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# SAMPLE PATH PROPERTIES OF THE STOCHASTIC FLOWS

BY DMITRY DOLGOPYAT,[1] VADIM KALOSHIN[2] AND LEONID KORALOV[3]

*University of Maryland, California Institute of Technology and Princeton University*

We consider a stochastic flow driven by a finite-dimensional Brownian motion. We show that almost every realization of such a flow exhibits strong statistical properties such as the exponential convergence of an initial measure to the equilibrium state and the central limit theorem. The proof uses new estimates of the mixing rates of the multi-point motion.

**1. Introduction.** The subject of this paper is the study of the long-time behavior of a passive substance (or scalar) carried by a stochastic flow. Motivation comes from applied problems in statistical turbulence and oceanography, Monin and Yaglom (1971), Yaglom (1987), Davis (1991), Isichenko (1992) and Carmona and Cerou (1999). The questions we discuss here are also related to the physical basis of the Kolmogorov model for turbulence, Molchanov (1996).

The physical mechanism of turbulence is still not completely understood. It was suggested in Ruelle and Takens (1971) that the appearance of turbulence could be similar to the appearance of chaotic behavior in finite-dimensional deterministic systems. Compared to other cases, the mechanism responsible for stochasticity in deterministic dynamical systems with nonzero Lyapunov exponents is relatively well understood. It is caused by a sensitive dependence on initial conditions, that is, by exponential divergence of nearby trajectories. It is believed that a similar mechanism can be found in many other situations, but mathematical results are scarce. Here we describe a setting where analysis similar to the deterministic dynamical systems with nonzero Lyapunov exponents can be used. In the paper we shall consider a flow of diffeomorphisms on a compact manifold, generated by solutions of stochastic differential equations driven by a finite-dimensional Brownian motion.

We show that *the presence of nonzero exponents combined with certain nondegeneracy conditions* (amounting roughly speaking to the assumption that the noise can move the orbit in any direction) *implies almost surely chaotic behavior* in the following sense:

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- Exponential, in time, decay of correlations between the trajectories with different initial data.
- Equidistribution of images of submanifolds.
- Central limit theorem, with respect to the measure on a "rich enough" subset, which holds for almost every fixed realization of the underlying Brownian motion.

In order to illustrate the last point, let us consider a periodic flow on $\mathbb{R}^n$, and let $\nu$ be a Lebesgue probability measure concentrated on an open subset. As a motivating example one may think of an oil spot on the surface of the ocean. The ultimate goal could be to remove the oil or at least to prevent it from touching certain places on the surface of ocean. Thus, we wish to predict the properties and the probability laws governing the dynamics of the spot in time. Let $\nu_t$ be the measure on $\mathbb{R}^n$ induced from $\nu$ by time $t$ map of the flow. We shall show that almost surely $\nu_t$ is asymptotically equivalent to a Gaussian measure with variance of order $t$. In other words, for a sufficiently large positive $R$ for large time, 99% of the oil spot is contained in the ball of radius $R\sqrt{t}$.

Even though we consider the random flows generated by SDEs, very little in our approach relies on the precise form of the noise, and in a future work we shall try to generalize our results to other random flows. Thus our work could be considered as a first step in extending deterministic dynamical system picture to a more general setup.

As a next step one may attempt to obtain the same results for the so-called isotropic Brownian flows introduced by Itô (1956) and Yaglom (1957). This is a class of flows for which the image of any simple point is a Brownian motion and the dependence (the covariance tensor) between two different points is a function of distance between these points. Related problems for this case have been studied by Harris (1981), Baxendale (1986), Le Jan (1985), Carmona and Cerou (1999), Cranston, Scheutzow and Steinsaltz (1999, 2000), Cranston and Scheutzow (2002), Lisei and Scheutzow (2001) and Scheutzow and Steinsaltz (2002). The authors have succesifully applied the results of this paper to the study of asymptotic properties of passive scalar trasport problems in Dolgopyat, Kaloshin and Karalov (2002, 2004).

The precise statements of our results are given in the next section. The proofs are carried out in Sections 3–5. Section 6 deals with dissipative flows.

REMARK 1. One of the key tools in our work is the exponential mixing estimate for the two-point motion (19) and almost sure equidistribution results of Sections 4 and 6 which follow from (19). After having submitted our paper we have learned that these mixing and equidistribution results were obtained earlier by Baxendale (1996) by slightly different methods. We are grateful to Professor Baxendale for telling us about his ideas from Baxendale (1996), providing references to the work of Meyn and Tweedie (1993a) and the book of Arnold

(1998). This way we were able to dramatically simplify our proofs, especially in Section 3. In addition, Professor Baxendale proposed the simple proof of Lemma 5 and suggested how to simplify proofs in Section 5.2 by introducing $\tilde{A}_t(x) = A_t(x) - f(x_t)$.

## 2. Central limit theorems and an application to periodic flows.

2.1. *Measure-preserving nondegenerate stochastic flows of diffeomorphisms.* Let $M$ be a $C^\infty$ smooth, connected, compact Riemannian manifold with a smooth Riemannian metric $d$. Consider on $M$ a stochastic flow of diffeomorphisms

$$dx_t = \sum_{k=1}^{d} X_k(x_t) \circ d\theta_k(t) + X_0(x_t)\,dt, \tag{1}$$

where $X_0, X_1, \dots, X_d$ are $C^\infty$-vector fields on $M$ and $\vec{\theta}(t) = (\theta_1(t), \dots, \theta_d(t))$ is a standard $\mathbb{R}^d$-valued Brownian motion. Since the differentials are in the sense of Stratonovich, the associated generator for the process is given by

$$\mathcal{L} = \frac{1}{2}\sum_{k=1}^{d} X_k^2 + X_0. \tag{2}$$

Let us impose additional assumptions on the vector fields $X_0, X_1, \dots, X_d$. All together we impose five assumptions, (A)–(E). All these assumptions, except (A) (measure preservation), are *nondegeneracy assumptions* and are satisfied for a generic set of vector fields $X_0, X_1, \dots, X_d$. Now we formulate them precisely.

(A) (*Measure preservation.*) The stochastic flow $\{x_t : t \geq 0\}$, defined by (1), almost surely w.r.t. the Wiener measure $\mathcal{W}$ of the Brownian motion $\vec{\theta}$ preserves a smooth measure $\mu$ on $M$ (this is equivalent to saying that each $X_j$ preserves $\mu$).

(B) (*Hypoellipticity for $x_t$.*) For all $x \in M$, we have

$$\mathrm{Lie}(X_1, \dots, X_d)(x) = T_x M, \tag{3}$$

that is, the linear space spanned by all possible Lie brackets made out of $X_1, \dots, X_d$ coincides with the whole tangent space $T_x M$ at $x$.

Denote by

$$\Delta = \{(x^1, x^2) \in M \times M : x^1 = x^2\} \tag{4}$$

the diagonal in the product $M \times M$.

(C) (*Hypoellipticity for the two-point motion.*) The generator of the two-point motion $\{(x_t^1, x_t^2) : t > 0\}$ is nondegenerate away from the diagonal $\Delta(M)$, that is, the Lie brackets made out of $((X_1(x^1), X_1(x^2)), \dots, (X_d(x^1), X_d(x^2))$ generate $T_{x^1}M \times T_{x^2}M$.

To formulate the next assumption we need additional notations. For $(t, x) \in [0, \infty) \times M$, let $Dx_t : T_{x_0}M \to T_{x_t}M$ be the linearization of $x_t$ at time $t$. We need an analog of hypoellipticity condition (B) for the process $\{(x_t, Dx_t): t > 0\}$. Denote by $TX_k$ the derivative of the vector field $X_k$, thought as the map on $TM$, and by $SM = \{v \in TM: |v| = 1\}$, the unit tangent bundle on $M$. If we denote by $\tilde{X}_k(v)$ the projection of $TX_k(v)$ onto $T_v SM$, then the stochastic flow (1) on $M$ induces by a stochastic flow on the unit tangent bundle $SM$ is defined by the following equation:

$$d\tilde{x}_t(v) = \sum_{k=1}^{d} \tilde{X}_k(\tilde{x}_t(v)) \circ d\theta_k(t) + \tilde{X}_0(\tilde{x}_t(v))\,dt \qquad \text{with } \tilde{x}_0(v) = v, \tag{5}$$

where $v \in SM$. With these notations we have condition

(D) (*Hypoellipticity for* $(x_t, Dx_t)$.) For all $v \in SM$, we have

$$\mathrm{Lie}(\tilde{X}_1, \dots, \tilde{X}_d)(v) = T_v SM.$$

The leading Lyapunov exponent $\lambda_1$ is defined by the following formula:

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log \|Dx_t(x)\|. \tag{6}$$

See Baxendale (1986), Carverhill (1985a, b) and Elworthy and Stroock (1986) for more information on Lyapunov exponents of stochastic flows. Our last assumption is:

(E) (*Positive Lyapunov exponent.*) $\lambda_1 > 0$.

For measure-preserving stochastic flows with conditions (D), Lyapunov exponents $\lambda_1, \dots, \lambda_{\dim M}$ do exist by *multiplicative ergodic theorem for stochastic flows* of diffeomorphisms [see Carverhill (1985b), Theorem 2.1]. Let us note that condition (E) follows from conditions (A)–(D) [we formulate it as a separate condition because most of our results are valid for dissipative flows satisfying (B)–(D), see Section 6]. Indeed, under condition (A) the sum of the Lyapunov exponents is equal to zero. On the other hand, Theorem 6.8 of Baxendale (1989) states that under condition (B) all of the Lyapunov exponents can be equal to zero only if, for almost every realization of the flow (1), one of the following two conditions is satisfied:

(a) there is a Riemannian metric $\rho'$ on $M$, invariant with respect to the flow (1) or

(b) there is a direction field $v(x)$ on $M$ invariant with respect to the flow (1).

However, (a) contradicts condition (C). Indeed, (a) implies that all the Lie brackets of $\{(X_k(x^1),\ X_k(x^2))\}_{k=1}^d$ are tangent to the leaves of the foliation

$$\{(x^1, x^2) \in M \times M : \rho'(x^1, x^2) = \mathrm{Const}\}$$

and do not form the whole tangent space. On the other hand, (b) contradicts condition (D), since (b) implies that all the Lie brackets are tangent to the graph of $v$.

This positivity of $\lambda_1$ is crucial for our approach.

2.2. *CLTs for nondegenerate stochastic flows of diffeomorphisms.* The first CLT for such flows is CLT for additive functionals of the two-point motion.

Denote by $\{A_t^{(2)}: t > 0\}$ an additive functional of the two-point motion $\{(x_t^1, x_t^2): t > 0\}$. Suppose $A_t^{(2)}$ is governed by the Stratonovich stochastic differential equation

$$dA_t^{(2)}(x_0^1, x_0^2) = \sum_{k=1}^{d} \alpha_k(x_t^1, x_t^2) \circ d\theta_k(t) + a(x_t^1, x_t^2)\, dt, \tag{7}$$

where $\{\alpha_k\}_{k=1}^d$ and $a$ are $C^\infty$-smooth functions, such that

$$\iint_{M\times M} \left[ a(x^1, x^2) + \tfrac{1}{2} \sum_k \bigl(L_{(X_k, X_k)}\alpha_k\bigr)(x^1, x^2) \right] dm_2(x^1, x^2) = 0, \tag{8}$$

where $m_2$ is the invariant measure of the two-point motion on $M \times M \setminus \Delta$ [which exists and is unique by Baxendale and Stroock (1988) under conditions (B)–(E)] and where $(L_{(X_k, X_k)}\alpha_k)(x^1, x^2)$ is the derivative of $\alpha_k$ along the vector field $(X_k, X_k)$ on $M \times M$ at the point $(x^1, x^2)$. This equality can be attained by subtracting a constant from $a(x^1, x^2)$.

THEOREM 1. *Let* $\{A_t^{(2)}: t > 0\}$ *be the additive functional of the two-point motion, defined by* (7), *and let the two-point motion* $\{(x_t^1, x_t^2): t > 0\}$ *for* $x_0^1 \neq x_0^2$ *be defined by the stochastic flow of diffeomorphisms* (1), *which in turn satisfies conditions* (B)–(E). *Then as* $t \to \infty$, *we have that* $A_t^{(2)}/\sqrt{t}$ *converges weakly to a normal random variable.*

Fix a positive integer $n > 2$. The second CLT for stochastic flows, defined by (1) and satisfying assumptions (B)–(E), is the CLT for additive functionals of the $n$-point motion. In other words, CLT for the two-point motion (Theorem 1) has a natural generalization to a CLT for the $n$-point motion. Denote by

$$\Delta^{(n)}(M) = \{(x^1, \dots, x^n) \in M \times \cdots \times M : \exists\, j \neq i \text{ such that } x^j = x^i\}$$

the generalized diagonal in the product $M \times \cdots \times M$ ($n$ times). Replace (C) by the following condition.

($C_n$) The generator of the $n$-point motion $\{(x_t^1, \dots, x_t^n): t > 0\}$ is nondegenerate away from the generalized diagonal $\Delta^{(n)}(M)$, meaning that Lie brackets made out of $(X_1(x^1), \dots, X_1(x^n)), \dots, (X_d(x^1), \dots, X_d(x^n))$ generate $T_{x^1}M \times \cdots \times T_{x^n}M$.

Similarly to the case of the two-point motion, denote by $\{A_t^{(n)}: t > 0\}$ an additive functional of the $n$-point motion $\{(x_t^1, \dots, x_t^n): t > 0\}$. Suppose $A_t^{(n)}$ is governed by the Stratonovich stochastic differential equation

$$dA_t^{(n)}(x_0^1, \dots, x_0^n) = \sum_{k=1}^{d} \alpha_k(x_t^1, \dots, x_t^n) \circ d\theta_k(t) + a(x_t^1, \dots, x_t^n)\, dt, \tag{9}$$

where $\{\alpha_k\}_{k=1}^d$ and $a$ are $C^\infty$-smooth functions, which satisfy

$$\int \cdots \int_{M\times\cdots\times M} \Bigg[ a(x^1,\dots,x^n) + \frac{1}{2}\sum_{k=1}^d \big(L_{(X_k,\dots,X_k)}\alpha_k\big)(x^1,\dots,x^n) \Bigg] dm_n(x^1,\dots,x^n), \tag{10}$$

where $m_n$ is the invariant measure for the $n$-point motion and where we denote by $(L_{(X_k,\dots,X_k)}\alpha_k)(x^1,\dots,x^n)$ the derivative of $\alpha_k$ along the vector field $(X_k,\dots,X_k)$ on $M\times\cdots\times M$ ($n$ times) at the point $(x^1,\dots,x^n)$. As in the two-point case, this equality can be attained by subtracting a constant from $a(x^1,\dots,x^n)$. For the $n$-point motion we have the following:

THEOREM 2. *Let $\{A_t^{(n)}: t>0\}$ be the additive functional of the $n$-point motion, defined by* (9), *and let the $n$-point motion $\{(x_t^1,\dots,x_t^n): t>0\}$ for pairwise distinct $x_0^i \neq x_0^j$ be defined by the stochastic flow of diffeomorphisms* (1), *which in turn satisfies conditions* (B), ($C_n$), (D), *and* (E). *Then as $t\to\infty$ we have that $A_t^{(n)}/\sqrt{t}$ converges weakly to a normal random variable.*

The third CLT for stochastic flows, defined by (1) and satisfying assumptions (A)–(E), is the main result of the paper. This CLT holds for a fixed realization of the underlying Brownian motion and is with respect to the randomness in the initial set, which is assumed to be a set of positive Hausdorff dimension in $M$.

Consider stochastic flow (1) and an additive functional $\{A_t: t>0\}$ of the one-point motion satisfying

$$dA_t(x_0) = \sum_{k=1}^d \alpha_k(x_t)\circ d\theta_k(t) + a(x_t)\,dt \tag{11}$$

with $C^\infty$-smooth coefficients. Define

$$\hat{a}(x) = \frac{1}{2}\sum_{k=1}^d \big(L_{X_k}\alpha_k\big)(x) + a(x), \tag{12}$$

where $(L_{X_k}\alpha_k)(x)$ is the derivative of $\alpha_k$ along the vector field $X_k$ at point $x$. Impose additional assumptions on the coefficients of (11).

(F) (*No drift or preservation of the center of mass.*)

$$\int_M \hat{a}(x)\,d\mu(x) = 0, \qquad \int_M \alpha_k(x)\,d\mu(x) = 0 \qquad \text{for } k=1,\dots,d. \tag{13}$$

This condition can be attained by subtracting appropriate constants from functions $\alpha_1,\dots,\alpha_d$, and $a$. For $A_t$ as above, when $t\to\infty$, we have that $A_t/\sqrt{t}$ converges to a normal random variable with zero mean and some variance $D(A)$. Our next result below shows how little randomness in initial condition is needed for the CLT to hold.

Let $\nu$ be a probability measure on $M$, such that for some positive $p$ it has a finite $p$-energy

$$I_p(\nu) = \iint \frac{d\nu(x)\,d\nu(y)}{d^p(x,y)} < \infty. \tag{14}$$

In particular, this means that the Hausdorff dimension of the support of $\nu$ on $M$ is positive [see Mattila (1995), Section 8]. Let $\mathcal{M}_t^\theta$ be the measure on $\mathbb{R}$ defined on Borel sets $\Omega \subset \mathbb{R}$ by

$$\mathcal{M}_t^\theta(\Omega) = \nu\left\{x \in M : \frac{A_t^\theta(x)}{\sqrt{t}} \in \Omega\right\}. \tag{15}$$

THEOREM 3. *Let* $\{x_t : t > 0\}$ *be a stochastic flow of diffeomorphisms* (1), *and let conditions* (A)–(F) *be satisfied. Then for almost every realization of the Brownian motion* $\{\theta(t)\}$ *as* $t \to \infty$ *the measure* $\mathcal{M}_t^\theta$ *converges weakly to the Gaussian measure with zero mean and some variance* $D(A)$.

2.3. *Application to periodic flows.* Consider the stochastic flow (1) on $\mathbb{R}^N$, with the periodic vector fields $X_k$. Application of Theorems 1–3 to the corresponding flow on the $N$-dimensional torus leads to the clear statements on the behavior of the flow on $\mathbb{R}^N$. We formulate those as Theorems $1'$–$3'$ below.

The usual CLT describes the distribution of the displacement of a single particle with respect to the measure of the underlying Brownian motion $\{\theta(t)\}$. The CLT formulated below (Theorem $3'$), on the contrary, holds for almost every realization of the Brownian motion and is with respect to the randomness in the initial condition.

THEOREM $1'$. *Let* $\{X_k\}_{k=0}^d$ *be* $C^\infty$ *periodic vector fields in* $\mathbb{R}^N$ *with a common period, and let conditions* (B)–(E) *be satisfied. Let* $x_t^1$ *and* $x_t^2$ *be the solutions of* (1) *with different initial data. Then for some value of the drift* $v$, *the vector* $\frac{1}{\sqrt{t}}(x_t^1 - vt, x_t^2 - vt)$ *converges as* $t \to \infty$ *to a Gaussian random vector with zero mean.*

THEOREM $2'$. *Let* $\{X_k\}_{k=0}^d$ *be* $C^\infty$ *periodic vector fields in* $\mathbb{R}^N$ *with a common period, and let the conditions* (B), ($C_n$), (D), *and* (E) *be satisfied. Let* $x_t^1, \dots, x_t^n$ *be the solutions of* (1) *with pairwise different initial data. Then for some value of the drift* $v$, *the vector* $\frac{1}{\sqrt{t}}(x_t^1 - vt, \dots, x_t^n - vt)$ *converges as* $t \to \infty$ *to a Gaussian random vector with zero mean.*

THEOREM 3′. *Let $\{X_k\}_{k=0}^d$ be $C^\infty$ periodic vector fields in $\mathbb{R}^N$ with a common period, let $\nu$ be a probability measure with finite $p$-energy for some $p > 0$ and with compact support, and let conditions* (A)–(F) *be satisfied. For the condition* (F) *we take $\alpha_k = X_k$ and $a = X_0$. Let $x_t$ be the solution of* (1) *with the initial measure $\nu$. Then for almost every realization of the Brownian motion $\{\theta(t)\}$ the distribution of $x_t/\sqrt{t}$ induced by $\nu$ converges weakly as $t \to \infty$ to a Gaussian random variable on $\mathbb{R}^N$ with zero mean and some variance $D$.*

PROOF OF THEOREMS 1′–3′. The functions $\alpha_k$ and $a$ in the formulas (7), (9) and (11) could be considered to be vector-valued, thus defining the vector-valued additive functionals. Any linear combination of the components of a vector-valued additive functional is a scalar additive functional, for which Theorems 1–3 hold. If any linear combination of the components of a vector is a Gaussian random variable, then the vector itself is Gaussian. Therefore, Theorems 1–3 hold for vector-valued additive functionals as well.

It remains to rewrite equation (1) in the integral form and apply Theorems 1, 2 or 3 to the vector-valued additive functional of the flow on the torus. □

The proofs of the CLT's occupy Sections 3–5. In the next section we prove CLT for additive functionals of the two-point and $n$-point motion. The proof relies on results of Baxendale–Stroock [Baxendale and Stroock (1988)] and the CLT for $V$-ergodic Markov processes of Meyn–Tweedie [Meyn and Tweedie (1993b)]. In Section 4 we prove that a set of positive Hausdorff dimension on $M$ becomes uniformly distributed by the flow (1) in the limit as time tends to infinity. Section 5 is devoted to the proof of CLT for measures (Theorem 3). In Section 6 we extend the above CLT to the dissipative case.

**3. Proof of the CLT for two-point and multi-point functionals.** We first state several lemmas, which will lead to the CLT for the two-point motion. In order to describe the two-point motion close to the diagonal $\Delta \subset M \times M$ we shall use the following result of Baxendale–Stroock [Baxendale and Stroock (1988)] showing, in particular, that a positive Lyapunov exponent for (1) implies that the exit time from the $r$ neighborhood of the diagonal has exponential moments.

LEMMA 1 [Baxendale and Stroock (1988), Theorem 3.19]. *There are positive constants $\alpha_0$ and $r_0$, which depend on the vector fields $X_0, \dots, X_d$, such that for any $0 < \alpha < \alpha_0$ and $0 < r < r_0$, there are $p(\alpha)$, $K(\alpha)$ with the property $p(\alpha) \to 0$ as $\alpha \to 0$, such that for any pair of distinct points $x^1$ and $x^2$ which are at most $r$ apart we have*

$$K^{-1} d^{-p}(x^1, x^2) \le E_z(e^{\alpha\tau}) \le K\, d^{-p}(x^1, x^2), \tag{16}$$

*where $\tau$ is the stopping time of $x_s^1$ and $x_s^2$ getting distance $r$ apart, that is, $\tau = \inf\{s > 0 : d(x_s^1, x_s^2) = r\}$ and $z = (x_0^1, x_0^2)$.*

The next result of Meyn–Tweedie will be used to demonstrate the exponential ergodicity of the two-point process on $M \times M \setminus \Delta$. We state it in slightly lesser generality than in Meyn and Tweedie (1993a).

For a diffusion process $z_t$ on a (possibly noncompact) manifold, we denote the stochastic transition function by $p_t(x, dy)$. For a measure $\mu$ and a positive measurable function $f$ we denote

$$\|\mu\|_f = \sup_{|g|\le f} |\mu(g)|.$$

LEMMA 2 [Meyn and Tweedie (1993a), Theorem 6.1]. *Let $z_t$ be a positive Harris recurrent diffusion process on a manifold with an invariant measure $\pi$. Assume that for any compact set $K$ the measures $p_1(x, dy), x \in K$ are uniformly bounded from below by a nontrivial positive measure. Suppose that there exists a continuous positive function $V(z)$, such that*:

(a) *For any $N > 0$ there is a compact set $K$, such that $V(z) \ge N$ for $z \notin K$.*
(b) *For some $c > 0$, $d < \infty$,*

$$\mathcal{L}V(z) \le -cV(z) + d, \tag{17}$$

*where $\mathcal{L}$ is the generator of the diffusion process.*

*Then there exist $\theta > 0$ and $C < \infty$, such that*

$$\|p_t(z, \cdot) - \pi\|_V \le CV(z)e^{-\theta t}. \tag{18}$$

Let us now apply Lemma 2 to the case of the two-point motion $z_t$ on $M \times M \setminus \Delta$.

Recall that by (C) the two-point process has the unique invariant measure $m_2$ on $M \times M \setminus \Delta$. Inside the $r$-neighborhood of the diagonal the function $V(z) = E_z(e^{\alpha\tau})$ satisfies the relation

$$\mathcal{L}V(z) = -\alpha V(z),$$

and, by Lemma 1 it satisfies (16). Extending $V(z)$ from the $r/2$ neighborhood of the diagonal to the rest of $M \times M \setminus \Delta$ as a smooth positive function, we find that Lemma 2 applies to the process $z_t$. Thus, we obtain the following:

COROLLARY 3. *Let $B \in C^\infty(M \times M)$ be a function with zero mean with respect to the invariant measure. For any point $z \in M \times M \setminus \Delta$, let $\rho_{z,B}(t) = E_z(B(z_t))$, where $z_t$ is the two-point motion with $z_0 = z = (x, y)$. Then for sufficiently small positive $\theta$, there are positive $C(\theta)$ and $p(\theta)$, with the property that $p(\theta) \to 0$ when $\theta \to 0$, such that*

$$|\rho_{z,B}(t)| \le Ce^{-\theta t} d^{-p}(x, y). \tag{19}$$

PROOF OF THEOREM 1. Consider the Markov chain $(z_n, \xi_n)$ on $M \times M \setminus \Delta$, where $\xi_n = A_n^{(2)} - A_{n-1}^{(2)}$. Let $P_n(z, \cdot)$ denote the stochastic transition function for this chain (note that it only depends on the first component of the original point). This chain is positive Harris recurrent, and its invariant measure $\Pi$ is given by

$$\Pi(f) = \int P_1(z, f)\, d\pi(z).$$

Let $W(z, \xi) = V(z) + \xi^2$, and let $f$ be a function on $(M \times M \setminus \Delta) \times \mathbb{R}$, such that $|f| \le |W|$. This choice of $W$ is due to the fact that we need the inequality $\xi^2 \le W$ for the results of Meyn and Tweedie below to be applicable. Then

$$\mathbb{E}_z\big(f(z_n, \xi_n)\big) = \mathbb{E}_z\big(P_1(z_{n-1}, f)\big).$$

Note that

$$|P_1(f)| \le P_1(V + \xi^2) \le C_1 V + C_2 \le C_3 V.$$

Therefore, from Lemma 2 it follows that the following estimate holds:

$$\|P_n - \Pi\|_W \le D W e^{-\theta n}. \tag{20}$$

Now by the general theory of $V$-ergodic Markov chains [Meyn and Tweedie (1993b), Theorem 17.0.1] $A_n/\sqrt{n}$ is asymptotically normal. To get Theorem 1 write

$$\frac{A_t}{\sqrt{t}} = \frac{A_{[t]}}{\sqrt{t}} + \frac{(A_t - A_{[t]})}{\sqrt{t}}$$

and notice that the second term converges to 0 in probability. $\square$

PROOF OF THEOREM 2. The proof is the same as for Theorem 2, except that we need a new proof of the existence of the function $V_n(x^1, x^2, \ldots, x^n)$ satisfying (17). Let $\mathcal{L}^n$ be the generator of the $n$-point motion, and let

$$V_n(x^1, x^2, \ldots, x^n) = \sum_{i,j} V(x^i, x^j).$$

Then

$$(\mathcal{L}^n V_n)(x^1, x^2, \ldots, x^n) = \sum_{i,j} (\mathcal{L} V)(x^i, x^j),$$

so the required inequality follows from (17) for the two-point motion. $\square$

**4. Equidistribution.** In this section we prove that images of smooth submanifolds become uniformly distributed over $M$ as $t \to \infty$. More generally, we prove that if a measure $\nu$ has finite $p$-energy, defined in (14), for some $p > 0$, then the image of this measure under the dynamics of stochastic flow (1), satisfying conditions (A)–(E), becomes uniformly distributed on $M$.

LEMMA 4. *Let $\nu$ be a measure on $M$ which has finite $p$-energy for some $p > 0$. Let $b \in C^\infty(M)$ satisfy $\int b(x)\,d\mu(x) = 0$. Then there exist positive $\gamma$ independent of $\nu$ and $b$, and $C$ independent of $\nu$ such that for any positive $t_0$*

$$\mathbb{P}\left\{\sup_{t\ge t_0}\left|\int b(x_t)\,d\nu(x)\right| > CI_p(\nu)^{1/2}e^{-\gamma t_0}\right\} \le Ce^{-\gamma t_0}. \tag{21}$$

PROOF. Without loss of generality, we may assume that $\nu(M) = 1$ (otherwise we multiply $\nu$ by a constant). By the exponential mixing of the two-point processes (Corollary 3) we have

$$\begin{aligned}\mathbb{E}\left(\int b(x_t)\,d\nu(x)\right)^2 &= \iint_{M\times M}\mathbb{E}_{(x,y)}\big(b(x_t)b(y_t)\big)\,d\nu(x)\,d\nu(y)\\ &\le C_1\|b\|^2e^{-\theta t}\iint\frac{d\nu(x)\,d\nu(y)}{d^p(x,y)}\\ &\le C_2I_p(\nu)e^{-\theta t}.\end{aligned} \tag{22}$$

Therefore,

$$\mathbb{E}\left|\int b(x_t)\,d\nu(x)\right| \le C_3I_p(\nu)^{1/2}e^{-\theta t/2}. \tag{23}$$

This shows that the expectation of the integral decays exponentially fast. Now we shall use standard arguments based on the Borel–Cantelli lemma to show that $\int b(x_t)\,d\nu(x)$ itself decays to zero exponentially fast.

Fix small positive $\alpha$ and $\kappa$, to be specified later. Let $\tau_{n,j} = n + je^{-\kappa n}$, $0 \le j \le e^{\kappa n}$. By the Chebyshev inequality, (23) implies

$$\mathbb{P}\left\{\left|\int b\big(x_{\tau_{n,j}}\big)\,d\nu(x)\right| > I_p(\nu)^{1/2}e^{-\alpha n}\right\} \le C_3e^{(\alpha-\theta/2)n}.$$

Taking the sum over $j$,

$$\mathbb{P}\left\{\max_{0\le j<e^{\kappa n}}\left|\int b\big(x_{\tau_{n,j}}\big)\,d\nu(x)\right| > I_p(\nu)^{1/2}e^{-\alpha n}\right\} \le C_3e^{(\kappa+\alpha-\theta/2)n}. \tag{24}$$

Next we consider the oscillation of $\int b(x_t)\,d\nu(x)$ on the interval $[\tau_{n,j}, \tau_{n,j+1}]$. By Itô's formula, due to (1),

$$\begin{aligned}&\int b(x_t)\,d\nu(x) - \int b\big(x_{\tau_{n,j}}\big)\,d\nu(x)\\ &\qquad= \sum_{k=1}^d\int_{\tau_{n,j}}^t\int\beta_k(x_s)\,d\nu(x)\,d\theta_k(s) + \int_{\tau_{n,j}}^t\int\beta(x_s)\,d\nu(x)\,ds,\end{aligned} \tag{25}$$

where $\beta_k$, $k = 0,\dots,d$, are some bounded functions on $M$. Each of the integrals $\int_{\tau_{n,j}}^t\int\beta_k(x_s)\,d\nu(x)\,d\theta_k(s)$ can be obtained from a Brownian motion by a random time change. This time change has a bounded derivative, since $\beta_k$'s are bounded.

The absolute value of the last term on the right-hand side of (25) is not greater than $e^{-\kappa n/4}/(2(d+1))$ for sufficiently large $n$. Therefore, for suitable $C_4, C_5, C_6 > 0$, which are independent of $\nu$ since $\nu(M) = 1$,

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{\tau_{n,j}\le t_1\le t_2\le \tau_{n,j+1}}\left|\int b(x_{t_1})\,d\nu(x) - \int b(x_{t_2})\,d\nu(x)\right| \ge e^{-\kappa n/4}\right\}\\
&\quad\le d\cdot\mathbb{P}\left\{\sup_{0\le t_1\le t_2\le C_4 e^{-\kappa n}} |w(t_1)-w(t_2)| \ge \frac{e^{-\kappa n/4}}{d+1}\right\}\\
&\quad\le C_5\exp\left(-\frac{(e^{-\kappa n/4}/(d+1))^2}{2C_4 e^{-\kappa n}}\right)\exp\left(\frac{\kappa n}{2}\right)\\
&\quad= C_5\exp(-C_6 e^{\kappa n/2})\exp\left(\frac{\kappa n}{2}\right).
\end{aligned}\tag{26}$$

Combining this with (24), we obtain

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{n\le t\le n+1}\left|\int b(x_t)\,d\nu(x)\right| > I_p(\nu)^{1/2}(e^{-\alpha n}+e^{-\kappa n/2})\right\}\\
&\qquad\le C_3 e^{(\kappa+\alpha-\theta/2)n} + C_5 e^{3\kappa n/2}\exp(-C_6 e^{\kappa n/2}).
\end{aligned}$$

This implies (21) if we take $\alpha = \kappa = \gamma = \theta/10$ in the last inequality and take the sum over all $n$ such that $n \ge t_0 - 1$. $\square$

## 5. CLT for measures.

5.1. *Energy estimate.* We prove the lemma which is needed in order to control the growth of the the $p$-energy of a measure. Let $\nu_t(\Omega) = \nu(x_t \in \Omega)$. Also we shall write $\nu(f) = \int f\,d\nu$.

LEMMA 5. *If $p$ is small enough, then for some $C_1, C_2, \theta > 0$,*

$$\mathbb{E}(I_p(\nu_t)) \le C_1 e^{-\theta t} I_p(\nu) + C_2.$$

PROOF. We apply Lemma 2 to the two-point process to obtain

$$\left|\mathbb{E}_{(x,y)}\, d^{-p}(x_t, y_t) - \int d^{-p}(x,y)\,d\mu(x)\,d\mu(y)\right| \le C_1\, d^{-p}(x,y)e^{-\theta t}.\tag{27}$$

[This formula follows from (18) once we recall that $V(x,y)$ is estimated by $d^{-p}$ from above and below.] From (27) it follows that

$$\mathbb{E}_{(x,y)}\, d^{-p}(x_t,y_t) \le C_1 e^{-\theta t}\, d^{-p}(x,y) + C_2.$$

Integrating with respect to $d\nu(x)\,d\nu(y)$, we obtain the lemma. $\square$

5.2. *Moment estimates and the proof of the main result.* This section is devoted to the proof of the main result of the paper: CLT for the passive tracer (Theorem 3). Recall that we start with a nonrandom measure $\nu$ of finite $p$-energy $I_p(\nu) < \infty$ for some $p > 0$, a stochastic flow of diffeomorphisms (1), and an additive functional $\{A_t : t > 0\}$ of the one-point motion, defined by the stochastic differential equation (11).

The assumption in (F) that $\hat{a}(x)$ has mean zero implies the existence of a smooth function $f(x)$ such that $Lf(x) = \hat{a}(x)$. Define the new additive functional $\widetilde{A}_t(x) = A_t(x) - f(x_t)$. Since $f$ is bounded, then $\widetilde{A}_t/\sqrt{t}$ has the same asymptotic properties as $A_t/\sqrt{t}$. Now by Itô's formula,

$$d\widetilde{A}_t(x) = \sum_{k=1}^{d} \big(\alpha_k(x_t) - \big(L_{X_k} f\big)(x_t)\big)\, d\theta_k(t).$$

Notice that since each $X_k$ preserves the measure $\mu$ we have $\int (L_{X_k} f)\, d\mu = 0$. Hence, the new function $\alpha_k - \int (L_{X_k} f)\, d\mu$ satisfies (F). Thus, without loss of generality, we can assume that $\hat{a} \equiv 0$ and, in particular, that $A_t$ is a martingale.

Let $\chi(t, \xi) = \nu(\exp\{\frac{i\xi}{\sqrt{t}} A_t\})$ be the characteristic function of the functional $A_t(x)$. Below we shall see that $\chi(t, \xi)$ is equicontinuous in $\xi$ when $\xi \in K$, $t \in \mathbb{N}$ on a set or realizations of randomness of measure $1 - \delta$, where $\delta > 0$ and a compact set $K$ are arbitrary. We shall further see that

$$\lim_{t \to \infty} \frac{\nu(|A_t - A_{[t]}|)}{\sqrt{t}} = 0 \qquad \text{almost surely.} \tag{28}$$

Finally, we shall see that there exists $D(A)$, such that for any $\xi$ fixed

$$\lim_{n \to \infty} \chi(n, \xi) = \exp\biggl(-\frac{D(A)\xi^2}{2}\biggr) \qquad \text{almost surely.} \tag{29}$$

Moreover, we shall prove:

LEMMA 6. *With the notations above for any positive $\rho$, $N$, there is $C > 0$ such that for any $t > 0$, we have*

$$\mathbb{P}\biggl\{\exp\biggl(-\frac{D(A)\xi^2 + \rho}{2}\biggr) \le \chi(t, \xi) \le \exp\biggl(-\frac{D(A)\xi^2 - \rho}{2}\biggr)\biggr\} \ge 1 - Ct^{-N}.$$

Combining (29) with (28) above, and with the equicontinuity of $\chi$, we obtain Theorem 3. Thus it remains to verify the equicontinuity of $\chi$ and formulas (28) and (29).

In the next three lemmas we estimate the growth rate of $A_t$ and of its moments.

LEMMA 7. *Let $\delta, N_0$ and $N$ be arbitrary positive numbers. There exists $C = C(\delta, N_0, N)$, such that for any $t > 0$, and any signed measure $\nu$, which satisfies $|\nu|(M) \le 1$, $I_p(|\nu|) \le t^{N_0}$, we have*

$$\mathbb{P}\{|\nu(A_t)| > t^{\delta}\} \le Ct^{-N}.$$

PROOF. Write the equation for $\nu(A_t)$ in Itô's form, (as shown above we can assume that $\hat{a} \equiv 0$)

$$\nu(A_t) = \sum_{k=1}^{d} \int_0^t \int \alpha_k(x_s)\, d\nu(x)\, d\theta_k(s).$$

By Lemma 4,

$$\mathbb{P}\left\{ \sup_{t^{\delta/3} \le s \le t} \left| \int \alpha_k(x_s)\, d\nu(x) \right| > e^{-\gamma t^{\delta/3}} t^{N_0/2} \right\} \le C e^{-\gamma t^{\delta/3}}. \tag{30}$$

The sum $\sum_{k=1}^{d} \int_0^{t^{\delta/3}} [\int \alpha_k(x_s)\, d\nu(x)]\, d\theta_k(s)$ is estimated using the facts that $\alpha_k$ are bounded and that the stochastic integrals can be viewed as Brownian motions with a random time change. The same integrals over the interval $[t^{\delta/3}, t]$ are similarly estimated using (30). □

The proof of (28) is similar to the proof of Lemma 7. Rewrite (28) as

$$\lim_{t \to \infty} \frac{\nu((A_t - A_{[t]})^2)}{t} = 0 \qquad \text{almost surely.}$$

Now one can write the expression for $\nu((A_t - A_{[t]})^2)$ in Itô's form, and then use the fact that the stochastic integral can be viewed as a time-changed Brownian motion. Alternatively, (28) follows from a more general result in Lisei and Scheutzow (2001).

LEMMA 8. *There exists a constant* $C > 0$, *such that for any* $k \ge 0$ *and any initial point* $x$,

$$\mathbb{P}_x\left\{ \frac{|A_t|}{\sqrt{t}} > k \right\} \le C \exp\left( -\frac{k^2}{C} \right). \tag{31}$$

PROOF. Recall that $A_t = \sum_{k=1}^{d} \int_0^t \alpha_k(x_s)\, d\theta_k(s)$. For each of the stochastic integrals, recall that $\int_0^t \alpha_k(x_s)\, d\theta_k(s)$ can be viewed as a time-changed Brownian motion, with the derivative of the time change bounded. Therefore,

$$\begin{aligned} &\mathbb{P}_x\left\{ \frac{|\int_0^t \alpha_k(x_s)\, d\theta_k(s)|}{\sqrt{t}} > k \right\} \\ &\quad \le \mathbb{P}\left\{ \frac{\sup_{s \le ct} |W_s|}{\sqrt{t}} > k \right\} \\ &\quad \le C \exp\left( -\frac{k^2}{C} \right) \qquad \text{for some } C > 0. \end{aligned} \tag{32}$$

Therefore the estimate (31) holds, with possibly a different constant $C$. □

LEMMA 9. *For any positive $\delta$ and any $N \in \mathbb{N}$, there exists a constant $C > 0$, such that for any measure $\nu$ with $|\nu|(M) \le 1$ and any $n \in \mathbb{N}$,*

$$\mathbb{P}\{|\nu|(|A_t|^n) > n! t^{(1/2+\delta)n}\} \le C t^{-N-\delta n/2}. \tag{33}$$

PROOF. Without loss of generality we may assume that $\nu$ is a probability measure. By Jensen's inequality, $(\int |A_t|^n \, d\nu)^l \le \int |A_t|^{nl} \, d\nu$ for $l \in \mathbb{N}$. It is therefore sufficient to estimate the probability $\mathbb{P}\{\nu(|A|^{nl}) > (n!)^l t^{(1/2+\delta)nl}\}$. By the Chebyshev inequality,

$$\mathbb{P}\{\nu(|A|^{nl}) > (n!)^l t^{(1/2+\delta)nl}\} \le \frac{\int \mathbb{E}_x |A|^{nl} \, d\nu(x)}{(n!)^l t^{(1/2+\delta)nl}}. \tag{34}$$

Take $l > \frac{N}{\delta} + \frac{1}{2}$. Then the right-hand side of (34) is not greater than $\sup_x \mathbb{E}_x |A|^{nl} / ((n!)^l t^{1/2nl}) t^{-N-\delta n/2}$. This is less than $C t^{-N-\delta n/2}$ by Lemma 8. $\square$

Put $n_t = [t^{1/3}]$, $\tau_t = t/n_t$ and for each $0 < s < t$ denote the increment of the functional $A_t(x)$ from time $s$ to time $t$ by

$$\Delta_{s,t}(x) = A_t(x) - A_s(x). \tag{35}$$

We split the time interval $[0, t]$ into $n_t$ equal parts and decompose

$$A_t(x) = \sum_{j=0}^{n_t - 1} \Delta_{j\tau_t, (j+1)\tau_t}(x). \tag{36}$$

The idea is to prove that this is a sum of weakly dependent random variables and that the CLT holds for almost every realization of the Brownian motion. We need an estimate on the correlation between the inputs from different time intervals. For any positive $\tau$, $s$ and $l$ with $\tau \le l$, we denote

$$\nu(\Delta_{l-\tau,l} \Delta_{l,l+s}) = \int \Delta_{l-\tau,l}(y) \Delta_{l,l+s}(y) \, d\nu(y).$$

LEMMA 10. *Let some positive $c_1, c_2, \gamma_1$ and $\gamma_2$ be fixed, and consider $s$ and $\tau$, which satisfy $c_1 l^{\gamma_1} \le s, \tau \le c_2 l^{\gamma_2}$. For any positive $\delta$, $N$ and $N_0$ there exists a constant $C$ such that for $l \ge C$, and any measure $\nu$, which satisfies $|\nu|(M) \le 1$, $I_p(|\nu|) \le l^{N_0}$, we have*

$$\mathbb{P}\{|\nu(\Delta_{l-\tau,l} \Delta_{l,l+s})| > l^\delta\} \le l^{-N}. \tag{37}$$

PROOF. Without loss of generality we may assume that $\nu$ is a probability measure. We start by decomposing each of the segments $[l - \tau, l]$, $[l, l + s]$ into two:

$$[l - \tau, l] = \Delta_1 \cup \Delta_2 = [l - \tau, l - \ln^2 l] \cup [l - \ln^2 l, l],$$
$$[l, l + s] = \Delta_3 \cup \Delta_4 = [l, l + \ln^2 l] \cup [l + \ln^2 l, l + s].$$

We denote

$$\bar{\Delta}_{a,b} = \Delta_{a,b}\chi_{\{\Delta_{a,b} \le (b-a)^2\}}.$$

By Lemma 8 there is a constant $C$, such that

$$(\mathbb{E}_x \Delta^2_{a,b})^{1/2} \le C(1+b-a)$$

and

$$\big(\mathbb{E}_x(\Delta_{a,b} - \bar{\Delta}_{a,b})^2\big)^{1/2} \le Ce^{-(b-a)/C}.$$

Therefore for two segments, $[a,b]$ and $[c,d]$, such that $b \le c$,

$$\begin{aligned} &|\mathbb{E}_x(\Delta_{a,b}\Delta_{c,d} - \bar{\Delta}_{a,b}\bar{\Delta}_{c,d})| \\ &\quad = \big|\mathbb{E}_x\big((\Delta_{a,b} - \bar{\Delta}_{a,b})\Delta_{c,d}\big) + \mathbb{E}_x\big(\bar{\Delta}_{a,b}(\Delta_{c,d} - \bar{\Delta}_{c,d})\big)\big| \\ &\quad \le C(1+|b-a|+|d-c|)\big(e^{-(b-a)/C} + e^{-(d-c)/C}\big). \end{aligned}$$

After using Chebyshev's inequality, we obtain that for any positive $k$,

$$\begin{aligned} &\mathbb{P}\{|\nu(\Delta_{a,b}\Delta_{c,d}) - \nu(\bar{\Delta}_{a,b}\bar{\Delta}_{c,d})| > k\} \\ &\quad \le \frac{C(1+|b-a|+|d-c|)(e^{-(b-a)/C} + e^{-(d-c)/C})}{k}. \end{aligned} \tag{38}$$

In the same way one obtains

$$\begin{aligned} &\mathbb{P}\{|\nu(\Delta_{a,b}\Delta_{c,d}) - \nu(\bar{\Delta}_{a,b}\Delta_{c,d})| > k\} \\ &\quad \le \frac{C(1+|b-a|+|d-c|)(e^{-(b-a)/C} + e^{-(d-c)/C})}{k}. \end{aligned} \tag{39}$$

The contribution from $\nu(\Delta_2\Delta_3)$ is estimated using estimate (38) with $k = l^\delta$: for any positive $\delta$ and $N$ for sufficiently large $l$,

$$\mathbb{P}\{|\nu(\Delta_2\Delta_3)| > l^\delta\} \le l^{-N}.$$

The contribution from each of the other three products is estimated using the fact that the segments are separated by a distance of order $\ln^2 l$. Let us, for example, prove that $\mathbb{P}\{|\nu(\Delta_1\Delta_3)| > l^\delta\} \le l^{-N}$. By taking $k = l^\delta$ in (39), we obtain

$$\mathbb{P}\{|\nu(\Delta_1\Delta_3) - \nu(\bar{\Delta}_1\Delta_3)| > l^\delta\} \le l^{-N}.$$

In order to estimate $\nu(\bar{\Delta}_1\Delta_3)$ we apply the change of measure

$$\nu(\bar{\Delta}_1\Delta_3) = \int \bar{\Delta}_{l-\tau, l-\ln^2 l}\Delta_{l, l+\ln^2 l}\, d\nu(x) = \int \Delta_{\ln^2 l, 2\ln^2 l}(x)\, d\hat{\nu}(x), \tag{40}$$

where

$$\hat{\nu}(A) = \int \chi_{\{x_{l-\ln^2 l} \in A\}}\bar{\Delta}_{l-\tau, l-\ln^2 l}(x)\, d\nu(x).$$

The measure $\hat{\nu}$ has a density with respect to $\nu_{l-\ln^2 l}$, which is bounded by $l^2$ since $\bar{\Delta}_{l-\tau,l-\ln^2 l}$ is bounded. Therefore, by Lemma 5, for any positive $N$, there exists $M$ such that

$$\mathbb{P}\{I_p(\hat{\nu}) > l^M\} \le l^{-N}.$$

The right-hand side of (40) is written as

$$\int \Delta_{\ln^2 l, 2\ln^2 l}(x)\, d\hat{\nu}(x) = \sum_{k=1}^{d} \int_{\ln^2 l}^{2\ln^2 l} \int \alpha_k(x_s)\, d\hat{\nu}(x)\, d\theta_k(s). \tag{41}$$

We now proceed as in the proof of Lemma 7. The contribution from $\nu(\Delta_1 \Delta_4)$ and $\nu(\Delta_2 \Delta_4)$ is estimated in exactly the same way. □

Our next statement concerns the asymptotic behavior of the second moment of the functional $A_t$.

LEMMA 11. *The following limit exists and the convergence is uniform in the initial point $x$:*

$$D(A) = \lim_{n\to\infty} \frac{\mathbb{E}_x(A_n^2)}{n}. \tag{42}$$

PROOF.

$$\lim_{n\to\infty} \frac{\mathbb{E}_x(A_n^2)}{n} = \lim_{n\to\infty} \frac{1}{n} \int_0^n \mathbb{E}_x\left(\sum_{k=1}^{d} \alpha_k(x_t)^2\right) dt = \int \left(\sum_{k=1}^{d} \alpha_k(x_t)^2\right) d\mu(y)$$

using the uniform ergodicity of the one-point motion. □

The next lemma provides a linear bound (in probability) on the growth of $\nu(A_t^2)$. Note that such a bound implies the equicontinuity of $\chi(t,\xi)$ in the sense discussed above.

LEMMA 12. *For any positive $N$, $N_0$ and $\rho > 0$, there is $C > 0$, such that for any measure $\nu$ which satisfies $|\nu|(M) \le 1$, $I_p(|\nu|) \le t^{N_0}$, we have*

$$\mathbb{P}\{|\nu(A_t^2) - \nu(M)D(A)t| \ge \rho t\} \le Ct^{-N}.$$

PROOF. Let us prove that $\mathbb{P}\{\nu(A_t^2) - \nu(M)D(A)t \ge \rho t\} \le Ct^{-N}$. The estimate from below can be proved similarly. Consider the event $Q_t$ that $I_p(\nu_{j\tau_t}) \le t^{N_1}$ for all $j < n_t$. By Lemma 5 we have $\mathbb{P}\{Q_t\} \ge 1 - t^{-N}$, if a sufficiently large $N_1$ is selected. Fix any $\delta$ with $0 < \delta < 1/20$. Let $R_t$ be the event that $\nu(\Delta^2_{j\tau_t,(j+1)\tau_t}) \le \tau_t^{1+\delta}$ for all $j < n_t$. By Lemma 9 we have $\mathbb{P}\{R_t\} \ge 1 - t^{-N}$. Let $\beta_j = \nu(\Delta^2_{(j-1)\tau_t,j\tau_t})\chi_{\{Q_t \cap R_t\}}$ and $\mathcal{B}_j = \sum_{k=1}^{j} \beta_k$. We shall prove that

$$\mathbb{E}\left\{\exp\left(\frac{\mathcal{B}_j - j(\nu(M)D(A)+\rho)\tau_t}{t^{5/6}}\right)\right\} \le (1 - \rho t^{-1/6}/2)^j. \tag{43}$$

The proof will proceed by induction on $j$. First we show that

$$\mathbb{E}\left\{\exp\left(\frac{\beta_j - (\nu(M)D(A)+\rho)\tau_t}{t^{5/6}}\right)\right\} \le (1 - \rho t^{-1/6}/2). \tag{44}$$

Indeed, using the Taylor expansion,

$$\begin{aligned} &\mathbb{E}\left\{\exp\left(\frac{\beta_j - (\nu(M)D(A)+\rho)\tau_t}{t^{5/6}}\right)\right\} \\ &\quad = 1 + \mathbb{E}\frac{\beta_j - (\nu(M)D(A)+\rho)\tau_t}{t^{5/6}} \\ &\qquad + \mathbb{E}\sum_{k=2}^{\infty}\left(\frac{\beta_j - (\nu(M)D(A)+\rho)\tau_t}{t^{5/6}}\right)^k \Big/ (k!). \end{aligned} \tag{45}$$

Since by definition $\beta_j \le \nu(\Delta^2_{j\tau_t,(j+1)\tau_t})$, we have $\mathbb{E}\beta_j \le \mathbb{E}\nu(\Delta^2_{j\tau_t,(j+1)\tau_t})$. From Lemma 11 it easily follows that $\mathbb{E}\nu(\Delta^2_{j\tau_t,(j+1)\tau_t}) \le (\nu(M)D(A) + \rho/4)\tau_t$ for large $t$. The expectation of the infinite sum is less than $\rho t^{-1/6}/4$, since $\beta_j \le \tau_t^{1+\delta}$. This proves (44).

Assume that (43) holds for some $j$. Then,

$$\begin{aligned} &\mathbb{E}\left\{\exp\left(\frac{\mathcal{B}_{j+1} - (\nu(M)D(A)+\rho)(j+1)\tau_t}{t^{5/6}}\right)\right\} \\ &\quad = \mathbb{E}\left\{\exp\left(\frac{\mathcal{B}_j - (\nu(M)D(A)+\rho)j\tau_t}{t^{5/6}}\right)\right. \\ &\qquad \left.\times\, \mathbb{E}\left(\exp\left(\frac{\beta_{j+1} - (\nu(M)D(A)+\rho)\tau_t}{t^{5/6}}\right)\Big|\mathcal{F}_{j\tau_j}\right)\right\}. \end{aligned} \tag{46}$$

Due to (44) and since $I_p(\nu_{j\tau_t}) \le t^{N_1}$ on $Q_t$ by the Markov property, the conditional expectation on the right-hand side of (46) is not greater than $1 - \rho t^{-1/6}/2$. Therefore,

$$\begin{aligned} &\mathbb{E}\left\{\exp\left(\frac{\mathcal{B}_{j+1} - (\nu(M)D(A)+\rho)(j+1)\tau_t}{t^{5/6}}\right)\right\} \\ &\quad \le (1 - \rho t^{-1/6}/2)\mathbb{E}\left\{\exp\left(\frac{\mathcal{B}_j - (\nu(M)D(A)+\rho)j\tau_t}{t^{5/6}}\right)\right\}. \end{aligned} \tag{47}$$

This proves (43). It follows from (43) with $j = n_t$ that

$$\mathbb{P}\{\mathcal{B}_{n_t} - \nu(M)D(A)t \ge \rho t\} \le Ct^{-N}.$$

Recall that

$$\mathbb{P}\left\{\mathcal{B}_{n_t} \neq \sum_{j=0}^{n_t-1} \nu\big(\Delta^2_{j\tau_t,(j+1)\tau_t}\big)\right\} \leq Ct^{-N}$$

by Lemmas 5 and 9.

Finally, direct application of Lemma 11 to pair products of Δ's at different time segments gives that for any positive $\delta$, we have

$$\mathbb{P}\left\{\left|\nu(A_t^2) - \sum_{j=0}^{n_t-1} \nu\big(\Delta^2_{j\tau_t,(j+1)\tau_t}\big)\right| > t^{1/3+\delta}\right\} \leq Ct^{-N} \tag{48}$$

for sufficiently large $t$. This completes the proof of the lemma. □

LEMMA 13. *Let $\rho$, $N_0$, and $N$ be arbitrary positive numbers. There exists a constant $C > 0$, such that for any $t > 0$, and any signed measure $\nu$, which satisfies $|\nu|(M) \leq 1$, $I_p(|\nu|) \leq t^{N_0}$ we have*

$$\mathbb{P}\left\{\left|\nu\left(\exp\left\{\frac{i\xi}{\sqrt{t}}\Delta_{0,\tau_t}\right\}\right) - \nu(M)\left(1 - \frac{D(A)\xi^2}{2t^{1/3}}\right)\right| \geq \rho t^{-1/3}\right\} \leq Ct^{-N}.$$

PROOF OF LEMMA 13. Consider the Taylor expansion of the function $\exp(\frac{i\xi}{\sqrt{t}}\Delta_{0,\tau_t}(x))$,

$$\begin{aligned}\nu&\left(\exp\left\{\frac{i\xi}{\sqrt{t}}\Delta_{0,\tau_t}\right\}\right)\\ &= \nu(M) + \frac{i\xi}{\sqrt{t}}\nu\big(\Delta_{0,\tau_t}\big) - \frac{\xi^2}{2t}\nu\big(\Delta^2_{0,\tau_t}\big) + \sum_{k=3}^{\infty}\left(\frac{i\xi}{\sqrt{t}}\right)^k \nu\big(\Delta^k_{0,\tau_t}\big).\end{aligned} \tag{49}$$

By Lemma 7 for any $\delta > 0$, we have

$$\mathbb{P}\left\{\left|\frac{i\xi}{\sqrt{t}}\nu\big(\Delta_{0,\tau_t}(x)\big)\right| > t^{-1/2+\delta}\right\} < Ct^{-N}. \tag{50}$$

By Lemma 12 almost certainly, we have

$$\big|\nu\big(\Delta^2_{0,\tau_t}(x)\big) - \nu(M)D(A)\tau_t\big| \leq \rho\tau_t. \tag{51}$$

To estimate the tail we apply Lemma 9. This proves the lemma. □

PROOF OF THEOREM 3. It remains to demonstrate that (29) holds. First we show that

$$\begin{aligned}\mathbb{P}\Bigg\{\Bigg|&\nu\left(\exp\left\{\frac{i\xi}{\sqrt{t}}\Delta_{0,(j+1)\tau_t}\right\}\right)\\ &- \left(1 - \frac{D(A)\xi^2}{2t^{1/3}}\right)\nu\left(\exp\left\{\frac{i\xi}{\sqrt{t}}\Delta_{0,j\tau_t}\right\}\right)\Bigg| \geq \rho t^{-1/3}\Bigg\} \leq Ct^{-N}.\end{aligned} \tag{52}$$

Write

$$\begin{aligned}
&\nu\left(\exp\left\{\frac{i\xi}{\sqrt{t}}\Delta_{0,(j+1)\tau_t}\right\}\right)\\
&\quad=\nu\left(\exp\left\{\frac{i\xi}{\sqrt{t}}\Delta_{0,j\tau_t}\right\}\exp\left\{\frac{i\xi}{\sqrt{t}}\Delta_{j\tau_t,(j+1)\tau_t}\right\}\right)\\
&\quad=\hat{\nu}\left(\exp\left\{\frac{i\xi}{\sqrt{t}}(\Delta_{0,\tau_t}\circ\mathcal{T}_{j\tau_t})\right\}\right),
\end{aligned}\tag{53}$$

where $\hat{\nu}$ is a random measure, defined by

$$\hat{\nu}(A)=\int\exp\left\{\frac{i\xi}{\sqrt{t}}\Delta_{0,j\tau_t}\right\}\chi_{\{x_{j\tau_t}\in A\}}\,d\nu(x)$$

and $\mathcal{T}_s$ is time shift by $s$. Note that by Lemma 5 for some $N_0$,

$$\mathbb{P}\{I_p(|\hat{\nu}|)>t^{N_0}\}\le Ct^{-N}.$$

Thus the right-hand side of (53) can be estimated with the help of Lemma 13:

$$\mathbb{P}\left\{\left|\hat{\nu}\left(\exp\left\{\frac{i\xi}{\sqrt{t}}\Delta_{0,\tau_t}\right\}\right)-\hat{\nu}(M)\left(1-\frac{D(A)\xi^2}{2t^{1/3}}\right)\right|\ge\rho t^{-1/3}\right\}\le Ct^{-N}.$$

This is exactly the same as (52). Applying (52) recursively for $j=n_t-1,\dots,1$ we obtain that for any positive $N$ and $\rho$ there is $C>0$ such that

$$\begin{aligned}
&\mathbb{P}\left\{\left(1-\frac{D(A)\xi^2+\rho}{2t^{1/3}}\right)^{n_t}\le\nu\left(\exp\left\{\frac{i\xi}{\sqrt{t}}\Delta_{0,j\tau_t}\right\}\right)\le\left(1-\frac{D(A)\xi^2-\rho}{2t^{1/3}}\right)^{n_t}\right\}\\
&\qquad\ge 1-Ct^{-N}.
\end{aligned}$$

This implies Lemma 6 and formula (29), which also completes the proof of the theorem. □

**6. The dissipative case.** In this section we extend our CLT for measures (Theorem 3), proved in the previous section for measure-preserving stochastic flows [see condition (A)], to the dissipative case. In other words, we consider stochastic flows defined by the stochastic differential equation (1) satisfying conditions (B)–(E). Notice that without measure-preservation assumption it is no longer true that generically the largest Lyapunov exponent is positive. However, the case when all the Lyapunov exponents are negative is well understood [Le Jan (1986b)], so we shall concentrate on the case with at least one positive exponent. The main result of this section is CLT for measures (Theorem 5).

Let $m$ be the invariant measure of the one-point process, which is unique by hypoellipticity assumption (B). Let $m_2$ be the invariant measure of the two-point process which is supported away from the diagonal. Such a measure exists and is unique for the processes with positive largest exponent by the results of Baxendale and Stroock (1988). Moreover, we have exponential convergence to $m_2$. Now the normalization condition becomes more subtle since the Lebesgue measure is no longer invariant under each realization of the stochastic flow. It is convenient to define statistical equilibrium measure by taking the Lebesgue measure and iterating it from $-\infty$. Note that this involves defining stochastic flow backward, as well as forward in time. The simplest way to do so is to take countably many independent copies of stochastic flows defined on time interval [0,1]. Denote by $\Theta$ the canonical space of the two-sided $d$-dimensional Brownian motion with the Wiener measure $P$. We denote by $\phi_{s,t}$ the diffeomorphism obtained by evolving our stochastic flow between times $s$ and $t$.

THEOREM 4. *With the notations above there is a family of probability measures* $\{\mu_t : t > 0\}$ *such that*:

(a) *For any measure* $\nu$ *of finite* $p$-*energy*,

$$\lim_{n\to\infty} \phi^*_{-n,t}(\nu) = \mu_t$$

*almost surely*.

(b) *The process* $t \to \mu_t$ *is Markovian and push forward* $\phi_t$ *by the time* $t$ *stochastic flow* (1) *satisfies* $\phi_t^*(\mu_0) = \mu_t$.

(c) *For any continuous function* $b$ *for any measure* $\nu$ *of positive* $p$-*energy*,

$$|\nu(b(x_t)) - \mu_t(b)| \to 0 \qquad \text{as } t \to +\infty$$

*almost surely* (*see Lemma* 17 *for a more precise statement*).

REMARK 2. The measure $\mu_t$ has been extensively studied and part (b) is well known, see Arnold (1998). Dimensional characteristics of $\mu_t$ have been studied in several papers [Ledrappier and Young (1988a, b) and Le Jan (1985, 1986a, b)]. The questions which we discuss here are different from the ones studied in the book and these papers and we will not use any of their results. Also, do not confuse $\mu_t$'s with the Riemannian measure $\mu$ we have on the Riemannian manifold $M$.

Let $\{A_t : t > 0\}$ be an additive functional of the one-point motion given by (11) and satisfying

$$\int_M \hat{a}(x)\,dm(x) = 0, \qquad \int_M \alpha_k(x)\,dm(x) = 0 \qquad \text{for } k = 1, \dots, d, \tag{54}$$

where $\hat{a}$ is given by (12).

Denote by $\bar{a}(t) = \mu_t(a)$, $\bar{\alpha}_k(t) = \mu_t(\alpha_k)$ for $k = 1, \dots, d$ averages with respect to $\mu_t$ and define two additive functionals

$$\begin{aligned} dC_t &= \sum_{k=1}^{d} \bar{\alpha}_k(t) \circ d\theta_k(t) + \bar{a}(t)\,dt, \\ dB_t(x) &= \sum_{k=1}^{d} \big(\alpha_k(x_t) - \bar{\alpha}_k(t)\big) \circ d\theta_k(t) + \big(a(x_t) - \bar{a}\big)\,dt. \end{aligned} \tag{55}$$

Functional $B_t(x)$ for each $t$ becomes a random variable on $\Theta \times M$ with the product measure $P \times \nu$. Denote

$$C_t^* = \frac{C_t}{\sqrt{t}}, \qquad B_t^* = \frac{B_t}{\sqrt{t}}.$$

THEOREM 5. (a) *Let $\mathcal{M}_t^{\theta,*}$ be the measure on $\mathbb{R}$ defined on Borel sets $\Omega \subset \mathbb{R}$ by*

$$\mathcal{M}_t^{\theta,*}(\Omega) = \nu\{x \in M : B_t^{\theta,*} \in \Omega\}. \tag{56}$$

*Then there is a constant $D'(A)$ such that almost surely $\mathcal{M}_t^{\theta,*}$ converges weakly to a Gaussian measure with zero mean and variance $D'(A)$.*

(b) *$C_t^*$ is asymptotically Gaussian with zero mean and some variance $D''(A)$.*

(c) *$B_t^*$ and $C_t^*$ are asymptotically independent.*

We can reformulate Theorem 5 as follows.

COROLLARY 14. (a) *Almost surely for large $t$ the measure $\mathcal{M}_t^{\theta}$ defined by* (15) *is asymptotically Gaussian with a random drift $C_t^*$ and deterministic variance $D'(A)$.*

(b) *As $t \to +\infty$ the distribution of the drift $C_t^*$ is asymptotically Gaussian with zero mean and the variance $D''(A)$.*

PROOF OF THEOREM 4. Given a measure $\nu$ of finite $p$-energy denote

$$\mu_t^{(n)}(\nu) = \phi_{-n,t}\nu. \qquad \square$$

LEMMA 15. *There is a constant $\rho < 1$ such that for each continuous function $b$ on $M$, each $t > 0$ and any pair of probability measures $\nu_1$ and $\nu_2$, each of finite $p$ energy, almost surely there exists a constant $C = C(\theta)$ such that*

$$\left|\mu_t^{(n)}(\nu_1)(b) - \mu_t^{(n)}(\nu_2)(b)\right| \le C\rho^n. \tag{57}$$

*Moreover, almost surely the limit $\lim_{n\to\infty} \mu_t^{(n)}(\nu_i)$ exists and is not dependent on $i$.*

REMARK 3. This proves part (a) of Theorem 4.

PROOF OF LEMMA 15. Notice that

$$\begin{aligned} &\left|\mathbb{E}\left(\mu_t^{(n)}(\nu_1)(b) - \mu_t^{(n)}(\nu_2)(b)\right)\right| \\ &\quad = \left|\mathbb{E}(b(x_t))\,d\nu_1(x_{-n}) - \mathbb{E}(b(x_t))\,d\nu_2(x_{-n})\right| \\ &\quad \le \operatorname{Const}\rho_1^n, \end{aligned} \tag{58}$$

since both terms are exponentially close to $m(b)$ by the exponential mixing of one-point process. Likewise,

$$\begin{aligned}\mathbb{E}\big([\mu_t^{(n)}(\nu_1)(b) - \mu_t^{(n)}(\nu_2)(b)]^2\big) \\ = \mathbb{E}\iint b(x_t)b(y_t)d(\nu_1\times\nu_1)(x_{-n},y_{-n}) \\ + \mathbb{E}\iint b(x_t)b(y_t)d(\nu_2\times\nu_2)(x_{-n},y_{-n}) \\ - 2\mathbb{E}\iint b(x_t)b(y_t)\,d(\nu_1\times\nu_2)(x_{-n},y_{-n}) \\ \le \mathrm{Const}\,\rho_2^n,\end{aligned} \tag{59}$$

since the first two terms are exponentially close to $m_2(b\times b)$ and the last term is exponentially close to $2m_2(b\times b)$. Thus, the Borel–Cantelli lemma implies the first part of the lemma. This, in turn, implies independence of the limit $\lim_{n\to\infty}\mu_t^{(n)}(\nu_i)$ from $i$.

To prove existence of this limit notice that almost surely there is a random constant $C = C(\theta)$ such that

$$\big|\mu_t^{(n+1)}(\nu)(b) - \mu_t^{(n)}(\nu)(b)\big| \le C\rho^n. \tag{60}$$

The proof of (60) is similar to the proof of (57) and can be left to the reader. □

By the construction we have part (b) of Theorem 4.

LEMMA 16. *For all $b_1 \in C^\infty(M)$, $b_2 \in C^\infty(M\times M)$,*

$$\mathbb{E}\big(\mu_t(b_1)\big) = m(b_1), \qquad \mathbb{E}\big(\mu_t\times\mu_t(b_2)\big) = m_2(b_2).$$

*For small $p$ we have $\mathbb{E}(I_p(\mu_t)) < \infty$ and is independent of $t$ by stationarity.*

PROOF. The first part is obtained by taking expectation in the limits

$$(\phi_{-n,t}^* m)(b_1) \to \mu_t(b_1), \qquad \big((\phi_{-n,t}\times\phi_{-n,t})^*(m\times m)\big)(b_2) \to (\mu_t\times\mu_t)(b_2).$$

The second follows from the first by Lemma 5. □

LEMMA 17. *Let $\nu$ be a measure on $M$ which has finite $p$-energy for some $p > 0$. Let $b \in C^\infty(M)$. Then there exist positive $\gamma$ independent of $\nu$ and $b$, and $C$ independent of $\nu$, such that for any positive $t_0$,*

$$\mathbb{P}\left\{\sup_{t\ge t_0}\left|\int b(x_t)\,d\nu(x) - \mu_t(b)\right| > C I_p(\nu)^{1/2} e^{-\gamma t_0}\right\} \le Ce^{-\gamma t_0}. \tag{61}$$

REMARK 4. This is part (c) of Theorem 4.

PROOF OF LEMMA 17. Following the argument of the proof of Lemma 15, we get, for any two measures $\nu_1$ and $\nu_2$ of finite $p$-energy,

$$\mathbb{P}\{|\nu_1(b(x_t)) - \nu_2(b(x_t))| \geq r\} \leq \frac{\mathrm{Const}[I_p(\nu_1)I_p(\nu_2)]^{1/2}e^{-\gamma_1 t}}{r}.$$

Taking $\nu_2 = \mu_0$, we get

$$\mathbb{P}\{|\nu(b(x_t)) - \mu_t(b)| \geq r\} \leq \frac{\mathrm{Const}[I_p(\nu_1)]^{1/2}e^{-\gamma_1 t}}{r}.$$

The rest of the proof is similar to the proof of Lemma 4. □

PROOF OF THEOREM 5. The proof of part (a) is the same as the proof of Theorem 3.

To prove part (b) observe that similarly to discussion at the beginning of Section 5.2 we can without the loss of generality assume that $A_t$ is a martingale. Thus, $dA_t = \sum_{k=1}^d \alpha_k(x_t)\,d\theta_k(t)$. Then

$$C_t = \sum_{k=1}^d \int_0^t \bar{\alpha}_k(s)\,d\theta_k(s)$$

is a martingale. So to prove (b) it suffices to show that for each $k$,

$$\frac{1}{t}\int_0^t \bar{\alpha}_k^2(s)\,ds \to c_k$$

in probability where

$$c_k = m_2(\alpha_k \times \alpha_k). \tag{62}$$

Now

$$\frac{1}{t}\mathbb{E}\biggl(\int_0^t \bar{\alpha}_k^2(s)\,ds\biggr) \to c_k$$

by Lemmas 15 and 16. So it is enough to show that

$$\frac{1}{t^2}\mathbb{E}\biggl(\int_0^t [\bar{\alpha}_k^2(s) - c_k]\,ds\biggr)^2 \to 0.$$

This follows from the estimate

$$\bigl|\mathbb{E}\bigl(\bigl(\bar{\alpha}_k^2(s_1) - c_k\bigr)\bigl(\bar{\alpha}_k^2(s_2) - c_k\bigr)\bigr)\bigr| \leq \mathrm{Const}\, e^{-\theta|s_1 - s_2|},$$

which can be proven by the arguments of Lemma 2.

Part (c) follows from the fact that $C_t^*$ depends only on the noise (and not on the initial conditions) and $B_t^*$ is asymptotically independent of the noise by part (a) of this theorem. □

REMARK 5. In the conservative case $\mu_t \equiv m$, so $\bar{a}$ and $\bar{\alpha}_k$ are nonrandom and Theorem 5 reduces to Theorem 3. Conversely, suppose that $D''(A) = 0$ for all additive functionals $A_t$. Let

$$A_t = \int_0^t a(x_s)\, d\theta_1(s),$$

where $m(a) = 0$. Then

$$C_t = \int_0^t \mu_s(a)\, d\theta(s).$$

By (62),

$$0 = D''(A) = m_2(a \times a).$$

This implies that for an arbitrary function [not assuming that $m(a) = 0$] we have $m_2(a \times a) = m(a)^2$. By polarization, for any pair of continuous functions $a, b$ on $M$, we have

$$\iint a(x)b(y)\, dm_2(x, y) = m(a)m(b).$$

Hence, $m_2 = m \times m$. By Kunita (1990) this implies that each $\phi_{s,t}$ preserves $m$. Hence, we can characterize the conservative case by the condition that the drift in Corollary 14 is nonrandom.

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D. DOLGOPYAT
DEPARTMENT OF MATHEMATICS
MATHEMATICS BUILDING
UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND 20742-4015
USA
E-MAIL: dmitry@math.umd.edu
URL: www.math.umd.edu/~dmitry/

V. KALOSHIN
DEPARTMENT OF MATHEMATICS
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA 91125
USA
E-MAIL: kaloshin@its.caltech.edu

L. KORALOV
DEPARTMENT OF MATHEMATICS
PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY 08544
USA
E-MAIL: koralov@math.princeton.edu