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# A $J$-SPECTRAL FACTORIZATION APPROACH TO $\mathscr{H}_\infty$ CONTROL*

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**Abstract.** Necessary and sufficient conditions for the existence of suboptimal solutions to the standard model matching problem associated with $\mathscr{H}_\infty$ control are derived using $J$-spectral factorization theory. The existence of solutions to the model matching problem is shown to be equivalent to the existence of solutions to two coupled $J$-spectral factorization problems, with the second factor providing a parametrization of all solutions to the model matching problem. The existence of the $J$-spectral factors is then shown to be equivalent to the existence of nonnegative definite, stabilizing solutions to two indefinite algebraic Riccati equations, allowing a state-space formula for a linear fractional representation of all controllers to be given. A virtue of the approach is that a very general class of problems may be tackled within a conceptually simple framework, and no additional auxiliary Riccati equations are required.

**Key words.** $\mathscr{H}_\infty$ control, $J$-spectral factorization, indefinite factorization, four block problems, Riccati equations, Nehari's Theorem

**AMS(MOS) subject classifications.** 93C35, 47A68

**Introduction.** Since their inception, $\mathscr{H}_\infty$ control problems have been amenable to a variety of solution techniques. These range from the complex function theory approaches based on Nevanlinna–Pick–Schur interpolation to operator theoretic and state space approaches to $\mathscr{L}_\infty$ extension problems. In the case of simple problems, like sensitivity minimization, the relationships between these various approaches are well understood [8], [10], [14], [18]. The considerable body of knowledge about $\mathscr{H}_\infty$ control problems and their solution has evolved from the interaction between these various approaches, all of which provide solutions to the simple "Nehari type" problems which are conceptually elegant and computationally tractable. Unfortunately, this class of problems is too special to be of general engineering significance. In the case of more general problems, such as the mixed sensitivity problem, the mathematical solution was until recently more complicated, the interconnections were not well understood, and the computational burden associated with the solution was all but prohibitive (see [8], [10], [20]).

The $J$-spectral factorization approach to the problem of finding all suboptimal controllers for the simple "Nehari type" problems is well documented [2], [4], [10] and the approach has also been used to solve the optimal case [3]. In a recent paper [1], a general class of $\mathscr{H}_\infty$ control problems is solved via several spectral and $J$-spectral factorizations. The resulting algorithm is far from computationally simple. The new solution to the $\mathscr{H}_\infty$ problem presented in [12], however, requires just two indefinite algebraic Riccati equations to be solved and it was observed that these were associated with two $J$-factorizations.

In this paper we re-analyze the work in [1], showing that all the spectral and $J$-spectral factorizations can be subsumed into just two $J$-spectral factorizations. The Bart, Gohberg, and Kaashoek factorization theory [6] can then be used to associate the existence of the appropriate $J$-spectral factors with the solvability of two indefinite algebraic Riccati equations, and these can then be used to construct a generator of all solutions.

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Concurrent with this work, several of the other approaches to $\mathcal{H}_\infty$ control have been generalized and entirely new connections have been uncovered. The following remarks, which are in no way a complete survey, are intended to connect this paper with these other developments.

The four block distance problem has been solved by Glover, Limebeer, Doyle, Kasenally, and Safonov [12], [13], [21] using all-pass embedding. In Glover and Doyle [12] the equivalence between maximum entropy $\mathcal{H}_\infty$ control and risk sensitive control was established, a connection observed also in [7]. Moreover, Doyle et al. [9] have developed a state-space approach with a separation argument reminiscent of classical linear quadratic Gaussian (LQG) theory. Khargonekar, Petersen, and Rotea [16] have also considered a state feedback approach, observing a connection with LQ game theory. The connection between game theory and $J$-spectral factorization is long standing [5]. Extensions to time-varying systems using the maximum principle [25] and LQ game theory [19] have also been made. A conjugation approach developed by Kimura [17] is related to the $J$-spectral factorization method pursued here.

Note, however, that the assumptions used in the various approaches above are not all equivalent. In particular, the assumptions used here are more general than [9], where stronger assumptions are used for expository reasons. The optimal case is considered only in [13], [21].

Section 1 contains preliminaries and the standard stabilizing controller parametrization theory. In § 2 we analyze model matching problems of Nehari, unilateral and bilateral type and solve these in turn via $J$-spectral factorization. In order to satisfy the stability requirements it is necessary to impose an additional hitherto "unnoticed" condition on the $J$-spectral factors. Specifically, we will require the (1, 1) block of the factors to be outer. We note that Petersen and Clements [22] have also recently and independently observed that a $J$-spectral factorization with outer (1, 1) block can be associated with an $\mathcal{H}_\infty$ state feedback problem.

The relationship between $J$-spectral factorization and indefinite algebraic Riccati equations is analyzed in § 3. The results are reminiscent of existing results relating spectral factorization and Riccati equations and are derived using canonical factorization theory [6]. These results provide a state-space solution of the model matching problem in § 4. Section 5 gives necessary and sufficient conditions for a solution to the $\mathcal{H}_\infty$ control problem to exist and a representation formula for all solutions.

## 1. Preliminaries.

### 1.1 Notation.

| | |
|---|---|
| $\mathbb{R}, \mathbb{C}$ | real and complex number fields |
| $\bar{s}$ | complex conjugate of $s \in \mathbb{C}$ |
| $\mathcal{R}$ | proper rational functions of a complex variable with complex coefficients |
| $\mathbb{C}^{m\times n}, \mathcal{R}^{m\times n}$ | $m \times n$ matrices with entries in $\mathbb{C}$, $\mathcal{R}$ |
| $A^*$ | complex conjugate transpose of $A \in \mathbb{C}^{m\times n}$ |
| $\lambda_i(A)$ | $i$th eigenvalue of $A \in \mathbb{C}^{n\times n}$ |
| $\lambda_{\max}(A)$ | largest eigenvalue of a matrix $A \in \mathbb{C}^{n\times n}$ |
| $\mathrm{In}(A)$ | inertia of $A \in \mathbb{C}^{n\times n}$: $\mathrm{In}(A) = (\pi(A), \nu(A), \delta(A))$ where $\pi(A)$, $\nu(A)$, and $\delta(A)$ are, respectively, the number of eigenvalues of $A$ in the open right and left half planes and on the imaginary axis |
| $A \geqq B, A > B$ | $A - B \in \mathbb{C}^{n\times n}$ symmetric and positive semidefinite, positive definite |
| $\mathbf{M} \geqq \mathbf{N}, \mathbf{M} > \mathbf{N}$ | $\mathbf{M} - \mathbf{N} \in \mathcal{R}^{n\times n}$ and $\mathbf{M}(j\omega) \geqq \mathbf{N}(j\omega), \mathbf{M}(j\omega) > \mathbf{N}(j\omega), \forall \omega \in \mathbb{R} \cup \infty$ |

| | |
|---|---|
| $\mathcal{RL}_\infty^{m\times n}$ | matrices in $\mathcal{R}^{m\times n}$ without imaginary axis poles |
| $\|\mathbf{M}\|_\infty$ | $\mathcal{RL}_\infty$ norm: for $\mathbf{M}\in\mathcal{RL}_\infty \|\mathbf{M}\|_\infty = \sup_\omega \{\lambda_{\max}[\mathbf{M}(j\omega)^*\mathbf{M}(j\omega)]\}^{1/2}$ |
| $\mathcal{RH}_\infty^{m\times n}$ | subspace of $\mathcal{RL}_\infty^{m\times n}$ matrices without poles in the right half plane |
| $\mathcal{GH}_\infty^n$: | units of $\mathcal{RH}_\infty^{n\times n}$: $\mathbf{M}\in\mathcal{GH}_\infty^n \Leftrightarrow \mathbf{M}, \mathbf{M}^{-1}\in\mathcal{RH}_\infty^{n\times n}$ |
| $\mathbf{M}^\sim$ | $\mathbf{M}^\sim(s)=\mathbf{M}(-\bar{s})^*$ |
| $\Gamma_\mathbf{M}$ | Hankel operator with symbol $\mathbf{M}\in\mathcal{RL}_\infty^{m\times n}$ |

Associated with a matrix $\mathbf{M}\in\mathcal{R}^{m\times n}$ is a state space realization:

$$\mathbf{M}(s)=D+C(sI-A)^{-1}B \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]\in\mathbb{C}^{(n+m)\times(n+p)}. \tag{1.1}$$

If $\mathbf{P}\in\mathcal{R}^{(l+m)\times(p+q)}$ is partitioned as

$$\mathbf{P}=\begin{array}{cc} \begin{array}{cc} p & q \end{array} & \\ \begin{bmatrix}\mathbf{P}_{11} & \mathbf{P}_{12}\\ \mathbf{P}_{21} & \mathbf{P}_{22}\end{bmatrix} & \begin{array}{c} l \\ m\end{array}\end{array} \tag{1.2}$$

then

$$\mathcal{F}(\mathbf{P},\mathbf{K})=\mathbf{P}_{11}+\mathbf{P}_{12}\mathbf{K}(I-\mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}.$$

We say $\mathbf{P}$ is stabilizable if there exists such a $\mathbf{K}$ for which $\mathcal{F}(\mathbf{P},\mathbf{K})$ is internally stable (see [10]). The $\mathcal{H}_\infty$ control problem we will be concerned with is to find necessary and sufficient conditions for the existence of an internally stabilizing controller $\mathbf{K}$ such that $\|\mathcal{F}(\mathbf{P},\mathbf{K})\|_\infty<\gamma$, and when such conditions hold, to parametrize all solutions.

Finally, define the indefinite matrix $J_{pq}(\gamma)\in\mathbb{C}^{p+q}$, $\gamma>0$, by

$$J_{pq}(\gamma)=\begin{bmatrix} I_p & 0\\ 0 & -\gamma^2 I_q\end{bmatrix}. \tag{1.3}$$

For convenience we will often abbreviate $J_{pq}(\gamma)$ to $J$.

**1.2. Parametrization of all stabilizing controllers and the model matching problem.** Suppose $\mathbf{P}\in\mathcal{R}^{(l+m)\times(p+q)}$ is partitioned as in (1.2) and is stabilizable. Suppose $\mathbf{P}_{22}$ has a doubly coprime factorization over $\mathcal{RH}_\infty$:

$$\mathbf{P}_{22}=\mathbf{N}_r\mathbf{D}_r^{-1}=\mathbf{D}_l^{-1}\mathbf{N}_l \tag{1.4a}$$

where

$$\begin{bmatrix}\mathbf{V}_r & \mathbf{U}_r\\ -\mathbf{N}_l & \mathbf{D}_l\end{bmatrix}\begin{bmatrix}\mathbf{D}_r & -\mathbf{U}_l\\ \mathbf{N}_r & \mathbf{V}_l\end{bmatrix}=\begin{bmatrix} I_q & 0\\ 0 & I_m\end{bmatrix} \tag{1.4b}$$

is the corresponding Bezout identity. Further

$$\begin{bmatrix}\mathbf{V}_r & \mathbf{U}_r\\ -\mathbf{N}_l & \mathbf{D}_l\end{bmatrix} \text{ and } \begin{bmatrix}\mathbf{D}_r & -\mathbf{U}_l\\ \mathbf{N}_r & \mathbf{V}_l\end{bmatrix}\in\mathcal{RH}_\infty^{m+q}. \tag{1.4c}$$

It is well known (see, e.g., [8], [10], [23]) that $\mathbf{K}$ is a stabilizing controller if and only if $\mathbf{K}$ is given by

$$\mathbf{K}=\mathbf{K}_1\mathbf{K}_2^{-1}, \begin{bmatrix}\mathbf{K}_1\\ \mathbf{K}_2\end{bmatrix}=\begin{bmatrix}\mathbf{D}_r & -\mathbf{U}_l\\ \mathbf{N}_r & \mathbf{V}_l\end{bmatrix}\begin{bmatrix}\mathbf{Q}\\ I_m\end{bmatrix} \quad \mathbf{Q}\in\mathcal{RH}_\infty^{q\times m}. \tag{1.5}$$

Substituting (1.4) and (1.5) into $\mathcal{F}(\mathbf{P},\mathbf{K})$ we obtain

$$\begin{aligned}\mathcal{F}(\mathbf{P},\mathbf{K})&=\mathbf{P}_{11}+\mathbf{P}_{12}\mathbf{K}(I-\mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\\ &=(\mathbf{P}_{11}-\mathbf{P}_{12}\mathbf{U}_l\mathbf{D}_l\mathbf{P}_{21})+(\mathbf{P}_{12}\mathbf{D}_r)\mathbf{Q}(\mathbf{D}_l\mathbf{P}_{21})\\ &=\mathbf{T}_{11}+\mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}.\end{aligned} \tag{1.6}$$

Thus, the $\mathcal{H}_\infty$ control problem can be posed as a model matching problem: Given the $\mathbf{T}_{ij}$'s, find necessary and sufficient conditions for the existence of $\mathbf{Q} \in \mathcal{RH}_\infty$ such that $\|\mathbf{T}_{11}+\mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\|_\infty < \gamma$ and, when such conditions hold, parametrize all solutions.

**2. Model matching theory.** In this section we solve a sequence model matching problem of increasing generality via $J$-spectral factorization. The existence of a solution to the model matching problem is shown to be equivalent to the existence of a $J$-spectral factor $\mathbf{W} \in \mathcal{GH}_\infty$ satisfying a relation of the form $\mathbf{G}^\sim \mathbf{J}\mathbf{G} = \mathbf{W}^\sim \mathbf{J}\mathbf{W}$ in which $\mathbf{W}_{11} \in \mathcal{GH}_\infty$, where $\mathbf{W}_{11}$ is the $(1, 1)$ block of $\mathbf{W}$. The $J$-spectral factor $\mathbf{W}$, when it exists, is shown to parametrize all solutions to the model matching problem.

**2.1. The Nehari problem.** The purpose of this section is to summarize the standard results [2], [10] relating the Nehari extension problem to $J$-spectral factorization. The condition $\mathbf{W}_{11} \in \mathcal{GH}_\infty$ is new, however, and is one that not only turns out to be particularly useful in the more general model matching problems we subsequently consider, but simplifies the proofs for the Nehari case as well.

THEOREM 2.1. *Let* $\mathbf{R} \in \mathcal{RL}_\infty^{p\times q}$. *The following are equivalent*:

1. $\|\Gamma_\mathbf{R}\| < \gamma$;
2. *There exists* $\mathbf{Q} \in \mathcal{RH}_\infty^{p\times q}$ *such that* $\|\mathbf{R}+\mathbf{Q}\|_\infty < \gamma$;
3. *There exists* $\mathbf{W} \in \mathcal{GH}_\infty^{p+q}$ *with* $\mathbf{W}_{11} \in \mathcal{GH}_\infty^p$ *satisfying*

$$\mathbf{G}^\sim J_{pq}(\gamma)\mathbf{G} = \mathbf{W}^\sim J_{pq}(\gamma)\mathbf{W}, \qquad \mathbf{G} = \begin{bmatrix} I_p & \mathbf{R} \\ 0 & I_q \end{bmatrix}. \tag{2.1}$$

*Proof.* $1 \Leftrightarrow 2$ is Nehari's Theorem. We shall prove that $1 \Rightarrow 3$ and that $3 \Rightarrow 2$.

$3 \Rightarrow 2$: Suppose a $\mathbf{W}$ with the required properties exists. Let $\mathbf{V} = \mathbf{W}^{-1}$ and partition $\mathbf{V}$ and $\mathbf{W}$ conformably with $\mathbf{G}$. Since $\mathbf{V}_{22}^{-1} = \mathbf{W}_{22} - \mathbf{W}_{21}\mathbf{W}_{11}^{-1}\mathbf{W}_{12}$ [15, p. 656] and $\mathbf{W}_{11} \in \mathcal{GH}_\infty^p$, it follows that $\mathbf{V}_{22} \in \mathcal{GH}_\infty^q$. Set $\mathbf{Q} = \mathbf{V}_{12}(\mathbf{V}_{22})^{-1} \in \mathcal{RH}_\infty$, giving

$$\begin{bmatrix} \mathbf{R}+\mathbf{Q} \\ I \end{bmatrix} = \begin{bmatrix} I & \mathbf{R} \\ 0 & I \end{bmatrix}\begin{bmatrix} \mathbf{Q} \\ I \end{bmatrix} = \mathbf{G}\mathbf{V}\begin{bmatrix} 0 \\ \mathbf{V}_{22}^{-1} \end{bmatrix}.$$

Hence

$$\begin{aligned}(\mathbf{R}+\mathbf{Q})^\sim(\mathbf{R}+\mathbf{Q}) - \gamma^2 I &= \begin{bmatrix} \mathbf{R}+\mathbf{Q} \\ I \end{bmatrix}^\sim J \begin{bmatrix} \mathbf{R}+\mathbf{Q} \\ I \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \mathbf{V}_{22}^{-1} \end{bmatrix}^\sim \mathbf{V}^\sim \mathbf{G}^\sim J \mathbf{G}\mathbf{V}\begin{bmatrix} 0 \\ \mathbf{V}_{22}^{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{V}_{22}^{-1} \end{bmatrix}^\sim J \begin{bmatrix} 0 \\ \mathbf{V}_{22}^{-1} \end{bmatrix} \quad \text{by (2.1)} \\ &= -\gamma^2(\mathbf{V}_{22}\mathbf{V}_{22}{}^\sim)^{-1} < 0.\end{aligned}$$

This implies 2.

$1 \Rightarrow 3$: Decompose $\mathbf{R}$ as $\mathbf{R} = \mathbf{R}_+ + \mathbf{R}_-$, with $\mathbf{R}_-^\sim \in \mathcal{RH}_\infty$ and strictly proper, $\mathbf{R}_+ \in \mathcal{RH}_\infty$. Suppose, following [10], that $\mathbf{R}_-$ has a minimal realization $\mathbf{R}_-(s) = C(sI-A)^{-1}B$ and $P$ and $Q$ satisfy the Lyapunov equations

$$AP + PA^* = BB^* \tag{2.2a}$$

$$QA + A^*Q = C^*C. \tag{2.2b}$$

Since $\|\Gamma_\mathbf{R}\| < \gamma$, $\lambda_{\max}(QP) < \gamma^2$. Define

$$N = (I - \gamma^{-2}QP)^{-1}. \tag{2.3}$$

Define $\mathbf{X}$ by

$$\mathbf{X} \stackrel{s}{=} \left[\begin{array}{c|cc} -A^* & C^* & -QB \\ \hline \gamma^{-2}CPN & I & 0 \\ \gamma^{-2}B^*N & 0 & I \end{array}\right].$$

It is readily verified (using the state transformation $\left[\begin{smallmatrix} \gamma^2 N & Q \\ PN & I \end{smallmatrix}\right]^{-1}$ on $G_-^\sim JG_-$) that

$$G_-^\sim JG_- = \mathbf{X}^\sim J\mathbf{X}, \qquad \mathbf{G}_- = \begin{bmatrix} I & \mathbf{R}_- \\ 0 & I \end{bmatrix}.$$

Since $-A^*$ is asymptotically stable, we see that $\mathbf{X} \in \mathcal{RH}_\infty$. It is also easy to verify using (2.2) and (2.3) that the "$A$" matrix of $\mathbf{X}^{-1} = -N^{-1}A^*N$, so $\mathbf{X} \in \mathcal{GH}_\infty$. The "$A$" matrix of $(\mathbf{X}_{11})^{-1}$ is given by

$$\hat{A} = -A^* - \gamma^{-2}C^*CPN.$$

Using (2.2) and (2.3) it is easy to establish that

$$\hat{A}N^{-1}P^{-1} + P^{-1}N^*\hat{A}^* = -[\gamma^{-1}C^* \quad BP^{-1}]\begin{bmatrix} \gamma^{-1}C \\ B^*P^{-1} \end{bmatrix}$$

which shows, since $N^{-1}P^{-1} > 0$, that $\hat{A}$ is asymptotically stable, and consequently $\mathbf{X}_{11} \in \mathcal{GH}_\infty$, provided $(\hat{A}, [\gamma^{-1}C^* \; BP^{-1}])$ is controllable [11, Thm. 3.3]. The required controllability is easily seen from

$$[\hat{A} \quad \gamma^{-1}C^*] = [-A^* \quad C^*]\begin{bmatrix} I & 0 \\ -\gamma^{-2}CPN & \gamma^{-1}I \end{bmatrix}.$$

Finally, observe that

$$\mathbf{G} = \begin{bmatrix} I & \mathbf{R} \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & \mathbf{R}_- \\ 0 & I \end{bmatrix}\begin{bmatrix} I & \mathbf{R}_+ \\ 0 & I \end{bmatrix},$$

so $\mathbf{W}$ given by

$$\mathbf{W} = \mathbf{X}\begin{bmatrix} I & \mathbf{R}_+ \\ 0 & I \end{bmatrix}$$

has the required properties. □

Note that, provided $\gamma$ is not in the spectrum of $\Gamma_{\mathbf{R}}$, the generalization to the AAK problem (where $\mathbf{Q}$ is allowed $k$ poles in the right half plane) is simply that $\mathbf{W}_{11}^{-1}$ is allowed $k$ poles in the right half plane.

Consider the factorization (2.1). As with spectral factorization, $\mathbf{W} \in \mathcal{GH}_\infty$ satisfying (2.1) is not unique, being determined only up to a $J$-unitary matrix (see the following lemma). Supposing that *one* of these solutions has the property $\mathbf{W}_{11} \in \mathcal{GH}_\infty$, it is important to establish whether or not *all* of the other possible solutions have this property as well. For unless the property $\mathbf{W}_{11} \in \mathcal{GH}_\infty$ is an all or none affair, Theorem 2.1 will be of little practical value, as one would have to look through the class of possible $\mathbf{W}$'s in search of one with the desired $\mathbf{W}_{11} \in \mathcal{GH}_\infty$ property. Fortunately, this is not necessary.

LEMMA 2.2. *Suppose* $\mathbf{W} \in \mathcal{GH}_\infty^{p+q}$. *Then*

1. $\mathbf{Y} \in \mathcal{GH}_\infty^{p+q}$ *satisfies* $\mathbf{Y}^\sim \mathbf{J}\mathbf{Y} = \mathbf{W}^\sim \mathbf{J}\mathbf{W}$ *if and only if* $\mathbf{Y} = \mathbf{A}\mathbf{W}$, *where* $\mathbf{A}$ *is a constant J-unitary matrix* (*i.e.*, $A^*JA = J$).

2. *If* $\mathbf{W}_{11}^{\sim}\mathbf{W}_{11}-\gamma^2\mathbf{W}_{21}^{\sim}\mathbf{W}_{21}\geqq 0$ *and* $\mathbf{Y}\in\mathcal{GH}_\infty^{p+q}$ *satisfies* $\mathbf{Y}^{\sim}\mathbf{J}\mathbf{Y}=\mathbf{W}^{\sim}\mathbf{J}\mathbf{W}$, *then* $\mathbf{Y}_{11}\in\mathcal{GH}_\infty^{p}$ *if and only if* $\mathbf{W}_{11}\in\mathcal{GH}_\infty^{p}$.

*Proof.* Suppose $\mathbf{Y}\in\mathcal{GH}_\infty$ satisfies $\mathbf{W}^{\sim}\mathbf{J}\mathbf{W}=\mathbf{Y}^{\sim}\mathbf{J}\mathbf{Y}$. Then

$$(\mathbf{Y}^{\sim})^{-1}\mathbf{W}^{\sim}\mathbf{J}=\mathbf{J}\mathbf{Y}\mathbf{W}^{-1}. \tag{2.4}$$

Since $\mathbf{Y}\mathbf{W}^{-1}\in\mathcal{GH}_\infty$, it follows that $\mathbf{Y}\mathbf{W}^{-1}=A$ is constant and is $J$-unitary by (2.4). The converse is obvious.

Observe that $\mathbf{W}_{11}^{\sim}\mathbf{W}_{11}-\gamma^2 W_{21}^{\sim}W_{21}\geqq 0$ and $\mathbf{W}_{11}\in\mathcal{GH}_\infty\Rightarrow\mathbf{W}_{21}\mathbf{W}_{11}^{-1}\in\mathcal{RH}_\infty$ and $\|\mathbf{W}_{21}\mathbf{W}_{11}^{-1}\|_\infty\leqq\gamma^{-1}$. Also $A^*JA=J\Rightarrow A^{-1}=J^{-1}A^*J\Rightarrow AJ^{-1}A^*=J^{-1}$, the (1, 1) block of which is $A_{11}A_{11}^*-\gamma^{-2}A_{12}A_{12}^*=I$. Hence $A_{11}$ is nonsingular and $\|A_{11}^{-1}A_{12}\|<\gamma$. Therefore $(I+A_{11}^{-1}A_{12}\mathbf{W}_{21}\mathbf{W}_{11}^{-1})\in\mathcal{GH}_\infty$ or, equivalently, $\mathbf{Y}_{11}=A_{11}\mathbf{W}_{11}+A_{12}\mathbf{W}_{21}\in\mathcal{GH}_\infty$. For the converse, interchange $\mathbf{Y}$ and $\mathbf{W}$ in the above argument. □

Note that if $\mathbf{G}^{\sim}J\mathbf{G}=\mathbf{W}^{\sim}J\mathbf{W}$ and $\mathbf{G}_{21}=0$ then the condition $\mathbf{W}_{11}^{\sim}\mathbf{W}_{11}-\gamma^2\mathbf{W}_{21}^{\sim}\mathbf{W}_{21}\geqq 0$ is satisfied. Thus, given *any* $\mathbf{W}\in\mathcal{GH}_\infty$ such that $\mathbf{G}^{\sim}J\mathbf{G}=\mathbf{W}^{\sim}J\mathbf{W}$ with $\mathbf{G}$ as in (2.1), the Nehari problem has a solution if and only if $\mathbf{W}_{11}\in\mathcal{GH}_\infty$. The point is that if $\mathbf{W}_{11}\notin\mathcal{GH}_\infty$, we do not have to worry about the possibility of some other solution $\mathbf{Y}\in\mathcal{GH}_\infty$ such that $\mathbf{G}^{\sim}J\mathbf{G}=\mathbf{Y}^{\sim}J\mathbf{Y}$ having the property $\mathbf{Y}_{11}\in\mathcal{GH}_\infty$.

The next result is also standard [2], [10] and provides a characterization of all solutions to suboptimal Nehari extension problems.

THEOREM 2.3. *Let* $\mathbf{R}\in\mathcal{RL}_\infty^{p\times q}$ *and suppose there exists* $\mathbf{W}\in\mathcal{GH}_\infty^{p+q}$ *with* $\mathbf{W}_{11}\in\mathcal{GH}_\infty^{p}$ *satisfying* (2.1), *i.e.*, $\mathbf{G}^{\sim}J\mathbf{G}=\mathbf{W}^{\sim}J\mathbf{W}$. *Then the set of all matrices* $\mathbf{Q}\in\mathcal{RH}_\infty^{p\times q}$ *such that* $\|\mathbf{R}+\mathbf{Q}\|_\infty\leqq\gamma$ *is given by*

$$\mathbf{Q}=\mathbf{Q}_1\mathbf{Q}_2^{-1},\begin{bmatrix}\mathbf{Q}_1\\ \mathbf{Q}_2\end{bmatrix}=\mathbf{W}^{-1}\begin{bmatrix}\mathbf{U}\\ I_q\end{bmatrix},\quad \mathbf{U}\in\mathcal{RH}_\infty^{p\times q}\ \textit{with}\ \|\mathbf{U}\|_\infty\leqq\gamma. \tag{2.5}$$

*Proof.* Let $\mathbf{V}=\mathbf{W}^{-1}$ and recall $\mathbf{V}_{22}\in\mathcal{GH}_\infty$. Suppose $\mathbf{U}\in\mathcal{RH}_\infty$, $\|\mathbf{U}\|_\infty\leqq\gamma$. To prove $\mathbf{Q}\in\mathcal{RH}_\infty$ we show that $\mathbf{Q}_2\in\mathcal{GH}_\infty$. By (2.1), $\mathbf{V}J^{-1}\mathbf{V}^{\sim}=\mathbf{G}^{-1}J^{-1}(\mathbf{G}^{-1})^{\sim}$, the 2, 2 block of which gives $\mathbf{V}_{21}\mathbf{V}_{21}^{\sim}-\gamma^{-2}\mathbf{V}_{22}\mathbf{V}_{22}^{\sim}=-\gamma^{-2}I$. Hence $\|\mathbf{V}_{22}^{-1}\mathbf{V}_{21}\|_\infty<\gamma^{-1}$. It follows that $(\mathbf{V}_{22}^{-1}\mathbf{V}_{21}\mathbf{U}+I)\in\mathcal{GH}_\infty$ and hence $\mathbf{Q}_2=\mathbf{V}_{22}(\mathbf{V}_{22}^{-1}\mathbf{V}_{21}\mathbf{U}+I)\in\mathcal{GH}_\infty$, for all $\mathbf{U}\in\mathcal{RH}_\infty$ with $\|\mathbf{U}\|_\infty\leqq\gamma$. Also, with $\mathbf{Q}$ defined by (2.5) we have

$$\begin{aligned}(\mathbf{R}+\mathbf{Q})^{\sim}(\mathbf{R}+\mathbf{Q})-\gamma^2 I&=(\mathbf{Q}_2^{-1})^{\sim}\begin{bmatrix}\mathbf{U}\\ I\end{bmatrix}^{\sim}\mathbf{V}^{\sim}\mathbf{G}^{\sim}J\mathbf{G}\mathbf{V}\begin{bmatrix}\mathbf{U}\\ I\end{bmatrix}\mathbf{Q}_2^{-1}\\ &=(\mathbf{Q}_2^{-1})^{\sim}[\mathbf{U}^{\sim}\mathbf{U}-\gamma^2 I]\mathbf{Q}_2^{-1}\leqq 0.\end{aligned}$$

Conversely, suppose $\mathbf{Q}\in\mathcal{RH}_\infty$ is such that $\|\mathbf{R}+\mathbf{Q}\|_\infty\leqq\gamma$. Define

$$\begin{bmatrix}\mathbf{U}_1\\ \mathbf{U}_2\end{bmatrix}=\mathbf{W}\begin{bmatrix}\mathbf{Q}\\ I\end{bmatrix}=\mathbf{W}\mathbf{G}^{-1}\begin{bmatrix}\mathbf{R}+\mathbf{Q}\\ I\end{bmatrix}\in\mathcal{RH}_\infty.$$

Observe that $\mathbf{U}_1$, $\mathbf{U}_2\in\mathcal{RH}_\infty$ are right coprime and that

$$\mathbf{U}_1^{\sim}\mathbf{U}_1-\gamma^2\mathbf{U}_2^{\sim}\mathbf{U}_2=\begin{bmatrix}\mathbf{R}+\mathbf{Q}\\ I\end{bmatrix}^{\sim}J\begin{bmatrix}\mathbf{R}+\mathbf{Q}\\ I\end{bmatrix}\leqq 0.$$

It follows that $\mathbf{U}_2$ is invertible in $\mathcal{RL}_\infty$, and that $\mathbf{U}=\mathbf{U}_1\mathbf{U}_2^{-1}\in\mathcal{RL}_\infty$ with $\|\mathbf{U}\|_\infty\leqq\gamma$. Hence (2.5) holds, with $\mathbf{Q}_2=\mathbf{U}_2^{-1}$, and $\mathbf{Q}_1=\mathbf{Q}\mathbf{Q}_2$ and it remains to show that $\mathbf{U}\in\mathcal{RH}_\infty$. This we do by showing that $\mathbf{U}_2\in\mathcal{GH}_\infty$. To see this, observe that, since $\|\mathbf{V}_{22}^{-1}\mathbf{V}_{21}\mathbf{U}\|_\infty\leqq\|\mathbf{V}_{22}^{-1}\mathbf{V}_{21}\|_\infty\ \|\mathbf{U}\|_\infty<1$, the winding number (around the origin) of $\det\{(\mathbf{V}_{22}^{-1}\mathbf{V}_{21}\mathbf{U}+I)(j\omega)\}$ is zero. Also $\mathbf{V}_{22}^{-1}=(\mathbf{V}_{22}^{-1}\mathbf{V}_{21}\mathbf{U}+I)\mathbf{U}_2\in\mathcal{GH}_\infty$. It follows that the winding number of $\det(\mathbf{U}_2(j\omega))$ is zero, giving $\mathbf{U}_2\in\mathcal{GH}_\infty$, since $\mathbf{U}_2\in\mathcal{RH}_\infty$. □

**2.2. The unilateral model matching problem.** In the last section we considered a factorization problem associated with the Nehari extension problem $\|\mathbf{R}+\mathbf{Q}\|_\infty<\gamma$. In this case the factorization problem is particularly easy because $\mathbf{G}$ is square and invertible in $\mathcal{RL}_\infty$, a fact used in the proof of Theorem 2.3. We now turn to the unilateral model matching problem where we seek $\mathbf{Q}\in\mathcal{RH}_\infty$ such that $\|\mathbf{A}+\mathbf{BQ}\|_\infty<\gamma$, where $\mathbf{B}$ is "tall" (i.e., has more rows than columns), and the relevant "$\mathbf{G}$" is now also "tall." A related theorem is given in [14, p. 58].

The "tall" $J$-spectral factorization problem is shown to be equivalent to two spectral factorization problems together with a "square" $J$-spectral factorization problem (i.e., one of Nehari type). The techniques are similar to those used elsewhere [8], [10], [20] to reduce "two-block" distance problems to Nehari problems, but here the interpretation is in terms of the existence of solutions to $J$-spectral factorization problems.

THEOREM 2.4. *Suppose*

$$\mathbf{G}=\begin{bmatrix}\mathbf{B} & \mathbf{A}\\ 0 & I_q\end{bmatrix}\in\mathcal{RL}_\infty^{(l+q)\times(p+q)}$$

*has a left inverse in $\mathcal{RL}_\infty$. The following are equivalent*:

1. *There exists a $\mathbf{Q}\in\mathcal{RH}_\infty^{p\times q}$ such that $\|\mathbf{A}+\mathbf{BQ}\|_\infty<\gamma$;*
2. *There exists a $\mathbf{W}\in\mathcal{GH}_\infty^{p+q}$ with $\mathbf{W}_{11}\in\mathcal{GH}_\infty^{p}$ satisfying*

$$\mathbf{G}^\sim J_{lq}(\gamma)\mathbf{G}=\mathbf{W}^\sim J_{pq}(\gamma)\mathbf{W}. \tag{2.6}$$

*Furthermore, if such a $\mathbf{W}$ exists, the set of all matrices $\mathbf{Q}\in\mathcal{RH}_\infty$ satisfying $\|\mathbf{A}+\mathbf{BQ}\|_\infty\leqq\gamma$ is given by*

$$\mathbf{Q}=\mathbf{Q}_1\mathbf{Q}_2^{-1},\quad \begin{bmatrix}\mathbf{Q}_1\\ \mathbf{Q}_2\end{bmatrix}=\mathbf{W}^{-1}\begin{bmatrix}\mathbf{U}\\ I_q\end{bmatrix},\quad \mathbf{U}\in\mathcal{RH}_\infty^{p\times q}\ \text{with}\ \|\mathbf{U}\|_\infty\leqq\gamma. \tag{2.7}$$

*Proof.* $\mathbf{G}$ left invertible in $\mathcal{RL}_\infty$ is equivalent to $\mathbf{B}$ full column rank on the imaginary axis, so there exists $\mathbf{B}_0\in\mathcal{GH}_\infty$ such that $\mathbf{B}_0^\sim\mathbf{B}_0=\mathbf{B}^\sim\mathbf{B}$. Reduce to the Nehari problem as follows:

Let $\mathbf{B}_i=\mathbf{B}\mathbf{B}_0^{-1}$ and note $\mathbf{B}_i^\sim\mathbf{B}_i=I$. Let $\mathbf{B}_\perp$ be such that $[\mathbf{B}_i\mathbf{B}_\perp]$ is all-pass. Then

$$\|\mathbf{A}+\mathbf{BQ}\|_\infty\gamma\Leftrightarrow\left\|\mathbf{A}+[\mathbf{B}_i\mathbf{B}_\perp]\begin{bmatrix}\mathbf{B}_0\mathbf{Q}\\ 0\end{bmatrix}\right\|_\infty<\gamma$$

$$\Leftrightarrow\left\|\begin{bmatrix}\mathbf{R}_1\\ \mathbf{R}_2\end{bmatrix}+\begin{bmatrix}\mathbf{B}_0\mathbf{Q}\\ 0\end{bmatrix}\right\|_\infty<\gamma,\qquad \mathbf{R}=[\mathbf{B}_i\mathbf{B}_\perp]^\sim\mathbf{A}$$

$$\Leftrightarrow\|\mathbf{R}_2\|_\infty<\gamma\ \text{and}\ (\mathbf{R}_1+\mathbf{B}_0\mathbf{Q})^\sim(\mathbf{R}_1+\mathbf{B}_0\mathbf{Q})+\mathbf{R}_2^\sim\mathbf{R}_2<\gamma^2I.$$

Thus, there exists $\mathbf{Q}\in\mathcal{RH}_\infty$ such that $\|\mathbf{A}+\mathbf{BQ}\|_\infty<\gamma$ if and only if:

$$\exists\mathbf{N}\in\mathcal{GH}_\infty\ \text{with}\ \gamma^2\mathbf{N}^\sim\mathbf{N}=\mathbf{\Phi}=\gamma^2I-\mathbf{R}_2^\sim R_2=\gamma^2I_q-\mathbf{A}^\sim[I-\mathbf{B}(\mathbf{B}^\sim\mathbf{B})^{-1}\mathbf{B}^\sim]\mathbf{A}; \tag{2.8a}$$

and

$$\exists\hat{\mathbf{Q}}(=\mathbf{B}_0\mathbf{Q}\mathbf{N}^{-1})\in\mathcal{RH}_\infty\ \text{such that}\ \|\mathbf{R}_1\mathbf{N}^{-1}+\hat{\mathbf{Q}}\|_\infty<\gamma. \tag{2.8b}$$

By Theorem 2.1, there exists $\hat{\mathbf{Q}}\in\mathcal{RH}_\infty$ such that

$$\|\mathbf{R}_1\mathbf{N}^{-1}+\hat{\mathbf{Q}}\|_\infty<\gamma\Leftrightarrow\exists\mathbf{X}\in\mathcal{GH}_\infty\quad\text{with}\ \mathbf{X}_{11}\in\mathcal{GH}_\infty$$

such that

$$\begin{bmatrix}I & 0\\ (N^{-1})^\sim\mathbf{R}_1^\sim & I\end{bmatrix}J\begin{bmatrix}I & \mathbf{R}_1\mathbf{N}^{-1}\\ 0 & I\end{bmatrix}=\mathbf{X}^\sim J\mathbf{X}.$$

Note also that $\mathbf{R}_1=(\mathbf{B}_0^\sim)^{-1}\mathbf{B}^\sim\mathbf{A}$. Now observe that

$$\mathbf{G}^\sim J\mathbf{G}=\begin{bmatrix}\mathbf{B}_0 & 0\\ \mathbf{A}^\sim\mathbf{B}\mathbf{B}_0^{-1} & I\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & -\mathbf{\Phi}\end{bmatrix}\begin{bmatrix}\mathbf{B}_0 & (\mathbf{B}_0^\sim)^{-1}\mathbf{B}^\sim\mathbf{A}\\ 0 & I\end{bmatrix}. \tag{2.9}$$

It follows that $\mathbf{W}$ exists$\Leftrightarrow\mathbf{X}$ and $\mathbf{N}$ exists ($\mathbf{X}=\mathbf{W}\left[\begin{smallmatrix}\mathbf{B}_0 & 0\\ 0 & \mathbf{N}\end{smallmatrix}\right]^{-1}$) and the theorem is proved.

That (2.7) gives all solutions now follows from Theorem 2.3. □

*Remark* 2.5. The condition that $\mathbf{G}$ (equivalently $\mathbf{B}$) has a left inverse in $\mathcal{RL}_\infty$ is not necessary for there to exist a solution to the model matching problem. It is, however, a necessary condition for the existence of $\mathbf{W} \in \mathcal{GH}_\infty$ such that $\mathbf{G}^\sim \mathbf{J}\mathbf{G} = \mathbf{W}^\sim \mathbf{J}\mathbf{W}$.

**2.3. The bilateral model matching problem.** We now extend the constructions of § 2.2 to the bilateral case. That is, we seek $\mathbf{Q} \in \mathcal{RH}_\infty$ such that $\|\mathbf{A}+\mathbf{BQC}\|_\infty < \gamma$, with $\mathbf{B}$ "tall" and $\mathbf{C}$ "wide." The technique is based on reduction to the unilateral case, and the result involves two $J$-spectral factorizations.

THEOREM 2.6. *Suppose* $\mathbf{A} \in \mathcal{RL}_\infty^{l\times p}$, $\mathbf{B} \in \mathcal{RL}_\infty^{l\times q}$ *and* $\mathbf{C} \in \mathcal{RL}_\infty^{m\times p}$. *Suppose also that* $\mathbf{B}$ *has a left inverse and* $\mathbf{C}$ *has a right inverse in the appropriate* $\mathcal{RL}_\infty$ *spaces. Let* $\mathbf{B} = \mathbf{B}_a\mathbf{B}_s$ *in which* $\mathbf{B}_a \in \mathcal{RL}_\infty^{l\times l}$ *is all-pass and* $\mathbf{B}_s \in \mathcal{RH}_\infty^{l\times q}$. *Then there exists a* $\mathbf{Q} \in \mathcal{RH}_\infty^{q\times m}$ *such that* $\|\mathbf{A}+\mathbf{BQC}\|_\infty < \gamma$ *if and only if*

1. *There exists a* $\mathbf{V} \in \mathcal{GH}_\infty^{m+l}$ *with* $\mathbf{V}_{11} \in \mathcal{GH}_\infty^{m}$ *satisfying*

$$\mathbf{H}J_{pl}(\gamma)\mathbf{H}^\sim = \mathbf{V}J_{ml}(\gamma)\mathbf{V}^\sim, \qquad \mathbf{H} = \begin{bmatrix} \mathbf{C} & 0 \\ \mathbf{B}_a^\sim \mathbf{A} & I_l \end{bmatrix} \tag{2.10}$$

*and*

2. *There exists a* $\mathbf{W} \in \mathcal{GH}_\infty^{q+m}$ *with* $\mathbf{W}_{11} \in \mathcal{GH}_\infty^{q}$ *satisfying*

$$\mathbf{G}^\sim J_{lm}(\gamma)\mathbf{G} = \mathbf{W}^\sim J_{qm}(\gamma)\mathbf{W}, \qquad \mathbf{G} = \hat{J}\mathbf{V}^{-1}\hat{J}^* \begin{bmatrix} \mathbf{B}_s & 0 \\ 0 & I_m \end{bmatrix} \tag{2.11}$$

*where*

$$\hat{J} = \begin{bmatrix} 0 & -I_l \\ I_m & 0 \end{bmatrix}. \tag{2.12}$$

*In this case, the set of all matrices* $\mathbf{Q} \in \mathcal{RH}_\infty^{q\times m}$ *such that* $\|\mathbf{A}+\mathbf{BQC}\|_\infty \leqq \gamma$ *is given by*

$$\mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2^{-1}, \quad \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{bmatrix} = \mathbf{W}^{-1}\begin{bmatrix} \mathbf{U} \\ I_m \end{bmatrix}, \quad \mathbf{U} \in \mathcal{RH}_\infty^{q} \text{ with } \|\mathbf{U}\|_\infty \leqq \gamma. \tag{2.13}$$

*Proof.* We may assume, without loss of generality, that $\mathbf{B} \in \mathcal{RH}_\infty$, since $\|\mathbf{A}+\mathbf{BQC}\|_\infty \leqq \gamma \Leftrightarrow \|\mathbf{B}_a^\sim \mathbf{A} + \mathbf{B}_s\mathbf{Q}C\|_\infty \leqq \gamma$.

With $\mathbf{B} \in \mathcal{RH}_\infty$ we see that 1 is necessary by applying Theorem 2.4 to the problem $\mathbf{A}^* + \mathbf{C}^*\hat{\mathbf{Q}}$, where $\hat{\mathbf{Q}} = (\mathbf{BQ})^*$.

Let $\mathbf{C}_0 \in \mathcal{GH}_\infty$ be such that $\mathbf{CC}^\sim = \mathbf{C}_0\mathbf{C}_0^\sim$ and define $\mathbf{C}_i = \mathbf{C}_0^{-1}\mathbf{C}$. Let $\mathbf{C}_\perp$ be such that $\left[\begin{smallmatrix} \mathbf{C}_i \\ \mathbf{C}_\perp \end{smallmatrix}\right]$ is all-pass. Define $\mathbf{R}$ by

$$\mathbf{R} = [\mathbf{R}_1\mathbf{R}_2] = \mathbf{A}\begin{bmatrix} \mathbf{C}_i \\ \mathbf{C}_\perp \end{bmatrix}^\sim.$$

As in the proof of Theorem 2.4, the existence of $\mathbf{V}$ satisfying (2.10) implies that there exists $\mathbf{M} \in \mathcal{GH}_\infty$ such that

$$\gamma^2\mathbf{MM}^\sim = \gamma^2 I - \mathbf{R}_2\mathbf{R}_2^\sim.$$

So $\mathbf{Q} \in \mathcal{RH}_\infty$ satisfies $\|\mathbf{A}+\mathbf{BQC}\|_\infty < \gamma \Leftrightarrow \mathbf{V}$ exists and $\|\mathbf{M}^{-1}\mathbf{R}_1 + \mathbf{M}^{-1}\mathbf{BQC}_0\|_\infty < \gamma$. Assuming that the necessary condition 1 holds, we therefore need to show that there exists $\mathbf{Q} \in \mathcal{RH}_\infty$ such that $\|\mathbf{M}^{-1}\mathbf{R}_1 + \mathbf{M}^{-1}\mathbf{BQC}_0\|_\infty < \gamma \Leftrightarrow$ there exists $\mathbf{W}$ satisfying (2.11). But, since $\mathbf{C}_0 \in \mathcal{GH}_\infty$, this is just a unilateral model matching problem. By Theorem 2.4 we know that $\mathbf{Q}$ exists if and only if there exists $\mathbf{Y} \in \mathcal{GH}_\infty$ with $\mathbf{Y}_{11} \in \mathcal{GH}_\infty$ such that

$$\mathbf{Y}^\sim J\mathbf{Y} = \mathbf{P}_1^\sim J\mathbf{P}_1, \qquad \mathbf{P}_1 = \begin{bmatrix} \mathbf{M}^{-1}\mathbf{B} & \mathbf{M}^{-1}\mathbf{R}_1 \\ 0 & I \end{bmatrix},$$

and that $\mathbf{Y}^{-1}$ "generates" all $\mathbf{QC}_0$'s. But such a $\mathbf{Y}$ exists if and only if there exists $\mathbf{W} \in \mathcal{GH}_\infty$ with $\mathbf{W}_{11} \in \mathcal{GH}_\infty$ satisfying

$$\mathbf{W}^\sim J\mathbf{W} = \mathbf{P}^\sim J\mathbf{P}, \qquad \mathbf{P} = \mathbf{P}_1 \begin{bmatrix} I & 0 \\ 0 & \mathbf{C}_0^{-1} \end{bmatrix},$$

and furthermore $\mathbf{W}^{-1}$ "generates" all $\mathbf{Q}$'s. It remains therefore to show that $\mathbf{P}^\sim J\mathbf{P} = \mathbf{G}^\sim J\mathbf{G}$, with $\mathbf{G}$ as in (2.11):

$$\begin{aligned} \mathbf{P} &= \begin{bmatrix} \mathbf{M}^{-1} & \mathbf{M}^{-1}\mathbf{R}_1\mathbf{C}_0^{-1} \\ 0 & \mathbf{C}_0^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{B} & 0 \\ 0 & I \end{bmatrix} \\ &= \hat{J} \begin{bmatrix} \mathbf{C}_0 & 0 \\ \mathbf{R}_1 & \mathbf{M} \end{bmatrix}^{-1} \hat{J}^* \begin{bmatrix} \boldsymbol{B} & 0 \\ 0 & I \end{bmatrix}. \end{aligned} \tag{2.14}$$

Now observe that $\hat{J}^* J \hat{J} = -\gamma^2 J^{-1}$, that $\hat{J}\hat{J}^* = I$ and that

$$\begin{bmatrix} \mathbf{C}_0 & 0 \\ \mathbf{R}_1 & \mathbf{M} \end{bmatrix} J \begin{bmatrix} \mathbf{C}_0 & 0 \\ \mathbf{R}_1 & \mathbf{M} \end{bmatrix}^\sim = \mathbf{HJH}^\sim = \mathbf{VJV}^\sim.$$

It is then easy to check that $\mathbf{G}^\sim \mathbf{JG} = \mathbf{P}^\sim \mathbf{JP}$. □

*Remark* 2.7. Suppose $\mathbf{V}$ as in part one of the Theorem exists and that $\mathbf{G}$ is as given in (2.11). Since $\mathbf{G}^\sim J\mathbf{G} = \mathbf{P}^\sim J\mathbf{P}$ with $\mathbf{P}$ as in (2.14), it follows that if $\mathbf{W}$ satisfies (2.11) then the condition $\mathbf{W}_{11}^*\mathbf{W}_{11} - \gamma^2\mathbf{W}_{21}^*\mathbf{W}_{21} \geqq 0$ of Lemma 2.2 part two will be satisfied. Hence if *any* $\mathbf{W} \in \mathcal{GH}_\infty$ satisfying (2.11) has the property $\mathbf{W}_{11} \in \mathcal{GH}_\infty$, then *all* do.

**3. *J*-spectral factorization theory.** In the last section we solved the model matching problem in terms of $J$-spectral factorization. For the most part, the arguments made no reference to state space ideas. It is this connection that we now investigate. Specifically, we will relate the existence of $J$-spectral factors to the existence of solutions to indefinite algebraic Riccati equations. The main tool for this work is the state space factorization theory of Bart, Gohberg, and Kaashoek [6]. We begin with a little notation.

DEFINITION 3.1. A matrix $H \in \mathbb{C}^{2n\times 2n}$ is a Hamiltonian matrix if $\hat{J}H = H^*\hat{J}^*$, $\hat{J} = \left[\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right]$. If $H \in \mathbb{C}^{2n\times 2n}$ is a Hamiltonian matrix, we say $H \in \operatorname{dom}(\operatorname{Ric})$ if there exists $Q \in \mathbb{C}^{n\times n}$ and $\Lambda \in \mathbb{C}^{n\times n}$ such that

$$H \begin{bmatrix} I_n \\ Q \end{bmatrix} = \begin{bmatrix} I_n \\ Q \end{bmatrix} \Lambda$$

with $\Lambda$ asymptotically stable (i.e., $\operatorname{In}(\Lambda) = (0, n, 0)$). If $H \in \operatorname{dom}(\operatorname{Ric})$, then $Q = \operatorname{Ric}(H)$ is Hermitian and satisfies the algebraic Riccati equation

$$QH_{11} + H_{11}^*Q + QH_{12}Q - H_{21} = 0$$

with

$$H_{11} + H_{12}Q = \Lambda \text{ asymptotically stable.}$$

We now prove the equivalence between $J$-spectral factorization and the solution of indefinite Riccati equations. A related result is in [5].

THEOREM 3.2. *Suppose* $\mathbf{G} \in \mathcal{RH}_\infty^{(p+q)\times(m+l)}$ *is given by the realization* $\mathbf{G}(s) = D + C(sI - A)^{-1}B$, *with* $A \in \mathbb{C}^{n\times n}$ *asymptotically stable* (*i.e.*, $\operatorname{In}(A) = (0, n, 0)$). *Then there exists a* $\mathbf{W} \in \mathcal{GH}_\infty$ *such that*

$$\mathbf{G}^\sim J_{pq}(\gamma)\mathbf{G} = \mathbf{W}^\sim \mathbf{J}_{ml}(\gamma)\mathbf{W} \tag{3.1}$$

*if and only if*:

1. *There exists a nonsingular matrix* $W_\infty \in \mathbb{C}^{(m+l)\times(m+l)}$ *such that*

$$D^*J_{pq}(\gamma)D = W_\infty^* J_{ml}(\gamma) W_\infty \tag{3.2}$$

*and*

2. $H \in \operatorname{dom}(\mathrm{Ric})$, *where*

$$H = \begin{bmatrix} A & 0 \\ -C^*JC & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C^*JD \end{bmatrix} (D^*JD)^{-1}[D^*JC \quad B^*]. \tag{3.3}$$

(*Here*, $J = J_{pq}(\gamma)$).

*In this case* $\mathbf{W} \in \mathcal{GH}_\infty$ *satisfies* (3.1) *if and only if, for some solution* $W_\infty$ *of* (3.2), $\mathbf{W}$ *is given by*

$$\mathbf{W}(s) = W_\infty + L(sI - A)^{-1}B \tag{3.4a}$$

*where*

$$L = J_{ml}^{-1}(\gamma) W_\infty^{-*}(D^*J_{pq}(\gamma)C + B^*Q) \tag{3.4b}$$

$$Q = \mathrm{Ric}\,(H). \tag{3.4c}$$

*Proof.* Suppose 1 and 2 hold. Then $Q = \mathrm{Ric}\,(H)$ implies that $A - B(D^*JD)^{-1}[D^*JC + B^*Q] = A - BW_\infty^{-1}L$ is asymptotically stable. It follows, with $\mathbf{W}$ defined by (3.4), that $\mathbf{W} \in \mathcal{GH}_\infty$. Now note that the Riccati equation for $Q$ can be written as

$$QA + A^*Q + C^*JC - L^*JL = 0 \tag{3.5}$$

with $L$ as in (3.4b). Hence

$$\begin{aligned} \mathbf{W}^\sim J\mathbf{W} &= [W_\infty^* + B^*(-sI - A^*)^{-1}L^*]J[W_\infty + L(sI - A)^{-1}B] \\ &= D^*JD + [D^*JC + B^*Q](sI - A)^{-1}B + B^*(-sI - A^*)^{-1}[C^*JD + QB] \\ &\quad - B^*(-sI - A^*)^{-1}[Q(sI - A) + (-sI - A^*)Q - C^*JC](sI - A)^{-1}B \\ &= [D^* + B^*(-sI - A^*)^{-1}C^*]J[D + C(sI - A)^{-1}B] \\ &= \mathbf{G}^\sim J\mathbf{G}. \end{aligned}$$

That (3.4) gives all $\mathbf{W}$ follows from Lemma 2.2.

Now suppose there exists $\mathbf{W} \in \mathcal{GH}_\infty$ such that $\mathbf{G}^\sim J\mathbf{G} = \mathbf{W}^\sim J\mathbf{W}$. It follows by evaluating (3.1) at $s = \infty$ that (3.2) has a solution $W_\infty = \mathbf{W}(\infty)$. Let $\mathbf{M} = \mathbf{G}^\sim J\mathbf{G}$, $\mathbf{M}_+ = \mathbf{W}^\sim J$ and $\mathbf{M}_- = \mathbf{W}$. We then have $\mathbf{M} = \mathbf{M}_+\mathbf{M}_-$, $\mathbf{M}_- \in \mathcal{GH}_\infty$, $\mathbf{M}_+^\sim \in \mathcal{GH}_\infty$, which is a canonical Wiener–Hopf factorization of $\mathbf{M}$. To establish that $H \in \operatorname{dom}(\mathrm{Ric})$, we use the factorization theorem of Bart, Gohberg, and Kaashoek [6] (see also [10, Chap. 7]). The relevant result is the following theorem.

THEOREM (BGK). *Suppose* $\mathbf{M} = \hat{D} + \hat{C}(sI - \hat{A})^{-1}\hat{B}$ *with* $(\hat{A}, \hat{B}, \hat{C})$ *minimal,* $\hat{A} \in \mathbb{C}^{n\times n}$. *Then* $\mathbf{M}$ *has a canonical Wiener–Hopf factorization if and only if* $\hat{D}$ *is invertible,* $\hat{A}$ *and* $\hat{A}^\times = \hat{A} - \hat{B}\hat{D}^{-1}\hat{C}$ *have no imaginary axis eigenvalues and* $X_+(\hat{A})$ *and* $X_-(\hat{A}^\times)$ *are complementary* (*i.e.,* $X_+(\hat{A}) \cap X_-(\hat{A}^\times) = \{0\}$ *and* $X_+(\hat{A}) \cup X_-(\hat{A}^\times) = \mathbb{C}^n$), *where* $X_+(\hat{A})$ (*respectively,* $X_-(\hat{A})$) *is the subspace of* $\mathbb{C}^n$ *spanned by the generalized eigenvectors of* $\hat{A}$ *corresponding to eigenvalues* $\lambda$ *of* $\hat{A}$ *such that* $\mathrm{Re}\,(\lambda) > 0$ (*respectively,* $\mathrm{Re}\,(\lambda) < 0$).

The problem in applying this theorem in our case is that the realization of $\mathbf{M} = \mathbf{G}^\sim J\mathbf{G}$ is not required to be minimal under our assumptions. The assumption that $A$ is asymptotically stable in the realization (3.1) allows us to avoid the minimality condition by applying the BGK theorem to a minimal realization of $\mathbf{M}=\mathbf{G}^{\sim}J\mathbf{G}$ and then showing that the dilation to the original realization does not destroy the complementarity of the subspaces. We are going to do this in two steps: First we assume that $(A, B)$ is controllable in the realization of $G$.

*Temporary assumption.* $(A, B)$ controllable.

Since $A$ is asymptotically stable, there exists $P=P^*$ (unique) such that

$$PA+A^*P+C^*JC=0.$$

It follows that $\mathbf{G}^{\sim}J\mathbf{G}$ is given by

$$\mathbf{G}^{\sim}J\mathbf{G} \stackrel{s}{=} \left[\begin{array}{cc|c} A & 0 & B \\ 0 & -A^* & -K^* \\ \hline K & B^* & D^*JD \end{array}\right], \quad K=D^*JC+B^*P. \tag{3.6}$$

Since $(A, B)$ is controllable, the unobservable (respectively, uncontrollable) modes of the realization (3.6) are the unobservable modes of (K, A) (respectively, uncontrollable modes of $(-A^*, -K^*)$).

Therefore, without loss of generality suppose A, B, C are such that

$$A=\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad K=[K_1 \quad 0] \quad (K_1, A_{11}) \text{ observable.} \tag{3.7}$$

A minimal realization of $\mathbf{G}^{\sim}J\mathbf{G}$ is given by

$$\mathbf{G}^{\sim}J\mathbf{G} \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ 0 & -A_{11}^* & -K_1^* \\ \hline K_1 & B_1^* & D^*JD \end{array}\right] = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]. \tag{3.8}$$

By the BGK theorem, since $\mathbf{G}^{\sim}J\mathbf{G}$ has a canonical factorization, the Hamiltonian matrix $\hat{A}^{\times}=\hat{A}-\hat{B}\hat{D}^{-1}\hat{C}$ has no imaginary axis eigenvalues. Hence there exists nonsingular matrix $\hat{X}$ such that

$$\hat{A}^{\times}\hat{X}=\hat{X}T, \quad T=\begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} \quad \operatorname{Re}\{\lambda_i(T_1)\}<0, \operatorname{Re}\{\lambda_i(T_3)\}>0 \; i=1,\cdots,n. \tag{3.9}$$

Partition $\hat{X}$ conformably with $T$. We see from (3.8) and (3.9) that

$$X_+(\hat{A})=\operatorname{Im}\begin{bmatrix} 0 \\ I \end{bmatrix} \quad \text{and} \quad X_-(\hat{A}^{\times})=\operatorname{Im}\begin{bmatrix} \hat{X}_{11} \\ \hat{X}_{21} \end{bmatrix}.$$

By the BGK theorem $X_+(\hat{A})$ and $X_-(\hat{A}^{\times})$ are complementary, i.e.,

$$\begin{bmatrix} \hat{X}_{11} & 0 \\ \hat{X}_{21} & I \end{bmatrix} \quad \text{nonsingular.} \tag{3.10}$$

Hence $\hat{Q}=\hat{X}_{21}\hat{X}_{11}^{-1}=\operatorname{Ric}(\hat{A}^{\times})$.

Now return to the realization (3.6) with $(A, B, K)$ as in (3.7). Consider

$$\begin{aligned} \tilde{H} &= \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -K^* \end{bmatrix} (D^*JD)^{-1}[K \quad B^*] \\ &= \begin{bmatrix} \hat{A}_{11}^{\times} & 0 & \hat{A}_{12}^{\times} & \tilde{H}_{14} \\ \tilde{H}_{21} & A_{22} & \tilde{H}_{23} & \tilde{H}_{24} \\ \hat{A}_{21}^{\times} & 0 & \hat{A}_{22}^{\times} & \tilde{H}_{34} \\ 0 & 0 & 0 & -A_{22}^* \end{bmatrix}. \end{aligned}$$

Observe that

$$\tilde{H}\begin{bmatrix} \hat{X}_{11} & 0 \\ 0 & I \\ \hat{X}_{21} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{X}_{11} & 0 \\ 0 & I \\ \hat{X}_{21} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 & 0 \\ \tilde{T}_{21} & A_{22} \end{bmatrix}$$

and furthermore, with $H$ as in (3.3) we have that $\left[\begin{smallmatrix} I & 0 \\ -P & I \end{smallmatrix}\right] H \left[\begin{smallmatrix} I & 0 \\ P & I \end{smallmatrix}\right] = \tilde{H}$. It follows that $H \in \operatorname{dom}(\mathrm{Ric})$ and $Q = \mathrm{Ric}(H) = \left[\begin{smallmatrix} \hat{Q} & 0 \\ 0 & 0 \end{smallmatrix}\right] + P$.

*Removal of the controllability assumption.* Suppose $(A, B, C)$ is in controllable canonical form:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad (A_{11}, B_1) \text{ controllable}, \qquad C = [C_1 \quad C_2] \tag{3.11}$$

and define $\tilde{H}$ by

$$\tilde{H} = \begin{bmatrix} A_{11} & 0 \\ -C_1^* J C_1 & -A_{11}^* \end{bmatrix} - \begin{bmatrix} B_1 \\ -C_1^* J D \end{bmatrix} (D^* J D)^{-1} [D^* J C_1 \ B_1^*]. \tag{3.12}$$

Applying the above result (i.e., with the controllability assumption), we have $\tilde{H} \in \operatorname{dom}(\mathrm{Ric})$ and so there exists $\tilde{Q}$ such that

$$\tilde{H}\begin{bmatrix} I \\ \tilde{Q} \end{bmatrix} = \begin{bmatrix} I \\ \tilde{Q} \end{bmatrix}\tilde{\Lambda} \quad \text{with } \tilde{\Lambda} \text{ asymptotically stable (i.e., } \mathrm{In}(\tilde{\Lambda}) = (0, n, 0)).$$

Now consider $H$ defined by (3.3). Since $(A, B, C)$ is in controllable canonical form, $H$ is as follows:

$$H = \begin{bmatrix} \tilde{H}_{11} & H_{12} & \tilde{H}_{12} & 0 \\ 0 & A_{22} & 0 & 0 \\ \tilde{H}_{21} & H_{32} & -\tilde{H}_{11}^* & 0 \\ H_{32}^* & H_{42} & -H_{12}^* & -A_{22}^* \end{bmatrix}.$$

Since $\tilde{\Lambda} = \tilde{H}_{11} + \tilde{H}_{12}\tilde{Q}$ and $A_{22}$ are asymptotically stable, there exist $Q_{12}$ and $Q_{22}$ such that

$$\begin{aligned} &Q_{12}A_{22} + \tilde{\Lambda}^* Q_{12} = H_{32} - \tilde{Q} H_{12} \\ &Q_{22}A_{22} + A_{22}^* Q_{22} = H_{42} - H_{12}^* Q_{12} - Q_{12}^*(H_{12} + \tilde{H}_{12} Q_{12}), \end{aligned}$$

it follows that

$$H\begin{bmatrix} I & 0 \\ 0 & I \\ \tilde{Q} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \\ \tilde{Q} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}\begin{bmatrix} \tilde{\Lambda} & H_{12} + \tilde{H}_{12} Q_{12} \\ 0 & A_{22} \end{bmatrix}$$

and we see that $H \in \operatorname{dom}(\mathrm{Ric})$.

**4. State-space solution of the model matching problem.** We are now ready to apply the $J$-spectral factorization results to the model matching problem associated via (1.6) with the standard $\mathcal{H}_\infty$ generalized regulator problem [8], [10], [23].

**4.1. State-space preliminaries.** Throughout the remainder of the paper we will assume that $\mathbf{P}(s)$ has state-space realization given by

$$\mathbf{P} \overset{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{4.1}$$

where we assume:

A1. $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable.

A2. $D_{12}^*D_{12}=I$ and $D_{21}D_{21}^*=I$. We will also denote the unitary completions of $D_{12}$ and $D_{12}$ as $D_\perp$ and $\tilde{D}_\perp$.

As has already been noted [24], [13], the assumption implicit in (4.1) that $D_{11}=0$, $D_{22}=0$ can be made without loss of generality—by using a loop shifting argument which in the present context amounts to solving the factorization at $\infty$ problem first (see (3.2)) and introducing a ($\gamma$-dependent) change of variables. It is of course also possible to directly tackle the factorizations without assuming any special structure for $D$, but this considerably increases the length of the calculations.

By A1, there exist state feedback and output injection matrices $F$ and $H$ such that $A-B_2F$ and $A-HC_2$ are asymptotically stable. A doubly coprime factorization of $\mathbf{P}_{22}$, i.e.,

$$\mathbf{P}_{22}=\mathbf{N}_r\mathbf{D}_r^{-1}=\mathbf{D}_l^{-1}\mathbf{N}_l$$

with

$$\begin{bmatrix} \mathbf{V}_r & \mathbf{U}_r \\ -\mathbf{N}_l & \mathbf{D}_l \end{bmatrix}\begin{bmatrix} \mathbf{D}_r & -\mathbf{U}_l \\ \mathbf{N}_r & \mathbf{V}_l \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

is given by

$$\begin{bmatrix} \mathbf{D}_r & -\mathbf{U}_l \\ \mathbf{N}_r & \mathbf{V}_l \end{bmatrix} \overset{s}{=} \left[\begin{array}{c|cc} A-B_2F & B_2 & H \\ \hline -F & I & 0 \\ C_2 & 0 & I \end{array}\right]. \tag{4.2}$$

We then get the $\mathbf{T}_{ij}$'s of the associated model matching problem as [8], [10], [23]

$$\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & 0 \end{bmatrix} \overset{s}{=} \left[\begin{array}{cc|cc} A-B_2F & B_2F & B_1 & B_2 \\ 0 & A-HC_2 & B_1-HD_{21} & 0 \\ \hline C_1-D_{12}F & D_{12}F & 0 & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right]. \tag{4.3}$$

LEMMA 4.1. $\mathbf{T}_{21}$ (*respectively*, $\mathbf{T}_{12}$) *has a right* (*respectively, left*) *inverse in* $\mathcal{RL}_\infty$ *if and only if* $\left[\begin{smallmatrix} A-\lambda I & B_1 \\ C_2 & D_{21} \end{smallmatrix}\right]$ *has full row rank* (*respectively*, $\left[\begin{smallmatrix} A-\lambda I & B_2 \\ C_1 & D_{12} \end{smallmatrix}\right]$ *has full column rank*) *for all* $\lambda+\bar{\lambda}=0$.

*Proof.* $\mathbf{T}_{21}$ right invertible in $\mathcal{RL}_\infty \Leftrightarrow \mathbf{T}_{21}(\lambda)$ full row rank for all $\lambda+\bar{\lambda}=0$. Since $A-HC_2$ is asymptotically stable, $(A-HC_2-\lambda I)$ is nonsingular for any $\lambda+\bar{\lambda}=0$. Hence for $\lambda+\bar{\lambda}=0$,

$$u^*\mathbf{T}_{21}(\lambda)=0, \qquad u\neq 0$$

$$\Leftrightarrow [x^* \quad u^*]\begin{bmatrix} A-HC_2-\lambda I & B_1-HD_{21} \\ C_2 & D_{21} \end{bmatrix}=0, \qquad x\neq 0, \qquad u\neq 0$$

$$\Leftrightarrow [x^* \quad u^*]\begin{bmatrix} I & -H \\ 0 & I \end{bmatrix}\begin{bmatrix} A-\lambda I & B_1 \\ C_2 & D_{21} \end{bmatrix}=0. \qquad \square$$

**4.2. A unilateral model matching problem.** We now derive necessary and sufficient conditions, in terms of a nonnegative definiteness condition on the solution of an indefinite Riccati equation, for the existence of $\mathbf{Q} \in \mathcal{RH}_\infty$ such that $\|\mathbf{T}_{11} + \mathbf{Q}\mathbf{T}_{21}\|_\infty < \gamma$. We do this via Theorems 2.4 and 3.2. Consider $\mathbf{H}$ defined by

$$\mathbf{H} = \begin{bmatrix} \mathbf{T}_{21} & 0 \\ \mathbf{T}_{11} & I_l \end{bmatrix}. \tag{4.4}$$

By Theorem 2.4, applied to the matrix $\mathbf{G}(s) = \mathbf{H}(\bar{s})^*$, we need to solve the following factorization problem.

FACTORIZATION PROBLEM P1. *With* $\mathbf{H} \in \mathcal{RH}_\infty^{(m+l)\times(p+l)}$ *defined by* (4.4), *find* $\mathbf{V} \in \mathcal{GH}_\infty^{m+l}$ *with* $\mathbf{V}_{11} \in \mathcal{GH}_\infty^m$ *such that*

$$\mathbf{H} J_{pl}(\gamma) \mathbf{H}^{\sim} = \mathbf{V} J_{ml}(\gamma) \mathbf{V}^{\sim}. \tag{4.5}$$

THEOREM 4.2. *Let* $\mathbf{H}$ *be as in* (4.4). *Then Problem* P1 *has a solution if and only if* $H_Y \in \operatorname{dom}(\operatorname{Ric})$ *and* $\operatorname{Ric}(H_Y) \geqq 0$, *where*

$$H_Y = \begin{bmatrix} A^* & 0 \\ -B_1 B_1^* & -A \end{bmatrix} - \begin{bmatrix} C_2^* & C_1^* \\ -B_1 D_{21}^* & 0 \end{bmatrix} J^{-1} \begin{bmatrix} D_{21} B_1^* & C_2 \\ 0 & C_1 \end{bmatrix}. \tag{4.6}$$

$(J = J_{pl}(\gamma))$. *In this case, a solution* $\mathbf{V}$ *to Problem* P1 *is given by*

$$\mathbf{V} = \begin{bmatrix} \mathbf{D}_l & 0 \\ -\mathbf{P}_{12}\mathbf{U}_l\mathbf{D}_l & I \end{bmatrix} \mathbf{V}_1 \tag{4.7}$$

*where*

$$\mathbf{V}_1 \overset{s}{=} \left[\begin{array}{c|cc} A & M_1 & M_2 \\ \hline C_2 & I_m & 0 \\ C_1 & 0 & I_l \end{array}\right] \tag{4.8a}$$

*and*

$$M = [M_1 \quad M_2] = [Y_\infty C_2^* + B_1 D_{21}^* \quad -\gamma^{-2} Y_\infty C_1^*] \tag{4.8b}$$

*with*

$$Y_\infty = \operatorname{Ric}(H_Y). \tag{4.8c}$$

*Proof.* Write $\mathbf{H}$ as

$$\mathbf{H} = \mathbf{H}_1 \mathbf{H}_2 \tag{4.9}$$

where

$$\mathbf{H}_1 \overset{s}{=} \left[\begin{array}{c|cc} A - B_2 F & H & 0 \\ \hline 0 & I & 0 \\ C_1 - D_{12} F & 0 & I \end{array}\right] \tag{4.10a}$$

$$\mathbf{H}_2 \overset{s}{=} \left[\begin{array}{c|cc} A - HC_2 & B_1 - HD_{21} & 0 \\ \hline C_2 & D_{21} & 0 \\ C_1 & 0 & I \end{array}\right]. \tag{4.10b}$$

Since $\mathbf{H}_1 \in \mathcal{GH}_\infty$ and has the particular form $\mathbf{H}_1 = \left[\begin{smallmatrix} I & 0 \\ \mathbf{X} & I \end{smallmatrix}\right]$, we see that $\mathbf{V}$ solves Problem P1 if and only if there exists $\mathbf{V}_2 \in \mathcal{GH}_\infty$ with $(\mathbf{V}_2)_{11} \in \mathcal{GH}_\infty$ such that $\mathbf{H}_2 J \mathbf{H}_2^{\sim} = \mathbf{V}_2 J \mathbf{V}_2^{\sim}$;

$\mathbf{V}$ and $\mathbf{V}_2$ are related via $\mathbf{V}=\mathbf{H}_1\mathbf{V}_2$. Applying Theorem 3.2, we see that $H_Y \in \mathrm{dom}\,(\mathrm{Ric})$ is necessary and sufficient for the existence of $\mathbf{V}_2 \in \mathcal{GH}_\infty$, and that $\mathbf{V}_2$ is given by

$$\mathbf{V}_2 \overset{s}{=} \left[\begin{array}{c|cc} A-HC_2 & M_1-H & M_2 \\ \hline C_2 & I_m & 0 \\ C_1 & 0 & I_l \end{array}\right] \tag{4.11}$$

with $M$ as in (4.8b).

We now claim $(\mathbf{V}_2)_{11} \in \mathcal{GH}_\infty \Leftrightarrow Y_\infty = \mathrm{Ric}\,(H_Y) \geqq 0$. Since $(A-HC_2)$ is asymptotically stable, it follows that $(\mathbf{V}_2)_{11} \in \mathcal{GH}_\infty \Leftrightarrow A-M_1C_2$ is asymptotically stable. We therefore need to show that $A-M_1C_2$ is asymptotically stable $\Leftrightarrow Y_\infty \geqq 0$. To see this, write the Riccati equation for $Y_\infty$ as

$$AY_\infty + Y_\infty A^* + B_1B_1^* - MJM^* = 0. \tag{4.12}$$

Since $M_1 = Y_\infty C_2^* + B_1 D_{21}^*$ we see that

$$M_1M_1^* = Y_\infty C_2^* M_1^* + M_1 C_2 Y_\infty - Y_\infty C_2^* C_2 Y_2 + B_1 D_{21}^* D_{21} B_1^*. \tag{4.13}$$

Substituting into (4.12) we obtain

$$(A-M_1C_2)Y_\infty + Y_\infty(A-M_1C_2)^* + [Y_\infty C_2^* \quad \gamma M_2 \quad B_1\tilde{D}_\perp^*] \begin{bmatrix} C_2 Y_\infty \\ \gamma M_2^* \\ \tilde{D}_\perp B_1^* \end{bmatrix} = 0. \tag{4.14}$$

Since $(A-M_1C_2-M_2C_1)$ is asymptotically stable, $(A-M_1C_2, M_2)$ is stabilizable. Hence [26, Lemma 12.2], $Y_\infty \geqq 0 \Leftrightarrow (A-M_1C_2)$ is asymptotically stable.

It remains to verify the formula (4.7) for $\mathbf{V}=\mathbf{H}_1\mathbf{V}_2$. This is easily done via a state space calculation. □

*Remark* 4.3. The decomposition of $\mathbf{V}$ in (4.7) is analogous to the decomposition of $\mathbf{H}$ as

$$\mathbf{H} = \begin{bmatrix} \mathbf{D}_l & 0 \\ -\mathbf{P}_{12}\mathbf{U}_l\mathbf{D}_l & I \end{bmatrix} \begin{bmatrix} \mathbf{P}_{21} & 0 \\ \mathbf{P}_{11} & I \end{bmatrix} \tag{4.15}$$

(see (1.6)). It follows that $\mathbf{V}_1$ is a solution to the $J$-factorization observed in [12], namely

$$\mathbf{V}_1 J \mathbf{V}_1^{\sim} = \begin{bmatrix} \mathbf{P}_{21} & 0 \\ \mathbf{P}_{11} & I \end{bmatrix} J \begin{bmatrix} \mathbf{P}_{21} & 0 \\ \mathbf{P}_{11} & I \end{bmatrix} 2. \tag{4.16}$$

*Remark* 4.4. A necessary condition for $H_Y \in \mathrm{dom}\,(\mathrm{Ric})$ is that $H_Y$ have no imaginary axis eigenvalues. It is not difficult to show that a necessary condition for this is that $\left[\begin{smallmatrix} A-\lambda I & B_1 \\ C_2 & D_{21} \end{smallmatrix}\right]$ be full row rank for all $\lambda+\bar{\lambda}=0$, since

$$[x_1^* \quad x_2^*] \begin{bmatrix} A-\lambda I & B_1 \\ C_2 & D_{21} \end{bmatrix} = 0 \Rightarrow x_1^*(A - B_1D_{21}^*C_2) = 0 \quad \text{and} \quad x_1^*B_1(I-D_{21}^*D_{21}) = 0$$

$$\Rightarrow [0 \quad x_1^*]H_Y = \lambda[0 \quad x_1^*].$$

An alternative view of this necessary condition is obtained by considering the $J$-spectral factorization directly, since a necessary condition for the factorization (4.5) to exist (with $\mathbf{V} \in \mathcal{GH}_\infty$) is that $\mathbf{H}$ (equivalently $\mathbf{T}_{21}$) be right invertible in $\mathcal{RL}_\infty$. This is equivalent to $\left[\begin{smallmatrix} A-\lambda I & B_1 \\ C_2 & D_{21} \end{smallmatrix}\right]$ full row rank for all $\lambda+\bar{\lambda}$ by Lemma 4.1.

*Remark* 4.5. The problem of finding $\mathbf{Q} \in \mathcal{RH}_\infty$ such that $\|\mathbf{T}_{11}+\mathbf{T}_{12}\mathbf{Q}\|_\infty < \gamma$ can be tackled in an entirely analogous way, applying Theorem 3.2 to the matrix

$$\mathbf{E} = \begin{bmatrix} \mathbf{T}_{12} & \mathbf{T}_{11} \\ 0 & I \end{bmatrix}. \tag{4.17}$$

The relevant conditions are:

1. $H_X \in \operatorname{dom}(\text{Ric})$, where

$$H_X = \begin{bmatrix} A & 0 \\ -C_1^*C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B_2 & B_1 \\ -C_1^*D_{12} & 0 \end{bmatrix} J^{-1} \begin{bmatrix} D_{12}^*C_1 & B_2^* \\ 0 & B_1^* \end{bmatrix}. \tag{4.18}$$

2. $X_\infty = \text{Ric}(H_X) \geqq 0$.

The factorization dual to (4.16), i.e.,

$$\begin{bmatrix} \mathbf{P}_{12} & \mathbf{P}_{11} \\ 0 & I \end{bmatrix}^{\sim} J \begin{bmatrix} \mathbf{P}_{12} & \mathbf{P}_{11} \\ 0 & I \end{bmatrix} = \mathbf{X}^{\sim} J \mathbf{X}, \qquad \mathbf{X} \in \mathcal{GH}_\infty, \qquad \mathbf{X}_{11} \in \mathcal{GH}_\infty \tag{4.19}$$

is the factorization associated with the $\mathcal{H}_\infty$ state feedback problem in [22], where $\mathbf{P}$ is assumed stable.

**4.3. A bilateral model matching problem.** We derive necessary and sufficient conditions, in terms of nonnegative definiteness conditions on the solutions of two indefinite Riccati equations, for the existence of $\mathbf{Q} \in \mathcal{RH}_\infty$ such that $\|\mathbf{T}_{11}+\mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\|_\infty < \gamma$. The first Riccati equation is associated with the factorization Problem P1 in § 4.2 (see (4.6)), which we will, in this section, assume has a solution. The second Riccati equation is associated with the factorization of the matrix

$$\mathbf{G} = \hat{J}\mathbf{V}^{-1}\hat{J}^* \begin{bmatrix} \mathbf{T}_{12} & 0 \\ 0 & I_m \end{bmatrix}. \tag{4.20}$$

By Theorem 2.6, we need to solve the following factorization problem.

FACTORIZATION PROBLEM P2. *With* $\mathbf{G}$ *defined by* (4.20), *find* $\mathbf{W} \in \mathcal{GH}_\infty^{q+m}$ *with* $\mathbf{W}_{11} \in \mathcal{GH}_\infty^{q}$ *such that*

$$\mathbf{G}^{\sim} J_{lm}(\gamma)\mathbf{G} = \mathbf{W}^{\sim} J_{qm}(\gamma)\mathbf{W}. \tag{4.21}$$

THEOREM 4.6. *Let* $\mathbf{G}$ *be as in* (4.20). *Then Problem* P2 *has a solution if and only if* $H_Z \in \operatorname{dom}(\text{Ric})$ *and* $\text{Ric}(H_Z) \geqq 0$, *where*

$$H_Z = \begin{bmatrix} A - M_2C_1 & 0 \\ -C_1^*C_1 & -(A-M_2C_1)^* \end{bmatrix} - \begin{bmatrix} B_2 - M_2D_{12} & M_1 \\ -C_1^*D_{12} & 0 \end{bmatrix} J^{-1} \times \begin{bmatrix} D_{12}^*C_1 & (B_2 - M_2D_{12})^* \\ 0 & M_1^* \end{bmatrix}. \tag{4.22}$$

$(J = J_{lm}(\gamma))$. *In this case,* $\mathbf{W}$ *is given by*

$$\mathbf{W} = \mathbf{W}_1 \begin{bmatrix} \mathbf{D}_r & -\mathbf{U}_l \\ \mathbf{N}_r & \mathbf{V}_l \end{bmatrix} \tag{4.23a}$$

*where*

$$\mathbf{W}_1 \overset{s}{=} \left[\begin{array}{c|cc} A - M_1C_2 - M_2C_1 & B_2 - M_2D_{12} & M_1 \\ \hline L_1 & I & 0 \\ L_2 & 0 & I \end{array}\right] \tag{4.23b}$$

*and*

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} D_{12}^* C_1 + (B_2 - M_2 D_{12})^* Z_\infty \\ -(C_2 + \gamma^{-2} M_1^* Z_\infty) \end{bmatrix} \tag{4.24a}$$

*with*

$$Z_\infty = \operatorname{Ric}(H_Z). \tag{4.24b}$$

*Proof.* First, consider the formula for **G** in light of the fact that **V** is given by (4.7).

$$\begin{aligned}
\mathbf{G} &= \hat{J}\mathbf{V}^{-1}\hat{J}^* \begin{bmatrix} \mathbf{T}_{12} & 0 \\ 0 & I \end{bmatrix} \\
&= \hat{J}\mathbf{V}_1^{-1} \begin{bmatrix} \mathbf{D}_l^{-1} & 0 \\ \mathbf{P}_{12}\mathbf{U}_l & I \end{bmatrix} \hat{J}^* \begin{bmatrix} \mathbf{T}_{12} & 0 \\ 0 & I \end{bmatrix} \quad \text{by (4.7)} \\
&= \hat{J}\mathbf{V}_1^{-1}\hat{J}^* \begin{bmatrix} I & -\mathbf{P}_{12}\mathbf{U}_l \\ 0 & \mathbf{D}_l^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{12}\mathbf{D}_r & 0 \\ 0 & I \end{bmatrix}, \quad \text{since } \mathbf{T}_{12} = \mathbf{P}_{12}\mathbf{D}_r \\
&= \hat{J}\mathbf{V}_1^{-1}\hat{J}^* \begin{bmatrix} \mathbf{P}_{12} & 0 \\ -\mathbf{P}_{22} & I \end{bmatrix} \begin{bmatrix} \mathbf{D}_r & -\mathbf{U}_l \\ \mathbf{N}_r & \mathbf{V}_l \end{bmatrix}, \quad \text{using (1.4).}
\end{aligned}$$

Thus,

$$\mathbf{G} = \mathbf{G}_1 \begin{bmatrix} \mathbf{D}_r & -\mathbf{U}_l \\ \mathbf{N}_r & \mathbf{V}_l \end{bmatrix} \tag{4.25}$$

where

$$\mathbf{G}_1 = \hat{J}\mathbf{V}_1^{-1}\hat{J}^* \begin{bmatrix} \mathbf{P}_{12} & 0 \\ -\mathbf{P}_{22} & I \end{bmatrix} \tag{4.26a}$$

$$\stackrel{s}{=} \left[\begin{array}{c|cc} A - M_1 C_2 - M_2 C_1 & B_2 - M_2 D_{12} & M_1 \\ \hline C_1 & D_{12} & 0 \\ -C_2 & 0 & I \end{array}\right]. \tag{4.26b}$$

Since $\left[\begin{smallmatrix} \mathbf{D}_r & -\mathbf{U}_l \\ \mathbf{N}_r & \mathbf{V}_l \end{smallmatrix}\right] \in \mathcal{GH}_\infty$ there exists $\mathbf{W} \in \mathcal{GH}_\infty$ such that $\mathbf{G}^\sim J \mathbf{G} = \mathbf{W}^\sim J \mathbf{W}$ if and only if **W** is given by (4.23a), where $\mathbf{W}_1 \in \mathcal{GH}_\infty$ satisfies

$$\mathbf{G}_1^\sim J \mathbf{G}_1 = \mathbf{W}_1 J \mathbf{W}_1. \tag{4.27}$$

Using the realization (4.26b) and Theorem 3.2, there exists $\mathbf{W}_1 \in \mathcal{GH}_\infty$ satisfying (4.27) if and only if $H_Z \in \operatorname{dom}(\operatorname{Ric})$, and in this case, $\mathbf{W}_1$ given by (4.23b) satisfies (4.27).

Let us now consider necessary and sufficient conditions for $\mathbf{W}_{11} \in \mathcal{GH}_\infty$.

Using (4.23), (4.2) and the state transformation $\left[\begin{smallmatrix} I & 0 \\ -I & I \end{smallmatrix}\right]$ the following realization for **W** is obtained:

$$\mathbf{W} \stackrel{s}{=} \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & I \end{array}\right] \tag{4.28a}$$

where

$$\hat{A} = \begin{bmatrix} A - M_2 C_1 - (B_2 - M_2 D_{12}) F & -(B_2 - M_2 D_{12}) F + M_1 C_2 \\ M_2 (C_1 - D_{12} F) & A - M_2 D_{12} F - M_1 C_2 \end{bmatrix} \tag{4.28b}$$

$$\hat{B} = \begin{bmatrix} B_2 - M_2 D_{12} & M_1 \\ M_2 D_{12} & H - M_1 \end{bmatrix} \tag{4.28c}$$

$$\hat{C} = \begin{bmatrix} L_1 - F & -F \\ L_2 + C_2 & C_2 \end{bmatrix}. \tag{4.28d}$$

The "$A$" matrix of $\mathbf{W}_{11}^{-1}$ is therefore

$$\tilde{A}=\begin{bmatrix} A-M_2C_1-(B_2-M_2D_{12})L_1 & M_1C_2 \\ M_2(C_1-D_{12}L_1) & A-M_1C_2 \end{bmatrix}. \tag{4.29}$$

Rewrite the Riccati equation for $Z_\infty$ as:

$$\begin{aligned} &Z_\infty[A-M_2C_1-(B_2-M_2D_{12})L_1]+[A-M_2C_1-(B_2-M_2D_{12})L_1]^*Z_\infty \\ &\quad+Z_\infty[(B_2-M_2D_{12})(B_2-M_2D_{12})^*+\gamma^{-2}M_1M_1^*]Z_\infty+C_1^*(I-D_{12}D_{12}^*)C_1=0. \end{aligned} \tag{4.30}$$

Using (4.14), (4.30), and $M_2=-\gamma^{-2}Y_\infty C_1^*$, we therefore have

$$\begin{aligned} &\begin{bmatrix} Z_\infty & 0 \\ 0 & \gamma^2 I\end{bmatrix}\tilde{A}\begin{bmatrix} I & 0 \\ 0 & Y_\infty\end{bmatrix}+\begin{bmatrix} I & 0 \\ 0 & Y_\infty\end{bmatrix}\tilde{A}^*\begin{bmatrix} Z_\infty & 0 \\ 0 & \gamma^2 I\end{bmatrix} \\ &\quad=-\begin{bmatrix} Z_\infty & 0 \\ 0 & Y_\infty\end{bmatrix}\begin{bmatrix} B_2-M_2D_{12} & \gamma^{-1}M_1 \\ -C_1^*D_{12} & -\gamma C_2^*\end{bmatrix} \\ &\quad\quad\times\begin{bmatrix} B_2-M_2D_{12} & \gamma^{-1}M_1 \\ -C_1^*D_{12} & -\gamma C_2^*\end{bmatrix}^*\begin{bmatrix} Z_\infty & 0 \\ 0 & Y_\infty\end{bmatrix} \\ &\quad\quad-\begin{bmatrix} C_1^* \\ Y_\infty C_1^*\end{bmatrix}D_\perp D_\perp^*[C_1 \quad C_1Y_\infty]-\begin{bmatrix} 0 \\ \gamma B_1\end{bmatrix}\tilde{D}_\perp^*\tilde{D}_\perp[0 \ \gamma B_1^*]. \end{aligned} \tag{4.31}$$

*Temporary assumption.* $Y_\infty$ nonsingular. With $Y_\infty$ nonsingular, define

$$\tilde{Z}_\infty=\begin{bmatrix} Z_\infty & 0 \\ 0 & \gamma^2 Y_\infty^{-1}\end{bmatrix}. \tag{4.32}$$

Since $\hat{A}-\hat{B}\hat{C}$ is asymptotically stable, $([L_2+C_2 \ \ C_2], \tilde{A})$ is detectable. Observing that $L_2+C_2=-\gamma^{-2}M_1^*Z_\infty$ it follows from (4.31) and [26, Lemma 12.2] that $\tilde{A}$ is asymptotically stable$\Leftrightarrow \tilde{Z}_\infty \geqq 0$.

*Removal of temporary assumption.* Suppose, without loss of generality, the realization $(A, B, C)$ is such that $Y_\infty$ is of the form

$$Y_\infty=\begin{bmatrix} \hat{Y}_\infty & 0 \\ 0 & 0\end{bmatrix}, \quad \hat{Y}_\infty \text{ nonsingular.}$$

It follows from (4.14) that $A-M_1C_2$ is upper triangular:

$$A-M_1C_2=\begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22}\end{bmatrix}, \quad X_{22} \text{ asymptotically stable.}$$

Furthermore, we see from (4.29), since $M_2=-\gamma^{-2}Y_\infty C_1^*$, that $\tilde{A}$ is also upper triangular:

$$\tilde{A}=\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & X_{22}\end{bmatrix}.$$

Applying the $Y_\infty$ nonsingular argument to the 1, 1 block gives $\tilde{A}_{11}$ asymptotically stable$\Leftrightarrow Z_\infty \geqq 0$, and hence $\tilde{A}$ is asymptotically stable$\Leftrightarrow Z_\infty \geqq 0$.

*Remark* 4.7. The structure (4.23a) of $\mathbf{W}$ is of great significance, as we now explain. Recall from Theorem 2.6 that all matrices $\mathbf{Q}\in\mathcal{RH}_\infty$ such that $\|\mathbf{T}_{11}+\mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\|_\infty \leqq \gamma$ are given by

$$\mathbf{Q}=\mathbf{Q}_1\mathbf{Q}_2^{-1}, \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2\end{bmatrix}=\mathbf{W}^{-1}\begin{bmatrix} \mathbf{U} \\ I\end{bmatrix} \quad \mathbf{U}\in\mathcal{RH}_\infty \text{ with } \|\mathbf{U}\|_\infty \leqq \gamma$$

where $\mathbf{W}$ solves Problem P2. Also recall, from (1.5), that all stabilizing controllers are given by

$$\mathbf{K}=\mathbf{K}_1\mathbf{K}_2^{-1}, \begin{bmatrix}\mathbf{K}_1\\ \mathbf{K}_2\end{bmatrix}=\begin{bmatrix}\mathbf{D}_r & -\mathbf{U}_l\\ \mathbf{N}_r & \mathbf{V}_l\end{bmatrix}\begin{bmatrix}\mathbf{Q}\\ I_m\end{bmatrix} \quad \mathbf{Q}\in\mathcal{RH}_\infty^{q\times m}.$$

It follows from (4.23a) that all stabilizing controllers $\mathbf{K}$ such that $\|\mathcal{F}(\mathbf{P},\mathbf{K})\|_\infty\leqq\gamma$ are given by

$$\mathbf{K}=\mathbf{K}_1\mathbf{K}_2^{-1}, \begin{bmatrix}\mathbf{K}_1\\ \mathbf{K}_2\end{bmatrix}=\mathbf{W}_1^{-1}\begin{bmatrix}\mathbf{U}\\ I\end{bmatrix}, \ \mathbf{U}\in\mathcal{RH}_\infty \text{ with } \|\mathbf{U}\|_\infty\leqq\gamma. \tag{4.33}$$

**5. The controller generator.** Theorem 4.6 gives necessary and sufficient conditions for internally stabilizing controllers $\mathbf{K}$ such that $\|\mathcal{F}(\mathbf{P},\mathbf{K})\|_\infty<\gamma$ to exist. Furthermore, (4.23b) and (4.33) provide a representation formula for all such controllers. The result we give in this section provides an alternative formula for controllers; there will be two changes. First, we will replace $Z_\infty$ by an equivalent expression, since $Z_\infty=X_\infty(I-\gamma^{-2}Y_\infty X_\infty)^{-1}$, and second, we will transform the formula (4.33) into an equivalent feedback form more typical in the engineering literature.

THEOREM 5.1. *Suppose* $\mathbf{P}(s)$ *is given by the realization* (4.1), *that assumptions* A1 *and* A2 *hold and that*

A3.

$$\begin{bmatrix}A-\lambda I & B_2\\ C_1 & D_{12}\end{bmatrix} \quad \textit{and} \quad \begin{bmatrix}A-\lambda I & B_1\\ C_2 & D_{21}\end{bmatrix}$$

*are, respectively, full column and row rank for all* $\lambda+\bar{\lambda}=0$. *Then there exists a rational matrix* $\mathbf{K}$ *such that* $\mathcal{F}(\mathbf{P},\mathbf{K})$ *is internally stable and* $\|\mathcal{F}(\mathbf{P},\mathbf{K})\|_\infty<\gamma$ *if and only if* $H_X\in\operatorname{dom}(\mathrm{Ric})$, $H_Y\in\operatorname{dom}\mathrm{Ric})$ *and*

$$X_\infty\geqq 0, \quad Y_\infty\geqq 0 \quad \text{and} \quad \lambda_{\max}(X_\infty Y_\infty)<\gamma^2 \tag{5.1a}$$

*where*

$$X_\infty=\mathrm{Ric}\,(H_X), \qquad Y_\infty=\mathrm{Ric}\,(H_Y) \tag{5.1b}$$

*with* $H_Y$ *and* $H_X$ *as in* (4.6) *and* (4.18).

*Furthermore, when the conditions* (5.1) *hold, all controllers* $\mathbf{K}$ *such that* $\mathcal{F}(\mathbf{P},\mathbf{K})$ *is internally stable and* $\|\mathcal{F}(\mathbf{P},\mathbf{K})\|_\infty\leqq\gamma$ *are given by*

$$\mathbf{K}=\mathcal{F}(\mathbf{K}_a,\mathbf{U}) \qquad \mathbf{U}\in\mathcal{RH}_\infty \quad \textit{with } \|\mathbf{U}\|_\infty\leqq\gamma \tag{5.2}$$

*where*

$$\mathbf{K}_a \stackrel{s}{=} \left[\begin{array}{c|cc} A_k & B_{k1} & B_{k2}\\ \hline C_{k1} & 0 & I\\ C_{k2} & I & 0\end{array}\right] \tag{5.3a}$$

*with*

$$B_k=[\,Y_\infty C_2^*+B_1D_{21}^* \quad B_2+\gamma^{-2}Y_\infty C_1^*D_{12}] \tag{5.3b}$$

$$C_k=\begin{bmatrix}-(D_{12}^*C_1+B_2^*X_\infty)\\ -(C_2+\gamma^{-2}D_{21}B_1^*X_\infty)\end{bmatrix}(I-\gamma^{-2}Y_\infty X_\infty)^{-1} \tag{5.3c}$$

$$A_k=A-B_{k1}C_2+\gamma^{-2}Y_\infty C_1^*C_1+B_{k2}C_{k1}. \tag{5.3d}$$

*Proof.* We have already proved that $\gamma$-suboptimal controllers $\mathbf{K}$ exist$\Leftrightarrow\mathbf{Q}\in\mathcal{RH}_\infty$ exists such that $\|\mathbf{T}_{11}+\mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\|_\infty<\gamma\Leftrightarrow H_Y$ and $H_Z\in\operatorname{dom}(\mathrm{Ric})$ with $Y_\infty\geqq 0$ and $Z_\infty\geqq 0$ (provided $\mathbf{T}_{21}$ and $\mathbf{T}_{12}$ have right and left inverses in $\mathcal{RL}_\infty$, which is assured by Lemma 4.1 and A3).

We need to show, given $H_Y \in \operatorname{dom}(\operatorname{Ric})$ and $Y_\infty = \operatorname{Ric}(H_Y) \geqq 0$, that $H_Z \in \operatorname{dom}(\operatorname{Ric})$ and $Z_\infty = \operatorname{Ric}(H_Z) \geqq 0 \Leftrightarrow H_X \in \operatorname{dom}(\operatorname{Ric})$. $X_\infty = \operatorname{Ric}(H_X) \geqq 0$ and $\lambda_{\max}(X_\infty Y_\infty) < \gamma^2$.

Observe that

$$\begin{bmatrix} I & \gamma^{-2} Y_\infty \\ 0 & I \end{bmatrix} H_Z \begin{bmatrix} I & -\gamma^{-2} Y_\infty \\ 0 & I \end{bmatrix} = H_X. \tag{5.4}$$

Suppose $H_X \in \operatorname{dom}(\operatorname{Ric})$, $X_\infty = \operatorname{Ric}(H_X) \geqq 0$ and $\lambda_{\max}(X_\infty Y_\infty) < \gamma^2$. Then $(I - \gamma^{-2} Y_\infty X_\infty)$ is nonsingular, and from (5.4) we see that $H_Z \in \operatorname{dom}(\operatorname{Ric})$, with $Z_\infty = \operatorname{Ric}(H_Z) = X_\infty (I - \gamma^{-2} Y_\infty X_\infty)^{-1}$. To see that $Z_\infty \geqq 0$, note that

$$Z_\infty(\gamma^{-2} Y_\infty X_\infty - I) + (\gamma^{-2} Y_\infty X_\infty - I)^* Z_\infty + (X_\infty + X_\infty) = 0.$$

It follows [11, Thm. 3.3, part 3] that $Z_\infty \geqq 0$, since $(\gamma^{-2} Y_\infty X_\infty - I)$ is asymptotically stable.

Conversely, suppose $H_Z \in \operatorname{dom}(\operatorname{Ric})$ and $Z_\infty = \operatorname{Ric}(H_Z) \geqq 0$. Hence $(I + \gamma^{-2} Z_\infty Y_\infty)$ is nonsingular and from (5.4), $H_X \in \operatorname{dom}(\operatorname{Ric})$ with

$$X_\infty = \operatorname{Ric}(H_X) = (I + \gamma^{-2} Z_\infty Y_\infty)^{-1} Z_\infty = Z_\infty (I + \gamma^{-2} Y_\infty Z_\infty)^{-1}.$$

Clearly $X_\infty \geqq 0$ and we see that $\lambda_{\max}(X_\infty Y_\infty) < \gamma^2$ since

$$\lambda_i(X_\infty Y_\infty) = \lambda_i\{(I + \gamma^{-2} Z_\infty Y_\infty)^{-1} Z_\infty Y_\infty\} = \gamma^2 \frac{\lambda_i(Z_\infty Y_\infty)}{\gamma^2 + \lambda_i(Z_\infty Y_\infty)}.$$

This concludes the proof of the necessary and sufficient conditions for the existence of $\mathbf{K}$.

By Remark 4.7, $\mathbf{K}$ is given by

$$\mathbf{K} = \mathbf{K}_1 \mathbf{K}_2^{-1}, \begin{bmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{bmatrix} = \mathbf{W}_1^{-1} \begin{bmatrix} \mathbf{U} \\ I \end{bmatrix}, \qquad \mathbf{U} \in \mathcal{RH}_\infty \text{ with } \|\mathbf{U}\|_\infty < \gamma. \tag{5.5}$$

Defining $\mathbf{X} = \mathbf{W}_1^{-1}$, we can equivalently write

$$\mathbf{K} = \mathcal{F}(\mathbf{K}_a, \mathbf{U}), \qquad \mathbf{U} \in \mathcal{RH}_\infty \quad \text{with} \quad \|\mathbf{U}\|_\infty < \gamma$$

where

$$\mathbf{K}_a = \begin{bmatrix} \mathbf{X}_{12} & \mathbf{X}_{11} \\ I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{22} & \mathbf{X}_{21} \\ 0 & I \end{bmatrix}^{-1}. \tag{5.6}$$

Rewrite $L$ in (4.24a) as

$$L = \begin{bmatrix} D_{12}^* C_1 + B_2^* X_\infty \\ -(C_2 + \gamma^{-2} D_{21} B_1^* X_\infty) \end{bmatrix} (I - \gamma^{-2} Y_\infty X_\infty)^{-1}. \tag{5.7}$$

A straightforward state space calculation using (4.23) and (5.7) will reveal that a realization for $\mathbf{K}_a$ in (5.6) is indeed given by (5.3). □

We note here that this theorem agrees with others derived recently, such as [9], [12], [13], [21].

**6. Conclusion.** In this paper the $J$-spectral factorization approach to suboptimal $\mathcal{H}_\infty$ control problems of "Nehari"/"one-block"/"first kind" type has been extended to the general case.

The existence of solutions was shown to be equivalent to the existence of solutions to two coupled $J$-spectral factorization problems with the additional property that the $(1,1)$ block of both $J$-spectral factors be outer. The second of these $J$-spectral factors was shown to generate all solutions to the $\mathscr{H}_\infty$ control problem.

The existence of the $J$-spectral factors was then shown to be equivalent to the existence of nonnegative definite, stabilizing solutions to two indefinite algebraic Riccati equations. This allowed an explicit state space formula for a generator of all solutions to the suboptimal $\mathscr{H}_\infty$ control problem to be given.

The approach in this paper can easily be extended to AAK type problems where $k$ poles are allowed in the right half plane, with the proviso that one avoids the singular points (i.e., $\gamma$-optimal, the spectrum of the underlying Hankel operator, etc.). The change is that, instead of being outer, the inverse of the $(1,1)$ block of the $J$-spectral factors is required to be in $\mathscr{R}\mathscr{H}_\infty(k)$ (i.e., no more than $k$ poles in the right half plane). The singular (optimal) case is, however, more involved, as a noncanonical factorization is required.

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