) = P_+ B_1' G_c^{\sim} [U \; U_\perp] \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

where $[U \; U_\perp]$ is from (20) and $G_c^{\sim} [U \; U_\perp] \in \mathcal{RH}_2$ has the realization in (21). Note that from (18) the adjoint operator $\Gamma^*: \mathcal{L}_{2+} \to \mathcal{W}$ is

$$\Gamma^* w = \begin{bmatrix} P_-(U^{\sim} G_c B_1 w) \\ U_\perp^{\sim} G_c B_1 w \end{bmatrix} = \begin{bmatrix} P_- U^{\sim} \\ U_\perp^{\sim} \end{bmatrix} G_c B_1 w.$$

Since $[U \; U_\perp]$ is square and inner by Lemma 17, $\|z\|_2 = \|[U \; U_\perp]^{\sim} z\|_2$, and

$$[U \; U_\perp]^{\sim} z = \begin{bmatrix} U^{\sim} G_c B_1 w + \nu \\ U_\perp^{\sim} G_c B_1 w \end{bmatrix}.$$

Together with (25), this implies that $\sup_{w \in \mathcal{BH}_2} \|\Gamma^* w\|_2 < 1$ or equivalently, that

$$\sup_{q \in \mathcal{BW}} \|\Gamma q\|_2 < 1.$$

This is just the condition (16), so from Lemmas 3 and 13 and (21) we have that

$$H_W := \begin{bmatrix} A_{F2} & X_2^{-1} C_1' C_1 X_2^{-1} \\ -X_2 B_1 B_1' X_2 & -A_{F2}' \end{bmatrix} \in \text{dom(Ric)}$$

and $W = \text{Ric}(H_W) \geq 0$. Furthermore, the controllability Gramian for (21) is X_2^{-1} since

$$A_{F2} X_2^{-1} + X_2^{-1} A_{F2}' + B_2 B_2' + X_2^{-1} C_1' C_1 X_2^{-1} = 0.$$

Lemma 13 also implies $\rho(W X_2^{-1}) < 1$, or equivalently $X_2 > W$. Using the Riccati equation for X_2, one can verify that

$$T := \begin{bmatrix} -I & X_2^{-1} \\ -X_2 & 0 \end{bmatrix}$$

provides a similarity transformation between H_∞ and H_W, i.e., $H_\infty = T H_W T^{-1}$, so that $H_\infty \in \text{dom(Ric)}$. Also,

$$\mathcal{X}_-(H_\infty) = T \mathcal{X}_-(H_W) = T \,\text{Im} \begin{bmatrix} I \\ W \end{bmatrix} = \text{Im} \begin{bmatrix} I - X_2^{-1} W \\ X_2 \end{bmatrix}$$

so $X_\infty = X_2 (X_2 - W)^{-1} X_2 > 0$. ■

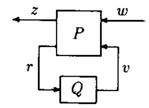
D. Item FI.4 and FI.5: Sufficiency

If $H_\infty \in \text{dom(Ric)}$ and $X_\infty = \text{Ric}(H_\infty) \geq 0$, then the set of all admissible controllers such that $\|T_{zw}\|_\infty < 1$ equals

$$K(s) = [F_\infty - Q(s) B_1' X_\infty \;\; Q(s)], \; Q \in \mathcal{RH}_\infty, \; \|Q\|_\infty < 1.$$

Note that this contains the *only if* part of FI.4.

Proof: We will again change variables to $v := u - F_\infty x$ and $r := w - B_1' X_\infty x$ with the corresponding controller $K_{tmp}(s) := K(s) - [F_\infty \; 0]$. The internal stability of the feedback systems with K and J_{tmp} are equivalent. Now suppose $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < 1$, and $K_{tmp}(s) = Q(s) [-B_1' X_\infty \; I]$ so $v = Qr$. This gives the feedback connection



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A_{F\infty} & B_1 & B_2 \\ \hline C_{1F\infty} & 0 & D_{12} \\ -B_1' X_\infty & I & 0 \end{array}\right]$$

By Lemma 18, P is inner and $P_{21}^{-1} \in \mathcal{RH}_\infty$. Hence, by Lemma 15 $T_{zw} \in \mathcal{RH}_\infty$ with $\|T_{zw}\|_\infty < 1$ iff $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < 1$. Hence, the stated class of controllers has the desired properties.

To verify that this class incorporates all admissible controllers such that $\|T_{zw}\|_\infty < 1$, let K be any such controller. Then $T_{vw} \in \mathcal{RH}_\infty$ and $T_{zw} = P_{11} + P_{12} T_{vw}$. Now define $Q = (I + T_{vw} P_{21}^{-1} P_{22})^{-1} T_{vw} P_{21}^{-1}$ so that $Q(I - P_{22} Q)^{-1} P_{21} = T_{vw}$ and $T_{zw} = P_{11} + P_{12} Q (I - P_{22} Q)^{-1} P_{21}$. Since P_{22} is strictly proper all the above are well-posed and Q is real-rational and proper. Hence, Lemma 15 implies that $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < 1$. ■

VIII. Proofs for Problem DF: Disturbance Feedforward

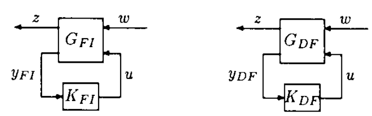
Given DF, the OE problem results follow by duality. We will show how to produce the DF results directly from the corresponding FI results. Specifically, we will prove two propositions that show that the two problems are equivalent. It is then a routine exercise to apply these propositions to obtain the DF results, and the details are omitted.

Denote the G for the FI problem in Section IV-A as G_{FI} and the G for the DF problem in Section IV-C as G_{DF}, so

$$G_{FI}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$$

$$G_{DF}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & I & 0 \end{array}\right].$$

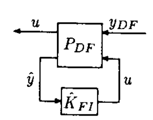
Recall that both have assumptions i)–iii) from Section IV-A but that the G_{DF} has the additional assumption that $A - B_1 C_2$ is stable. Suppose that we have controllers K_{FI} and K_{DF} and let T_{FI} and T_{DF} denote the closed-loop T_{zw} in



We will assume throughout that any controller realizations are stabilizable and detectable. This is easily verified for all the DF controllers. The following proposition is obvious.

Proposition 3: The controller K_{DF} internally stabilizes G_{DF} iff $K_{FI} = K_{DF}[C_2 \; I]$ internally stabilizes G_{FI}. Furthermore, in this case $T_{FI} = T_{DF}$.

To complete the equivalence, suppose that we have a controller for the FI problem, denoted by $\hat{K}_{FI}$ and let $\hat{K}_{DF}$ be the transfer function generated by



$$P_{DF} = \left[\begin{array}{c|cc} A - B_1 C_2 & B_1 & B_2 \\ \hline 0 & 0 & I \\ \begin{bmatrix} I \\ -C_2 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$$

Proposition 4: The controller $\hat{K}_{FI}$ internally stabilizes G_{FI} iff $\hat{K}_{DF}$ internally stabilizes G_{DF}. Furthermore, in this case $T_{FI} = T_{DF}$.

Proof: Apply $\hat{K}_{DF}$ to G_{DF} and let x and $\hat{x}$ denote the state of G_{DF} and P_{DF}, respectively. Then the overall equations in terms of $e := x - \hat{x}$ and $\hat{x}$ are

$$\dot{e} = (A - B_1 C_2)e$$

$$\dot{\hat{x}} = A\hat{x} + B_1 w + B_2 u + B_1 C_2 e \qquad u = K_{FI}\hat{y} = K_{FI}\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}$$

$$\hat{w} = y - C_2\hat{x} = w + C_2 e.$$

The proposition is then easily seen by comparing these equations to the corresponding equations when $\hat{K}_{FI}$ is applied to G_{FI}. ■

IX. Additional Interpretations

This section considers some additional connections with the 1984 approach and with the work of Whittle, and will be of interest primarily to readers already familiar with them. Section IX-A presents a 2×2-block generalization of Lemma 13, and gives some indication of how it could be used in the 1984 procedure to provide alternative proofs of Theorems 3 and 4. For further discussion of 2×2-block mixed Hankel–Toeplitz operators, see [12] and the references therein.

Section IX-B gives another separation interpretation of the central $\mathcal{H}_\infty$ controller of Theorem 3 in the spirit of Whittle [35]. It has been shown in [14] that the central controller corresponds exactly to the steady-state version of the optimal risk sensitive controller, derived by [35], who also derives a separation result and a certainty equivalence principle (see also [36]).

A. Mixed Hankel–Toeplitz Operators: The 2 × 2-Block Case

Given the historical role that mixed Hankel–Toeplitz operators have played in $\mathcal{H}_\infty$ theory, especially within the context of the 1984 approach, it is interesting to consider the 2×2-block generalization of Lemma 13. The proof of Lemma 19 below is completely straightforward and fairly short, given the other results in Section VI-A, and is omitted. Suppose that

$$G(s) := \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{array}\right]$$

$$H_X := \begin{bmatrix} A & B_2 B_2' \\ -C'C & -A' \end{bmatrix} \quad H_Y := \begin{bmatrix} A' & C_2' C_2 \\ -BB' & -A \end{bmatrix}.$$

Define $\mathcal{W} = \mathcal{H}_2^\perp \oplus \mathcal{L}_2$, $\mathcal{Z} = \mathcal{H}_2 \oplus \mathcal{L}_2$, and $\Gamma: \mathcal{W} \to \mathcal{Z}$ as

$$\Gamma\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} P_+ & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Lemma 19: $\sup_{w \in \mathcal{BW}} \|\Gamma w\|_2 < 1$ holds iff the following three conditions hold.

i) $H_X \in \mathrm{dom(Ric)}$.
ii) $H_Y \in \mathrm{dom(Ric)}$.
iii) $\rho(XY) < 1$ for $X = \mathrm{Ric}(H_X)$ and $Y = \mathrm{Ric}(H_Y)$.

To see how this lemma might be used in the 1984 procedure to prove Theorems 3 and 4, suppose we begin with G as in Section III with state dimension n. If we then use M_2 from Theorem 2 we can obtain a parametrization of all admissible controllers in terms of $Q \in \mathcal{RH}_\infty$ and the affine parametrization $T_{zw}(Q) = T_o + UQV$. It is easily shown that T_o is the $\mathcal{H}_2$ optimal closed-loop system from Theorem 1, U (from Sections V-E and VI-D) is inner, and V (the dual of U) is coinner (V' is inner).

Finding $Q \in \mathcal{RH}_\infty$ such that $\|T_{zw}(Q)\|_\infty < \gamma$ is called the $\mathcal{H}_\infty$ model-matching problem and is clearly a special case of the problem considered in this paper. The next step in the 1984 approach is to form $[U \; U\perp]$ from Section VI-D and the dual for V and transform to the 2×2 block general distance problem.

$$[U^\sim \; U_\perp^\sim](T_o + UQV)\begin{bmatrix} V^\sim \\ V_\perp^\sim \end{bmatrix}$$

$$= [U^\sim \; U_\perp^\sim]T_o\begin{bmatrix} V^\sim \\ V_\perp^\sim \end{bmatrix} + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

$$= R + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix}. \tag{26}$$

Note that $\|T_{zw}(Q)\|_\infty$ is the same as the $\mathcal{H}_\infty$ norm of the quantities in (26). It can be shown with a little algebra that R in (26) is antistable and has state dimension $2n$. We may now use Lemma 19 and some additional arguments to construct a $Q \in \mathcal{RH}_\infty$ from X and Y such that $\|T_{zw}(Q)\|_\infty < \gamma$. In fact, it turns out that X in Lemma 19 for R is exactly W in the FI proof in Section VII-C.

The final step is to obtain the controller from M_2 and Q. Since M_2 has state dimension n and Q has $2n$, the apparent state dimension of K is $3n$, but some tedious state space manipulations produce cancellations resulting in the n-dimensional controller formulas in Theorems 3 and 4. This approach is exactly the 1984 procedure with Lemma 19 used to solve the general distance problem. Although this approach is conceptually straightforward, and was in fact used to obtain the first proof of the results in this paper, it seems unnecessarily cumbersome and indirect. The simplicity of the resulting formulas suggested the more elegant separation argument that is used in this paper.

B. Relations with Separation in Risk Sensitive Control

Although [35] treats a finite horizon, discrete-time, stochastic control problem, his separation result has a clear interpretation for the present infinite horizon, continuous-time, deterministic control problem, as given below; and it is an interesting exercise to compare the two separation statements. This discussion will be entirely in the time-domain.

We will consider the system at time, $t = 0$, and evaluate the past stress $\mathcal{S}_-$, and future stress $\mathcal{S}_+$, as functions of the current state x. Firstly define the future stress as

$$\mathcal{S}_+(x) := \sup_w \inf_u (\|P_+ z\|^2 - \gamma^{-2}\|P_+ w\|^2)$$

then by the completion of the squares and saddle point argument of Section V-E, where u is not constrained to be a function of the measurements (FI case), we obtain

$$\mathcal{S}_+(x) = x' X_\infty x.$$

The past stress $\mathcal{S}_-(x)$ is a function of the past inputs and observations $u(t)$, $y(t)$ for $-\infty < t < 0$, and the present state x and is produced by the worst case disturbance w that is consistent with the given data

$$\mathcal{S}_-(x) := \sup (\|P_- z\|^2 - \gamma^2\|P_- w\|^2).$$

In order to evaluate $\mathcal{S}_-$ we see that w can be divided into two components, $D_{21}w$ and $D_{21}^\perp w$, with x only dependent on $D_{21}^\perp w$ (since $B_1 D_{21}^{\perp\prime} = 0$) and $D_{21}w = y - C_2 x$. The past stress is then calculated by a completion of the square and in terms of a filter output. In particular, let $\bar{x}$ be given by the stable differential equation

$$\dot{\bar{x}} = A\bar{x} + B_2 u + L_\infty(C_2\bar{x} - y) + Y_\infty C_1' C_1 \bar{x} \qquad \text{with } \bar{x}(-\infty) = 0.$$

Then it can be shown that the worst case w is given by

$$D_{21}^\perp w = D_{21}^\perp B_1' Y_\infty^{-1}(x(t) - \bar{x}(t)) \qquad \text{for } t < 0$$

and that this gives, with $e := x - \bar{x}$,

$$\mathcal{S}_-(x) = -\gamma^2 e(0)' Y_\infty^{-1} e(0) - \gamma^2\|P_-(y - C_2\bar{x})\|^2 + \|P_-(C_1\bar{x})\|^2 + \|P_- u\|^2.$$

The worst case disturbance will now reach the value of x to maximize the total stress, $\mathcal{S}_-(x) + \mathcal{S}_+(x)$, and this is easily shown to be achieved at the current state

$$\hat{x} = Z_\infty \bar{x}(0).$$

The definitions of X_∞ and Y_∞ can be used to show that the state equations for the central controller can be rewritten with state $\bar{x} := Z_\infty \hat{x}$, and with $\bar{x}$ as defined above. The control signal is then

$$u = F_\infty \hat{x} = F_\infty Z_\infty \bar{x}.$$

The *separation* is between the evaluation of future stress, which is a control problem with an unconstrained input, and the past stress, which is a filtering problem with known control input. The central controller then combines these evaluations to give a worst case estimate $\hat{x}$ and the control law acts as if this were the perfectly observed state.

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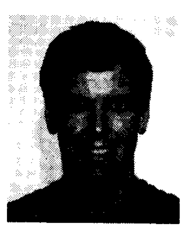

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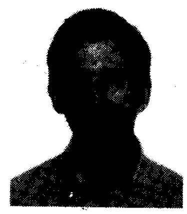
Keith Glover (S'71–M'73), for a photograph and biography, see this issue, p. 830.

Pramod P. Khargonekar (S'81–M'81), for a photograph and biography, see p. 618 of the June 1989 issue of this TRANSACTIONS.



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