

Analytic and numerical demonstration of quantum self-correction in the 3D Cubic Code

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A big open question in the quantum information theory concerns feasibility of a self-correcting quantum memory. A quantum state recorded in such memory can be stored reliably for a macroscopic time without need for active error correction if the memory is put in contact with a cold enough thermal bath. In this paper we derive a rigorous lower bound on the memory time T_{mem} of the 3D Cubic Code model which was recently conjectured to have a self-correcting behavior. Assuming that dynamics of the memory system can be described by a Markovian master equation of Davies form, we prove that $T_{mem} \geq L^c$ for some constant $c > 0$, where L is the lattice size and β is the inverse temperature of the bath. However, this bound applies only if the lattice size does not exceed certain critical value $L^* \sim e^{\beta/3}$. We also report a numerical Monte Carlo simulation of the studied memory indicating that our analytic bounds on T_{mem} are tight up to constant coefficients. In order to model the readout step we introduce a new decoding algorithm which might be of independent interest. Our decoder can be implemented efficiently for any topological stabilizer code and has a constant error threshold under random uncorrelated errors.

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I. INTRODUCTION

Any practical memory device must function reliably in a presence of small hardware imperfections and protect the recorded data against thermal noise. Building a memory capable of storing quantum information is particularly challenging since a quantum state must be protected against both bit-flip and phase-flip errors. Furthermore, in contrast to classical bits, quantum states form a continuous set thus being more vulnerable to small hardware imperfections.

Ground states of topologically ordered many-body systems, such as fractional quantum Hall liquids [30, 37] and unpaired Majorana fermions in nanowires and 2D heterostructures [1, 20, 39, 41, 43] were proposed as a natural quantum data repository insensitive to local imperfections. A qubit encoded into the ground subspace of a topologically ordered system is almost perfectly decoupled from any local perturbation due to local indistinguishability of the ground states [8, 9, 34, 36].

To undo the effect of noise, a user of any memory, either classical or quantum, must invoke some form of error correction. It was shown by Dennis et al [18] that topological memory based on the Kitaev's 2D toric code model [34] can tolerate stochastic local noise provided that error correction is performed frequently enough to prevent errors from accumulating. However, such active error correction would require an extensive and fast classical input/output to the quantum hardware which might pose a challenge for building large-scale memory devices.

An intriguing open question raised in [5, 18] is whether topological memories can be *self-correcting*, that is, whether active error correction can be imitated by the natural dynamics of the memory system coupled to a thermal bath. The physical mechanism behind self-correction envisioned in [5] relies on a presence of “energy barriers” separating distinct ground states and energy dissipation. If the energy barriers are high enough and the evolution time is not too long, the memory system will be locked in the “energy valley” surrounding the initial ground state. A user of a self-correcting memory would only be responsible for preparation of the initial ground state and performing one final round of active error correction at the readout step. The latter is required to clean up the residual low-energy errors. In contrast to the active error correction approach of [18], the storage itself would require no action from the user whatsoever.

The nature of excitations in a topological memory plays a crucial role in assessing its self-correcting capability. Anyons in the 2D toric code [34] provide a paradigmatic example of such excitations. A fundamental flaw of 2D topological memories that rules out self-correction is the lack of energy barriers that could suppress diffusion of anyons over large distances [18]. For instance, consider a process that involves a creation of anyon pair from the ground state, a transport of one

anyon along a non-contractible loop on the torus, and a final annihilation of the pair. This process enacts a non-trivial transformation on the ground subspace. However, it can be implemented by a stream of local errors at a constant energy cost. The lack of self-correction for the 2D toric code model was rigorously confirmed by Alicki et al [2] who showed that the relaxation time towards the equilibrium state is a constant independent of the lattice size. A more general no-go result for quantum self-correction based on arbitrary 2D stabilizer code Hamiltonians was derived in [12, 32].

While anyons have a rich algebraic structure that depends on a particular model, their common property is that anyons are *topological defects*. More broadly, a topological defect is a point-like excitation that cannot be created from the ground state by a local operator without creating other excitations. A domain wall in the 1D ferromagnetic Ising chain provides the simplest example of a topological defect. The second property that applies to all models with anyonic excitations is that anyons are *mobile* topological defects. In other words, once an anyon has been created at some spatial location, a local operator can move it to another location nearby without creating any other excitations. As we argued above, the latter property is the key reason why self-correction is hard, if not impossible, to achieve in 2D. It raises a question whether the laws of physics permit topological phases of matter without mobile topological defects? (In the coding language it translates to existence of topological codes without string-like logical operators.) Quite surprisingly, the answer turns out to be yes. A family of 3D topological memory models without mobile topological defects was recently discovered by one of the authors [26]. One of these models, which we call the 3D Cubic Code, obeys the so called no-strings rule [7]. It establishes a constant upper bound α on the mobility range of topological defects. More precisely, given any isolated topological defect occupying a spatial region of size R , no local operator can move this defect distance αR or more away without creating extra excitations, see [26] for details. A strong evidence in favor of self-correcting properties of the 3D Cubic Code has been recently obtained in [7] by proving that the energy barrier separating distinct ground states grows as a logarithm of the lattice size.

The main goal of the present paper is to assess self-correcting properties of the 3D Cubic Code in a more direct way by calculating its memory time as a function of the lattice size and the temperature. For the sake of concreteness, we shall state our results only for the 3D Cubic Code, but our analysis applies to any quantum memory model based on a stabilizer code with local generators which obeys the topological order condition and the no-strings rule as stated in [7].

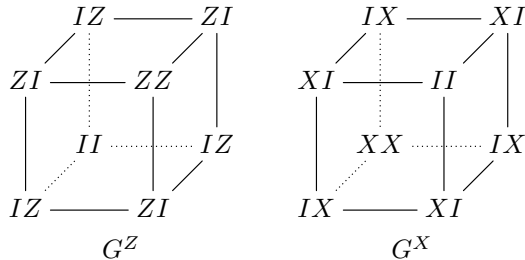


FIG. 1. Stabilizer generators of the 3D Cubic Code. Here $X \equiv \sigma^x$ and $Z \equiv \sigma^z$ represent single-qubit Pauli operators, while I is the identity operator. Double-letter indices represent two-qubit Pauli operators, for example, $IZ \equiv I \otimes Z$, $ZZ \equiv Z \otimes Z$, $II \equiv I \otimes I$ etc.

II. MEMORY HAMILTONIAN, THERMAL NOISE, AND DECODING

In order to use any memory, either classical or quantum, a user must be able to write, store, and read information. In this section we describe these steps formally for a topological quantum memory based on the 3D Cubic Code. Definition of the code Hamiltonian and its basic properties are summarized in Section II A. Our formal model of the thermal noise based on Davies weak-coupling limit is presented in Section II B. The readout step consists of a syndrome measurement and an error correction, see Section II C.

A. 3D Cubic Code Hamiltonian

Let $\Lambda = \mathbb{Z}_L \times \mathbb{Z}_L \times \mathbb{Z}_L$ be the regular 3D Cubic lattice of linear size L with periodic boundary conditions. Each site of the lattice is occupied by two qubits. The 3D Cubic Code Hamiltonian introduced in [26] has the following form:

$$H = -J \sum_c G_c^X + G_c^Z,$$

where the sum runs over all L^3 elementary cubes c and the operators G_c^X, G_c^Z act on the qubits of c as shown on Fig. 1. The positive coupling constant J will be set to $J = \frac{1}{2}$ for simplicity. We shall refer to operators G_c^X, G_c^Z as *stabilizer generators*, or simply stabilizers. Note that each stabilizer acts non-trivially only on 8 qubits. Let us recall some basic properties of the 3D Cubic Code, see [26] for details. First, one can easily check that the stabilizers $G_c^X, G_{c'}^Z$ commute with each other for all c, c' . A ground state of H is a common +1 eigenstate of all stabilizers. The degeneracy of the ground states is $2^{k(L)}$ for some integer $2 \leq k(L) \leq 4L$. The ground subspace of H has topological order, that is, different ground states cannot be distinguished locally. More

precisely, if O is any operator whose support can be bounded by a cubic box of size $< L$ then the restriction of O onto the ground subspace is proportional to the identity operator. Excited states of H can be described by configurations of defects, that is, stabilizers whose eigenvalue is -1 . Each defect costs one unit of energy.

B. Thermal noise

Suppose at time $t = 0$ the memory system is initialized in some ground state $\rho(0)$ encoding a quantum state to be stored. We shall model interaction between the memory system and the thermal bath using the Davies weak coupling limit [17]. It provides a Markovian master equation of the following form:

$$\dot{\rho}(t) = -i[H, \rho(t)] + \mathcal{L}(\rho(t)), \quad t \geq 0. \quad (1)$$

Here $\rho(t)$ is the state of the memory system at time t and \mathcal{L} is the Lindblad generator describing dissipation of energy. To define \mathcal{L} , let us choose some set of self-adjoint operators $\{A_\alpha\}$ through which the memory can couple to the bath. We shall assume that each A_α acts non-trivially on a constant number of qubits. For example, $\{A_\alpha\}$ could be the set of all single-qubit Pauli operators. Let $A_\alpha = \sum_\omega A_{\alpha,\omega}$, where $A_{\alpha,\omega}$ is the spectral component of A_α that maps eigenvectors of H with energy E to eigenvectors with energy $E - \omega$. Then

$$\mathcal{L}(\rho) = \sum_\alpha \sum_\omega h(\alpha, \omega) \left(A_{\alpha,\omega} \rho A_{\alpha,\omega}^\dagger - \frac{1}{2} \{ \rho, A_{\alpha,\omega}^\dagger A_{\alpha,\omega} \} \right). \quad (2)$$

The coefficient $h(\alpha, \omega)$ is the rate of quantum jumps caused by A_α transferring energy ω from the memory to the bath. It must obey the detailed balance condition

$$h(\alpha, -\omega) = e^{-\beta\omega} h(\alpha, \omega), \quad (3)$$

where β is the inverse bath temperature. The detailed balance condition Eq.(3) is the only part of our model that depends on the bath temperature. It guarantees that the Gibbs state $\rho_\beta \sim e^{-\beta H}$ is the fixed point of the dynamics, $\mathcal{L}(\rho_\beta) = 0$. This is a unique fixed point under certain natural ergodicity conditions [42]. Furthermore, we shall assume that $\|A_\alpha\| \leq 1$ and

$$\max_{\alpha,\omega} h(\alpha, \omega) = O(1). \quad (4)$$

Let us remark that the Davies weak coupling limit was adopted as a model of the thermal dynamics in most of the previous works with a rigorous analysis of quantum self-correction; see for instance [2, 3, 15, 16].

C. Decoding

The final state $\rho(t)$ generated by the Davies dynamics can be regarded as a corrupted version of the initial encoded state $\rho(0)$. A decoder retrieves the encoded information from $\rho(t)$ by performing a syndrome measurement and an error correction. A syndrome measurement involves a non-destructive eigenvalue measurement of all stabilizer generators G_c^X, G_c^Z . The measured syndrome S can be regarded as a classical bit string that assigns an eigenvalue ± 1 to each generator. Let Π_S be the projector onto the subspace with syndrome S .

The error correction step is specified by an algorithm that takes as input the measured syndrome S and returns a correcting Pauli operator $P_{ec}(S)$. A good choice of the error correction algorithm is a highly non-trivial task which we discuss in Section VI. The net action of the decoder on states can be described by a trace preserving completely positive (TPCP) linear map Φ_{ec} such that

$$\Phi_{ec}(\rho) = \sum_S P_{ec}(S) \Pi_S \rho \Pi_S P_{ec}(S)^\dagger, \quad (5)$$

where the sum runs over all possible syndromes.

III. MAIN RESULTS AND SKETCH OF THE PROOF

Our first result is an upper bound on the storage error — the trace distance between the initial encoded state and the final error corrected state.

Theorem 1. *There exists a decoder Φ_{ec} and a constant $c > 0$ such that for any inverse temperature $\beta > 0$, any state $\rho(0)$ supported on the ground subspace of H , any evolution time $t \geq 0$, and any constant $0 < a < 1$ one has*

$$\epsilon(t) \equiv \|\rho(0) - \Phi_{ec}(\rho(t))\|_1 \leq O(t) \cdot 2^{k(L)} \cdot L^{3-ac\beta} \quad (6)$$

as long as $L \leq e^{(1-a)\beta/3}$. The error correction algorithm used by the decoder has running time $\text{poly}(L)$.

Clearly, our bound is most useful when $k(L)$ is small. In the following we shall mostly be interested in the smallest ground state degeneracy, $k(L) = 2$. This happens for any odd $3 \leq L \leq 200$ such that L is not a multiple of 15 or 63, see [26]. One can also show [25] that there exists an infinite sequence of lattice sizes such that $k(L) = 2$, for example, $k(2^p + 1) = 2$ for all $p \geq 1$. Unfortunately, a general explicit formula for $k(L)$ is unknown.

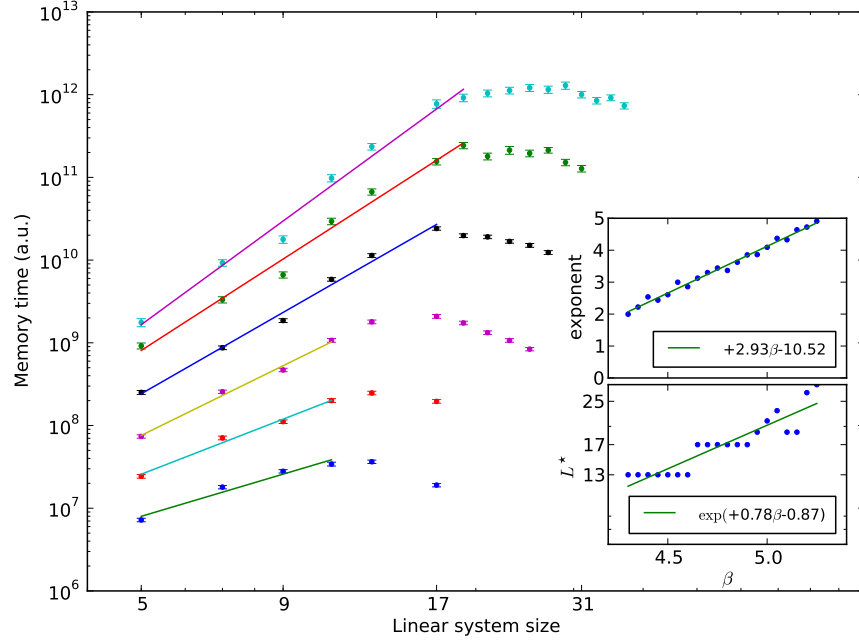


FIG. 2. The memory time T_{mem} vs. the system size L . In the upper inset is shown the exponent of the power law fit of T_{mem} for the first a few system sizes. It is clear that $T_{mem} \propto L^{2.93\beta-10.5}$ when $L < L^*$, where L^* is the optimal system size where T_{mem} reaches maximum. The data for $\beta = 4.3, 4.5, 4.7, 4.9, 5.1, 5.25$ are shown.

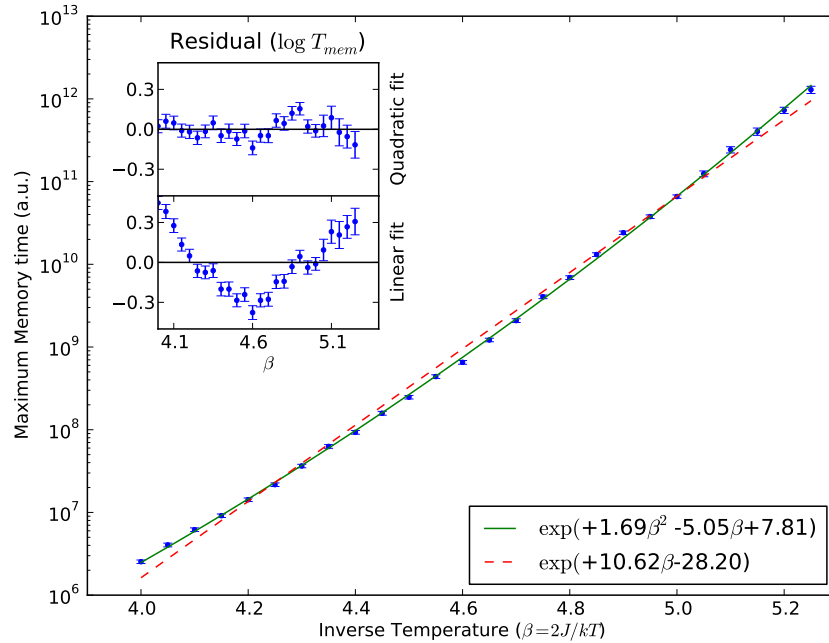


FIG. 3. The maximum memory time T_{mem} vs. the inverse temperature β . The memory time is maximized with respect to the system size. The logarithm of T_{mem} clearly follows a quadratic relation with β as oppose to a linear one.

The upper bound on the storage error can be easily translated to a lower bound on the memory time. Indeed, if one is willing to tolerate a fixed storage error ϵ , say $\epsilon = 0.01$, the memory time T_{mem} can be defined as the smallest $t \geq 0$ such that $\epsilon(t) \geq \epsilon$. Assuming that the lattice size is chosen such that $k(L) = 2$, Theorem 1 implies that

$$T_{mem} \geq L^{ac\beta-3} \quad \text{for any } L \leq e^{(1-a)\beta/3}. \quad (7)$$

Here we neglected the overall constant coefficient. It shows that for low temperatures, $\beta \gg 1$, and sufficiently small system size, $L \ll e^{\beta/3}$, the memory time grows polynomially with L , while the degree of the polynomial is proportional to β . To the best of our knowledge, this provides the first realistic example of a topological memory with a self-correcting behavior. Unfortunately, the studied model has no self-correction in the thermodynamic limit $\beta = O(1)$ and $L \rightarrow \infty$. In this sense, it is only partially self-correcting. At the optimally chosen lattice size, the maximum memory time $T_{mem}(\beta)$ achievable at a given temperature β is easily found from Eq. (7):

$$T_{mem}(\beta) \geq e^{c\beta^2/12} \quad (8)$$

for $\beta \gg 1$. For comparison, the memory time of the 2D toric code model grows only exponentially with β , see [2, 16]. Depending on the value of the constant c and the temperatures realizable in experiments, the scaling Eq. (8) may be favorable enough to achieve macroscopic memory times.

The restriction $L \ll e^{\beta/3}$ implies that the average number of defects (flipped stabilizers) in the equilibrium Gibbs state $\rho_\beta \sim e^{-\beta H}$ is less than one, that is, the Gibbs state has most of its weight on the ground subspace of H . This might suggest that the thermal noise is irrelevant in the studied regime. However, this is not the case. If the evolution time is large enough, so that $\rho(t) \approx \rho_\beta$, the encoded information cannot be retrieved from $\rho(t)$, since ρ_β does not depend on the initial state. If a time $t \sim e^{\Omega(\beta^2)}$ has elapsed, the system would have accommodated approximately $tL^3e^{-\beta} \sim e^{\Omega(\beta^2)}$ defects during the evolution. This implies in particular that the system has endured $e^{\Omega(\beta^2)}$ errors which becomes significant for low temperatures.

Since Theorem 1 only provides a lower bound on the memory time, a natural question is whether this bound is tight and, if so, what is the exact value of the constant coefficient c ? To answer this question, the memory time of the 3D Cubic Code has been computed numerically for a range of β 's and L 's. It should be emphasized that both Theorem 1 and our numerical simulation use the same decoder at the readout step (see below). Our simulation, described in detail in Section VIII, provides a Monte Carlo estimate of the successful decoding probability $p(t)$ as a function of the evolution time t . The decay of $p(t)$ was observed to follow an exponential law, $p(t) = \exp(-t/\tau)$.

The characteristic decay time τ was chosen as our numerical estimate of the memory time, that is, we set $T_{mem} = \tau$. The scaling of T_{mem} as a function of L is shown on Fig. 2. It suggests that $T_{mem} \approx L^{2.93\beta-10.52}$ as long as $L \leq L^* \approx e^{0.78\beta-0.87}$. The numerical results strongly suggest that our analytical bound Eq. (7) is tight up to constant coefficients (for the chosen decoder). A comparison between the numerical data for $T_{mem}(\beta)$ and the analytical bound Eq. (8) is shown on Fig. 3. The numerics suggests a scaling $T_{mem}(\beta) \approx e^{1.69\beta^2+O(\beta)}$. The numerical data were obtained for 26 values of β in the interval $4.0 \leq \beta \leq 5.25$ for each lattice size $5 \leq L \leq 33$ satisfying $k(L) = 2$ (that is, L is odd and $L \neq 15$). For each pair (L, β) the memory time was estimated using a few hundred Monte Carlo samples. Details on interpretation of the numerical data and the error analysis can be found in Section VIII.

The numerical computation of the memory time has become possible due to our second result, which is an efficient error correction algorithm applicable to any topological stabilizer code. We call it a Renormalization Group (RG) decoder since it falls into a larger family of error correction algorithms in which a syndrome is processed in a hierarchical way using real-space renormalization methods [19, 28]. Our algorithm and its rigorous analysis borrow many ideas from [28]. A detailed description of the RG decoder and its efficient implementation are described in Section VI. Let us briefly sketch the main idea of the algorithm. The RG decoding is a sequence of simple subroutines parameterized by integer levels $p = 0, 1, \dots, \log_2 L$. At any given level p , the decoder decomposes a syndrome into disjoint connected clusters, where the connectivity is defined using 2^p as a unit of length. This step can be implemented in time $O(N)$, where N is the volume of the lattice. The decoder then examines each cluster individually and tries to ‘annihilate’ it by a local Pauli operator. Clusters that cannot be annihilated are passed to the next level (that is, $p + 1$). The time needed to test whether a given connected cluster can be annihilated depends on a particular code. For the 3D Cubic Code we show how to perform this test in time $O(V)$, where V is the volume of the smallest rectangular box enclosing the cluster. Once the decoder reaches the highest level, $p = \log_2 L$, it returns the product of annihilation operators over all clusters that have been successfully annihilated. The overall running time of the RG decoder is $poly(N)$ for any topological stabilizer code.

The applicability of the RG decoder is by no means restricted to errors generated by thermal noise. A natural question is how well the RG decoder performs against local stochastic noise. In Appendix A we numerically benchmark the RG decoder for the 2D toric code, where we consider random independent X (or equivalently Z) errors with a uniform rate p . Our numerical estimate of the error threshold is $p_c \approx 6.7\%$ which is reasonably close to the threshold $p_c \approx 11\%$ obtained

using the minimum weight matching decoding [18]. Perhaps more importantly, we prove that the RG decoder has a constant error threshold against local stochastic errors for *any* topological stabilizer code, see Theorem 2 in Appendix B. To the best of our knowledge, the RG decoder is the first error correction algorithm applicable to any topological stabilizer code that combines good practical performance with a rigorous lower bound on the error threshold and polynomial worst-case running time.

Theorem 1: sketch of the proof

In Section V we prove some basic facts about “thermal errors” generated by the dissipative Lindblad dynamics. We define an energy barrier $\Delta(P)$ of a Pauli operator P as the smallest integer m such that P can be implemented by a stream of single-qubit errors that does not create more than m excitations during the process, see Section V A. The detailed balance condition and locality of quantum jump operators in the Lindblad equation imply that thermal errors are unlikely to have a high energy barrier. More precisely, Lemma 1 in Section V B provides an upper bound on the storage error $\epsilon(t)$ assuming that any error with energy barrier $\Delta(P) \leq m$ will be corrected at the decoding step. Here m is any fixed “energy cutoff”. The upper bound on $\epsilon(t)$ involves the Boltzmann suppression factor $e^{-\beta m}$, and an entropic factor that grows exponentially with the volume N of the lattice. We show that the entropic factor can be neglected in the regime $N e^{-\beta} \leq 1$ which yields a bound

$$\epsilon(t) \leq O(t) 2^{k(L)} N e^{-\beta m}, \quad (9)$$

see Section V C. Clearly, this bound is only useful if m grows at least logarithmically with N . At this point we must invoke the special properties of the 3D Cubic Code. Our key technical ingredient is Theorem 2 of Ref. [7]. It asserts (loosely) that any Pauli operator P creating a cluster of defects that contains a “topologically charged” sub-cluster separated from its complement by distance R , has energy barrier at least $\Delta(P) \geq c \log R$ for some constant coefficient c . Choosing the energy cutoff $m = c' \log L$ for some constant $c' < c$, we can guarantee that a syndrome $S(P)$ caused by any Pauli operator P satisfying $\Delta(P) \leq m$, consists of connected “topologically neutral” clusters of defects, where the connectivity is defined using $R \sim L^{c'/c}$ as a unit of length. The RG decoder will identify each of these clusters as a connected component of the syndrome for a sufficiently high RG level $p_0 \approx \log_2 R$. Since the clusters are topologically neutral, the RG decoder will be able to annihilate each of them locally. This implies that the correcting operator $P_{ec}(S)$ will annihilate all defects in S . Finally, we invoke the “localization of high-level errors lemma” (Lemma 2 in Ref. [7]) to show that, in fact, $P_{ec}(S)$ coincides with P modulo stabilizers, see Section VII B. Combining

this observation and Eq. (9) we prove Theorem 1.

It should be noted that the full hierarchy of the RG decoder is not needed for the proof of the theorem; one can run the decoder at a single level ($p = p_0$), and skip all the other levels. But, in practice, the hierarchical decoder performs better; it is able to correct errors with slightly higher energy barriers. Furthermore, the modified decoder using only one RG level would not have a constant threshold against local stochastic errors, see Appendix B.

IV. PREVIOUS WORK

Here we briefly review alternative routes towards quantum self-correction in topological memories proposed in the literature. Most of them focus on finding new mechanisms for suppressing diffusion of topological defects (here and below we only consider zero-dimensional defects). Arguably, the simplest of such mechanisms would be to have no topological defects in the first place. Unfortunately, this seems to require four spatial dimensions. The 4D toric code [18] provides the only known example of a truly self-correcting quantum memory. As was shown by Alicki and Horodecki's [3], the memory time of the 4D toric code grows exponentially with the lattice size for small enough bath temperature. The first 3D topological memory in which diffusion of defects is constrained by superselection rules was proposed by Chamon [14], see also [11]. Topological defects in this model have a limited mobility restricted to certain subspaces of \mathbb{R}^3 or have no mobility at all. However, the model has no macroscopic energy barrier that could suppress the diffusion. 2D topological memories in which diffusion of anyons is suppressed by effective long-range interactions were studied by Chesi et al [16] and Hamma et al [27]. A quenched disorder and Anderson localization were proposed as a means of suppressing propagation of defects at the zero temperature by Wootton and Pachos [46] and, independently, by Stark et al [44], see also [10]. A no-go theorem for quantum self-correction based on 3D stabilizer Hamiltonians in which ground state degeneracy does not depend on lattice dimensions was proved by Yoshida [47]. A different, very promising, line of research initiated by Pastawski et al [40] focuses on quantum memories in which active error correction is imitated by engineered dissipation driving the memory system towards the ground state (as opposed to the Gibbs state). Finally, let us emphasize that quantum self-correction is technically different from the thermal stability of topological phases, see for instance [13, 29, 31, 38]. While the latter attempts to establish the presence (or absence) of topological order in the equilibrium thermal state, quantum self-correction is mostly concerned with the relaxation time towards the equilibrium state.

V. ANALYSIS OF THERMAL ERRORS

We begin by setting up some terminology and notations. A ground state of the memory Hamiltonian will be referred to as a *vacuum*. It will be convenient to perform an overall energy shift such that the vacuum has zero energy. A *Pauli operator* is an arbitrary tensor product of single-qubit Pauli operators X, Y, Z and the identity operators I . We will say that a Pauli operator creates m *defects* iff P anticommutes with exactly m stabilizer generators G_c^X, G_c^Z . Equivalently, applying P to the vacuum one obtains an eigenvector of H with energy m . For example, using the explicit form of the generators, see Fig. 1, one can check that single-qubit X or Z errors create 4 defects, while Y errors create 8 defects. We will say that a Pauli error P is corrected by the decoder iff $P_{ec}(S(P)) = \gamma PG$, where $S(P)$ is the syndrome of P , G is a product of stabilizer generators and γ is an overall phase factor. Let $N = 2L^3$ be the total number of qubits. Note that N is also the number of stabilizer generators.

A. Energy barrier

Let $\Gamma = (P_0, P_1, \dots, P_t)$ be a finite sequence of Pauli operators such that the operators P_i and P_{i+1} differ on at most one qubit for all $0 \leq i < t$. We say that Γ is an *error path* implementing a Pauli operator P iff $P_0 = I$ and $P_t = P$. Let m_i be the number of defects created by P_i . The maximum number of defects

$$m(\Gamma) = \max_{0 \leq i \leq t} m_i$$

will be called an *energy cost* of the error path Γ . Given a Pauli operator P , we define its *energy barrier* $\Delta(P)$ as the minimum energy cost of all error paths implementing P ,

$$\Delta(P) = \min_{\Gamma} m(\Gamma).$$

Although the set of error paths is infinite, the minimum always exists because the energy cost is a non-negative integer. In fact, it suffices to consider paths in which P_i are all distinct. The number of such paths is finite since there are only finitely many Pauli operators for a given system size.

It is worth emphasizing that an operator P may have a very large energy barrier even though P itself creates only a few defects or no defects at all. Consider as an example the 2D Ising model, $H = -(1/2) \sum_{(u,v)} Z_u Z_v$, where the sum runs over all pairs of nearest neighbor sites on the square lattice of size $L \times L$ with open boundary conditions. Then the logical- X operator $P = \bigotimes_u X_u$ has

an energy barrier $\Delta(P) = L$ since any sequence of bit-flips implementing P must create a domain wall across the lattice.

It is clear that a Pauli operator P acting non-trivially on n qubits has energy barrier at most $O(n)$.

B. Lower bound on the memory time

A naive intuition suggests that a stabilizer code Hamiltonian is a good candidate for being a self-correcting memory if there exists an error correction algorithm, or a decoder, that corrects all errors with a sufficiently small energy barrier. Errors with a high energy barrier can confuse the decoder and cause it to make wrong decisions, but we expect that such errors are unlikely to be created by the thermal noise. The following lemma makes this intuition more rigorous. Let f be the maximum energy barrier of Pauli operators that appear in the expansion of the quantum jump operators $A_{\alpha,\omega}$ or $A_{\alpha,\omega}^\dagger A_{\alpha,\omega}$, see Section II B. We will see later that $A_{\alpha,\omega}$ act on a constant number of qubits (see Proposition 4 below). It implies that $f = O(1)$. Below m is an arbitrary energy cutoff.

Lemma 1. *Suppose an error correction algorithm $s \mapsto P_{ec}(s)$ corrects any Pauli error P with the energy barrier smaller than $m + 2f$. Let Φ_{ec} be the corresponding decoder defined by Eq.(5). Let Q_m be the projector onto the subspace with at least m defects. Then*

$$\epsilon(t) \equiv \|\Phi_{ec}(\rho(t)) - \rho(0)\|_1 \leq O(tN) \text{Tr} Q_m e^{-\beta H} \quad (10)$$

for any initial state $\rho(0)$ supported on the ground subspace of H . The time evolution of $\rho(t)$ is governed by the Lindblad equation, Eq. (1), with the inverse temperature β of the bath.

Proof. We begin with the following simple observation. Let \hat{H} be the linear map taking the commutator with H , that is, $\hat{H}(X) = HX - XH$.

Proposition 1. *The map \hat{H} commutes with the Lindblad generator \mathcal{L} . The time-evolved state $\rho(t)$ commutes with the memory Hamiltonian H . In addition, $\rho(t) = e^{\mathcal{L}t}(\rho(0))$.*

Proof. Since $A_{\alpha,\omega}$ transfers energy ω from the memory to the bath, we have the following identity:

$$A_{\alpha,\omega}H = HA_{\alpha,\omega} + \omega A_{\alpha,\omega} \quad \text{and} \quad A_{\alpha,\omega}^\dagger H = HA_{\alpha,\omega}^\dagger - \omega A_{\alpha,\omega}^\dagger. \quad (11)$$

From this one easily gets $\hat{H}\mathcal{L} = \mathcal{L}\hat{H}$. It follows that

$$\rho(t) = e^{-i\hat{H}t + \mathcal{L}t}(\rho(0)) = e^{\mathcal{L}t} \circ e^{-i\hat{H}t}(\rho(0)) = e^{\mathcal{L}t}(\rho(0)),$$

since the initial state $\rho(0)$ commutes with H . It also implies that $\rho(t)$ commutes with H . \square

Let $\mathcal{D} = \ker \hat{H}$ be the set of all operators that are commuting with the Hamiltonian H . Every operator in \mathcal{D} is block-diagonal in the energy eigenstate basis. Since $\rho(0)$ is supported on the ground subspace of H , we have $\rho(0) \in \mathcal{D}$. Below we only consider states from \mathcal{D} and linear maps preserving \mathcal{D} . Define an orthogonal identity decomposition

$$I = \Pi_- + \Pi_+$$

where Π_- projects onto the subspace with energy $< m + f$ and Π_+ projects onto the subspace with energy $\geq m + f$. Introduce auxiliary Lindblad generators

$$\mathcal{L}_-(\rho) = \sum_{\alpha} \sum_{\omega} h(\alpha, \omega) \left(B_{\alpha, \omega} \rho B_{\alpha, \omega}^{\dagger} - \frac{1}{2} \{ \rho, B_{\alpha, \omega}^{\dagger} B_{\alpha, \omega} \} \right), \quad \text{where } B_{\alpha, \omega} = \Pi_- A_{\alpha, \omega} \quad (12)$$

and

$$\mathcal{L}_+(\rho) = \sum_{\alpha} \sum_{\omega} h(\alpha, \omega) \left(C_{\alpha, \omega} \rho C_{\alpha, \omega}^{\dagger} - \frac{1}{2} \{ \rho, C_{\alpha, \omega}^{\dagger} C_{\alpha, \omega} \} \right), \quad \text{where } C_{\alpha, \omega} = \Pi_+ A_{\alpha, \omega}. \quad (13)$$

Simple algebra shows that \mathcal{L}_- and \mathcal{L}_+ preserve \mathcal{D} and their restrictions on \mathcal{D} satisfy

$$\mathcal{L} = \mathcal{L}_- + \mathcal{L}_+. \quad (14)$$

It is useful to note that any $X \in \mathcal{D}$ commutes with Π_{\pm} . By abuse of notations, we shall apply Eq.(14) as though it holds for all operators.

Proposition 2. *For any time $t \geq 0$ and for any state ρ_0 supported on the ground subspace of H one has*

$$\Phi_{ec}(e^{\mathcal{L}t}(\rho_0)) = \rho_0. \quad (15)$$

Proof. Since all maps are linear, we may assume $\rho_0 = |g\rangle\langle g|$ is a pure state. Then, $e^{\mathcal{L}t}(\rho_0)$ is in the span of states of form $|\psi\rangle = \Pi_- E_n \cdots \Pi_- E_2 \Pi_- E_1 |g\rangle$, where E_i are Pauli operators that appears in the expansion of $A_{\alpha, \omega}$ or $A_{\alpha, \omega}^{\dagger} A_{\alpha, \omega}$. This follows from the Taylor expansion of $e^{\mathcal{L}t}$. Since Pauli errors map eigenvectors of H to eigenvectors of H , we conclude that either $|\psi\rangle = 0$, or $|\psi\rangle = E_n \cdots E_2 E_1 |g\rangle$. Furthermore, the latter case is possible only if the Pauli operator $E \equiv E_n \cdots E_2 E_1$ has energy barrier smaller than $m + 2f$. Indeed, definition of Π_- implies that $E_j E_{j-1} \cdots E_1$ creates at most $m + f - 1$ defects for all $j = 1, \dots, n$. By assumption, each operator E_j can be implemented by an error path with energy cost at most f . Taking the composition of all such error paths one obtains an error path for E with energy cost at most $m + 2f - 1$ and thus Φ_{ec} will correct E . Since $\Phi_{ec} \circ e^{\mathcal{L}t}$ is a TPCP map, we must have $\Phi_{ec}(e^{\mathcal{L}t}(\rho_0)) = \rho_0$. \square

Proposition 3. For any decomposition $\mathcal{L} = \mathcal{L}_- + \mathcal{L}_+$ one has the following identity:

$$e^{\mathcal{L}t} = e^{\mathcal{L}_-t} + \int_0^t ds e^{\mathcal{L}_-(t-s)} \mathcal{L}_+ e^{\mathcal{L}_+s}. \quad (16)$$

Proof. Use the identity

$$\frac{d}{ds} e^{\mathcal{L}_-(t-s)} e^{\mathcal{L}_+s} = e^{\mathcal{L}_-(t-s)} (-\mathcal{L}_- + \mathcal{L}_+) e^{\mathcal{L}_+s} = e^{\mathcal{L}_-(t-s)} \mathcal{L}_+ e^{\mathcal{L}_+s}.$$

□

Proposition 4 ([15]). Each operator $A_{\alpha,\omega}$ acts non-trivially only on $O(1)$ qubits. Furthermore,

$$\|\mathcal{L}_+\|_1 \equiv \sup_X \frac{\|\mathcal{L}_+(X)\|_1}{\|X\|_1} = O(N).$$

Proof. Let $A_\alpha(t) = e^{iHt} A_\alpha e^{-iHt}$ such that $A_{\alpha,\omega}$ is the Fourier component of $A_\alpha(t)$ with the Bohr frequency ω , that is, $A_\alpha(t) = \sum_\omega e^{-i\omega t} A_{\alpha,\omega}$. Recall that each operator A_α acts on $O(1)$ qubits. Since H is a sum of pairwise commuting terms, we can represent $A_\alpha(t)$ as $A_\alpha(t) = e^{iH_\alpha t} A_\alpha e^{-iH_\alpha t}$ where H_α is obtained from H by retaining only those stabilizer generators that do not commute with A_α . All such generators must share at least one qubit with A_α . Therefore $A_\alpha(t)$ and $A_{\alpha,\omega}$ act non-trivially only on $O(1)$ qubits. Furthermore, since H_α has $O(1)$ distinct integer eigenvalues, $A_\alpha(t)$ has only $O(1)$ distinct Bohr frequencies. The bound $\|A_{\alpha,\omega}\| \leq 1$ follows trivially from our assumption $\|A_\alpha\| \leq 1$. The norm of \mathcal{L}_+ is then bounded from Eq. (2) using triangle inequality, $\|XY\|_1 \leq \|X\| \cdot \|Y\|_1$, and Eq. (4). □

Now we are ready to bound $\epsilon(t)$. Let $\rho_0 \equiv \rho(0)$. First, using Proposition 1 one gets $\rho(t) = e^{\mathcal{L}t}(\rho_0)$. Applying Propositions 2,3 one arrives at

$$\epsilon(t) \leq \int_0^t ds \|e^{\mathcal{L}_-(t-s)} \mathcal{L}_+ e^{\mathcal{L}_+s}(\rho_0)\|_1 \leq t \cdot \max_{0 \leq s \leq t} \|\mathcal{L}_+ e^{\mathcal{L}_+s}(\rho_0)\|_1. \quad (17)$$

We shall use an identity

$$\mathcal{L}_+(X) = \mathcal{L}_+(Q_m X Q_m) \quad (18)$$

valid for any $X \in \mathcal{D}$. Indeed, any quantum jump operator $A_{\alpha,\omega}$ changes the energy at most by f , so that $\mathcal{L}_+(Q_m^\perp X) = \mathcal{L}_+(X Q_m^\perp) = 0$ for any $X \in \mathcal{D}$ (note that any $X \in \mathcal{D}$ commutes with Q_m).

We arrive at

$$\epsilon(t) \leq t \cdot \|\mathcal{L}_+(Q_m e^{\mathcal{L}_+s}(\rho_0) Q_m)\|_1 \leq t \cdot \|\mathcal{L}_+\|_1 \cdot \|Q_m e^{\mathcal{L}_+s}(\rho_0) Q_m\|_1 \leq O(tN) \text{Tr} Q_m e^{\mathcal{L}_+s}(\rho_0), \quad (19)$$

where the maximization over s is implicit. Since the ground state energy of H is zero, one has

$$\rho_0 = \mathcal{Z}_\beta \rho_\beta \rho_0, \quad (20)$$

where \mathcal{Z}_β is the partition function. It yields

$$\text{Tr } Q_m e^{\mathcal{L}s}(\rho_0) = \text{Tr } \rho_0 e^{\mathcal{L}^*s}(Q_m) = \mathcal{Z}_\beta \text{Tr } \rho_\beta \rho_0 e^{\mathcal{L}^*s}(Q_m) = \mathcal{Z}_\beta \langle \rho_0, e^{\mathcal{L}^*s}(Q_m) \rangle_\beta, \quad (21)$$

where we introduced the Liouville inner product $\langle X, Y \rangle_\beta \equiv \text{Tr } \rho_\beta X^\dagger Y$. We denoted by \mathcal{L}^* the linear map adjoint to \mathcal{L} (the one describing time-evolution in the Heisenberg picture) with respect to the Hilbert-Schmidt inner product. The detailed balance condition Eq. (3) implies that the linear map \mathcal{L}^* is self-adjoint with respect to the Liouville inner product,

$$\langle X, \mathcal{L}^*(Y) \rangle_\beta = \langle \mathcal{L}^*(X), Y \rangle_\beta, \quad (22)$$

see [3]. It can be easily checked using the identities

$$A_{\alpha,\omega} \rho_\beta = e^{-\beta\omega} \rho_\beta A_{\alpha,\omega} \quad \text{and} \quad A_{\alpha,-\omega} = A_{\alpha,\omega}^\dagger \quad (23)$$

and Eq. (3). It follows that the map $e^{\mathcal{L}^*s}$ is also self-adjoint with respect to the Liouville inner product. Hence, we have

$$\text{Tr } Q_m e^{\mathcal{L}s}(\rho_0) = \mathcal{Z}_\beta \langle \rho_0, e^{\mathcal{L}^*s}(Q_m) \rangle_\beta = \mathcal{Z}_\beta \langle e^{\mathcal{L}^*s}(\rho_0), Q_m \rangle_\beta = \text{Tr } e^{\mathcal{L}^*s}(\rho_0) Q_m e^{-\beta H} \leq \text{Tr } Q_m e^{-\beta H}. \quad (24)$$

Here the last inequality follows from the fact that $e^{\mathcal{L}^*s}$ is a unital CP map and from $\rho_0 \leq I$. \square

C. Choosing the optimal number of qubits

Let us fix the inverse temperature β and ask what is the optimal system size N that minimizes the upper bound on the storage error $\epsilon(t)$ derived in Lemma 1. The dimension of the subspace with exactly n defects equals $2^{k(L)}$ times the binomial N choose n coefficient since the total number of potential defects locations is N (recall that the 3D Cubic Code has exactly N stabilizer generators). Choosing any constant $0 < a < 1$ and setting $k \equiv k(L)$ we obtain

$$\epsilon(t) \leq O(t) N 2^k e^{-a\beta m} \sum_{n \geq m} \binom{N}{n} e^{-(1-a)\beta n} \leq O(t) N 2^k e^{-a\beta m} (1 + e^{-(1-a)\beta})^N. \quad (25)$$

The Boltzmann factor $e^{-a\beta m}$ is responsible for the self-correcting behavior. It supports our initial intuition that errors with a high energy barrier (at least m) are unlikely to be generated by the thermal noise. On the other hand, the factor $(1 + e^{-(1-a)\beta})^N$ represents the entropy contribution. Loosely speaking, it takes into account the number of error paths with a given energy cost. The optimal choice of N is determined by the tradeoff between the Boltzmann factor and the entropy

factor which depends on the scaling of the energy barrier m . For example, suppose one chooses $N \leq e^{(1-a)\beta}$ such that the entropy factor is upper bounded by a constant. Then Eq. (25) yields

$$\epsilon(t) \leq O(t)N2^k e^{-a\beta m} \quad \text{for all } N \leq e^{(1-a)\beta}. \quad (26)$$

In Section VII, we will argue that the 3D Cubic Code (and any other stabilizer code having the topological order and obeying the no-strings rule as stated in [7]) admits an efficient error correction algorithm that corrects all errors with the energy barrier smaller than $m = c \log(N)$ for some constant $c > 0$. For such codes one gets

$$\epsilon(t) \leq O(t) \cdot 2^k \cdot N^{1-ac\beta} \quad \text{for all } N \leq e^{(1-a)\beta}. \quad (27)$$

This is equivalent to the statement of Theorem 1.

VI. RENORMALIZATION GROUP DECODER

Recall that the syndrome measurement reveals locations of defects (flipped stabilizer generators) created by an unknown error. The renormalization group (RG) decoder attempts to annihilate the defects comprising the syndrome S by dividing them into disjoint connected clusters $S = C_1 \cup \dots \cup C_m$ and then trying to annihilate each cluster C_a individually. More specifically, the decoder checks whether C_a can be annihilated by a Pauli operator P_a supported on a sufficiently small spatial region $b(C_a)$ enclosing C_a . If such local annihilation operator P_a exists, the decoder updates the syndrome by erasing all the defects comprising C_a , records the operator P_a , and moves on to the next cluster. If C_a cannot be annihilated, the decoder skips it. In general, the annihilation operator P_a might not be unique. However, if the enclosing region $b(C_a)$ is small enough to ensure that no logical operator can be supported on $b(C_a)$, all annihilation operators P_a must be equivalent modulo stabilizers and the choice of P_a does not matter.

After all clusters C_a have been examined, the decoder is left with a new configuration of defects S' , which is typically smaller than the original one. If no defects are left, i.e., $S' = \emptyset$, the decoder stops and returns the product of all recorded Pauli operators P_a . If $S' \neq \emptyset$, the decoder applies a scale transformation increasing the unit of length by some constant factor and repeats all the above steps starting from the syndrome S' . The scale transformation potentially merges several unerased clusters C_a into a single connected cluster whereby giving the decoder one more attempt to annihilate them.

The full decoding algorithm is the iteration of partitioning the defects into the connected clusters

and calculating the annihilation operators. It declares failure and aborts if the recorded operator cannot annihilate all the defects before the rescaled unit length is comparable to the lattice size.

A detailed implementation of the RG decoder must be tailored to a specific lattice geometry and a stabilizer code under consideration. It must include a precise definition of the connected clusters of defects C_a and the enclosing regions $b(C_a)$. It must also include an algorithm for choosing the annihilation operators P_a , a schedule for increasing the unit of length, and clearly stated conditions under which the decoder aborts. In the rest of this section we describe an efficient implementation of the RG decoder for arbitrary 3D stabilizer codes satisfying topological order conditions defined in [7]. The only part of this implementation specialized for the 3D cubic code is the “broom algorithm” of Section VI E.

A. Metric convention and terminology

Let Λ be the regular 3D cubic lattice of linear size L with periodic boundary conditions along all coordinates x, y, z . We shall label sites of Λ by triples of integers (i, j, k) defined modulo L and measure the distance between sites using the ℓ_∞ -metric. In other words, the distance $d(u, v)$ between a pair of sites u and v is the smallest integer r such that u and v can be enclosed by a cubic box with dimensions $r \times r \times r$. For example, $d(u, v) = 1$ whenever u and v belong to the same edge, plaquette, or elementary cube of the lattice. Each site of Λ represents one or several physical qubits (two qubits for the 3D Cubic Code). Each elementary cube c represents a spatial location of one or several stabilizer generators (two generators for the 3D Cubic Code). A generator located at cube c may act only on qubits located at vertices of c . We shall label each elementary cube by coordinates of its center, the triple of half-integers (i, j, k) defined modulo L . The distance $d(c, c')$ between a pair of cubes c and c' is the distance between their centers. For example, $d(c, c') = 1$ whenever c and c' share a vertex, an edge, or a plaquette.

A defect is a stabilizer generator whose eigenvalue has been flipped as a result of the error. We shall use a term *cluster of defects*, or simply *cluster* for any set of defects. Define the *diameter* of a cluster $d(C)$ as the maximum distance $d(c, c')$ where $c, c' \in C$. Here and below the distance between defects is defined as the distance between the cubes occupied by these defects. Given two non-empty clusters C and C' , define a distance $d(C, C')$ as the minimum distance $d(c, c')$ where $c \in C$ and $c' \in C'$. Given an integer r , we shall say that a cluster C is *connected at scale r* , or simply *r -connected*, if C cannot be partitioned into two proper subsets $C = C' \cup C''$ such that $d(C', C'') > r$. A maximal r -connected subset of a cluster C is called a *r -connected component* of

C . The *minimal enclosing box* $b(C)$ of a cluster C is the smallest rectangular box B enclosing all defects of C such that all vertices of B are dual sites of Λ . Note that the minimal enclosing box $b(C)$ is unique as long as $d(C) < L/2$ (if a cluster C has diameter $L/2$, one may have two boxes with the same dimensions enclosing C that ‘wrap’ around the lattice in two different ways).

B. Topological stabilizer codes

The RG decoder can be applied to any stabilizer code satisfying topological order conditions defined in [7]. For the sake of completeness, we state these conditions below. Let \mathcal{G} be the Abelian group generated by the stabilizer generators. Elements of \mathcal{G} will be called stabilizers. Let $S(P)$ be the syndrome of a Pauli operator P , that is, the set of all stabilizer generators anticommuting with P . The syndrome can be viewed as a cluster of defects.

Definition 1. A stabilizer code has *topological order at scale* L_{tqo} if any Pauli operator supported on a cube of linear size L_{tqo} satisfies the following conditions:

- (1) If P commutes with all stabilizers then P is stabilizer itself;
- (2) If $S(P) \neq \emptyset$ then there is a stabilizer $G \in \mathcal{G}$ such that PG is supported on the 1-neighborhood of the minimum enclosing box $b(S(P))$.

A family of stabilizer codes $\{\mathcal{C}(L)\}_L$ is called topological if there exists a constant $\gamma > 0$ such that each code $\mathcal{C}(L)$ has topological order at scale $L_{tqo} \geq L^\gamma$.

The 3D Cubic Code satisfies this topological order condition with $L_{tqo} = \frac{1}{2}L$, see [26]. Strictly speaking $L_{tqo} = L - 2$ for the 3D Cubic code, but in order to avoid unnecessary complications due to boundaries, we always assume that $L_{tqo} \leq \frac{1}{2}L$. Below we consider only topological stabilizer codes.

Definition 2. A cluster of defects C is called *neutral* if it can be created from the vacuum by a Pauli operator P supported on a cube of linear size L_{tqo} . Otherwise, the cluster is said to be *charged*.

For example, the 2D toric code [34] has two types of defects: magnetic charges (flipped plaquette operators) and electric charges (flipped star operators). A cluster of defects C is neutral iff C contains even number of magnetic charges and even number of electric charges.

It follows from Definition 1 that any neutral cluster of defects C can be annihilated by a Pauli operator supported on the 1-neighborhood of the minimum enclosing box $b(C)$.

C. RG decoder

We are now ready to define our RG decoder precisely. Recall that $d(C)$ is the diameter of a cluster C , and $L_{tqo} \leq \frac{1}{2}L$ by convention.

TestNeutral

Input S : a set of defects, **Output** P : a Pauli operator.

1. Compute the minimal enclosing box B of S .
2. **if** $d(B) > L_{tqo}$, **then return** I .
3. Try to compute a Pauli P supported on the 1-neighborhood of B such that $S(P) = S$.
4. **if** a consistent P is found **then return** P **else return** I .

The topological order condition of Definition 1 implies that TestNeutral computes the correcting Pauli operator for any neutral cluster. Step 1 is easy as we discuss in the end of Section VID. The specification of Step 3 depends on the code, but it always has an efficient implementation using the standard stabilizer formalism [22]. In general, the condition $S(P) = S$ can be described by a system of $O(V)$ linear equations over $O(V)$ binary variables parameterizing P , where V is the volume of B . The running time is then $O(V^3)$ by the Gauss elimination. In the special case of the 3D Cubic Code, there is a much more efficient algorithm running in time $O(V)$ which we describe in Section VIE.

Let p_M be the largest integer such that $2^{p_M} < \min(\frac{1}{2}L, L_{tqo})$. For any integer $0 \leq p \leq p_M$, we define the *level- p error correction* $\mathbf{EC}(p)$:

$\mathbf{EC}(p)$

Input S : a syndrome, **Output** P : a Pauli operator.

1. Partition S into 2^p -connected components: $S = C_1 \cup \dots \cup C_m$.
2. For each component, compute $P_a = \text{TestNeutral}(C_a)$.
3. **return** the product $P_1 P_2 \dots P_m$.

The overall running time of $\mathbf{EC}(p)$ is polynomial in the number of qubits N . Step 1 can be done, for example, by examining the distance between all pairs of defects, forming a graph whose edges connect pairs of defects separated by distance $\leq 2^p$, and finding connected components of this graph. A more efficient algorithm with the running time $O(N)$ is described in Section VID. Since $V = O(N)$ and $m = O(N)$, we see that the worst-case running time of $\mathbf{EC}(p)$ is $O(N^2)$. For instance, one can consider nested boxes, near the faces of which many defects lie. However, clusters

are created from the vacuum with some probability which we expect to be smaller than, say, $\frac{1}{2}$. So the number of clusters that have overlapping minimal enclosing box appears with exponentially small probability. On average, the running time of $\text{EC}(p)$ will be $O(N)$.

The full RG decoding algorithm is as follows.

RG Decoder

Input S : the syndrome, **Output** P_{ec} : a Pauli operator.

1. Set $P_{ec} = I$.
2. **for** $p = 0$ **to** p_M **do**
 - Let $Q = \text{EC}(p)(S)$.
 - Update $P_{ec} \leftarrow P_{ec} \cdot Q$ and $S \leftarrow S \oplus S(Q)$.
- end for**
3. **if** $S = 0$ **then** return P_{ec} **else** declare failure.

Here the notation $S \oplus S(Q)$ stands for the symmetric difference of the sets S and $S(Q)$ or addition modulo two, if the syndromes are represented by binary strings. To conform with conventions made in Section II C, we may let the decoder on failure return an arbitrary Pauli operator that is consistent with the measured syndrome. Of course, this is no better than initializing the memory with an arbitrary state. The discussion above implies that the RG decoder has running time $O(N^2 \log N)$ in the worst case, and $O(N \log N)$ in the case of sparse syndromes.

D. Cluster decomposition

Given a length scale r , the cluster decomposition of defects is to partition the defects into maximally connected subsets (connected components) of the syndrome at scale r . Naively, the task to compute the decomposition of all the defects into clusters will take time $O(m^2)$ where m is the total number of defects. The density of defects will typically be constant irrespective of the system size, and the computation time for decomposition will be $O(N^2)$, where N is the volume of the system. However, by exploiting the geometry of simple cubic lattice, we can do it in time $O(N)$. This is the optimal scaling since we have to sweep through the whole system anyway to identify the position of defects.

If $r = 1$, the problem is to label the connected components of binary array [4, 33]. Given a defect u_0 , we can compute the connected component containing u_0 in time $O(m)$ where m is the number of defects in the component. One prepares an empty queue (first-in-first-out data structure), and

puts u_0 into it. The subsequent computing is as follows: (i) Pop out the first element u from the queue, and of the neighborhood put the *unlabeled* defects into the queue and label them. (ii) Repeat until the queue becomes empty. Every defect in a connected component j is stored in the queue only once. Hence, this process computes the component j of a given defect in time proportional to the number m_j of defects in j . One examines the whole system in some order and finds the connected component whenever there is an unlabeled defect. The total computation time is proportional to $N + O(1) \sum_j m_j = O(N)$, since the connected components are disjoint.

For $r > 1$, the algorithm begins by dividing the whole lattice into boxes of linear size r or smaller. The defects in a box certainly belong to a single connected component (recall that we use the l_∞ metric). The defects in the boxes B, B' belong to the same component if and only if there is a pair $u \in B, v \in B'$ of defects such that $d(u, v) \leq r$. In other words, we evaluate the binary function

$$\delta(B, B') = \begin{cases} 1 & \text{if there are } u \in B, v \in B' \text{ such that } d(u, v) \leq r, \\ 0 & \text{otherwise} \end{cases}$$

for each neighbor B' of B ; if B does not meet B' , we know that $\delta(B, B') = 0$.

Given the table of δ , we can finish computing the decomposition in time $O((L/r)^D)$ as in the $r = 1$ case. We show that the computation of $\delta(B, B')$ can be done in time $O(r^D)$ where D is the dimension of the lattice. Then, the total time to compute the table of δ for all adjacent boxes will be $O(r^D(L/r)^D) = O(N)$. Let B and B' be adjacent. For clarity of presentation, we restrict to $D = 2$. Suppose B and B' meet along an edge parallel to x -axis. Since any difference $|x - x'|$ of x -coordinates of the defects in $B \cup B'$ is at most r , we only need to compare y -coordinates. That is, the problem is reduced to one dimension. It suffices to pick two defects from B, B' respectively that are the closest to the x -axis. If the y -coordinates differ at most by r , then $\delta(B, B') = 1$; otherwise, $\delta(B, B') = 0$.

Suppose B and B' meet at a vertex. Without loss of generality, we assume B is in the third quadrant, and B' is in the first quadrant. Define a binary function δ' on B' as

$$\delta'(i, j) = \begin{cases} 1 & \text{if there is a defect } (x, y) \in B' \text{ where } x \leq i \text{ and } y \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

The function table of δ' is computed in time $O(r^2)$. It is important to note that $\delta'(i, j) = 1$ implies $\delta'(i + 1, j) = \delta'(i, j + 1) = 1$. One starts from the origin and sets $\delta'(\frac{1}{2}, \frac{1}{2}) = 1$ if there is a defect at $(\frac{1}{2}, \frac{1}{2})$; otherwise $\delta'(\frac{1}{2}, \frac{1}{2}) = 0$. Here, $(\frac{1}{2}, \frac{1}{2})$ means the elementary square in the first quadrant

that is the closest to the origin. Proceeding by a lexicographic order of the coordinates, one sets $\delta'(i, j) = 1$ if $\delta'(i - 1, j) = 1$, or $\delta'(i, j - 1) = 1$, or there is a defect at (i, j) ; otherwise $\delta'(i, j) = 0$. It is readily checked that this procedure correctly computes δ' . Equipped with this δ' table, we can immediately test for each defect in B whether there is a defect in B' within distance r . Thus, we have computed $\delta(B, B')$ in time $O(r^2) + O(m)$ where m is the number of defects in B , which is at most $O(r^2)$. The computation of δ in higher dimensions is similar.

The computation of the minimal enclosing box for each cluster is also efficient. Given the coordinates of the m points in the cluster, we read out, say, x -coordinates x_1, \dots, x_m . The minimal enclosing interval B_x of x_1, \dots, x_m under periodic boundary conditions, is the complement of the longest interval between consecutive points x_i and x_{i+1} which can be computed in time $O(m)$. B_x is unambiguous if the diameter of the cluster is smaller than $L/2$. The minimal enclosing box is the product set $B_x \times B_y \times B_z$, whose vertices are computed in time $O(m)$.

E. Broom algorithm

Now we describe an efficient algorithm for the 3D cubic code that tests whether a cluster is neutral. If the test is positive, the algorithm also returns a Pauli operator E that annihilates the cluster. Recall that the 3D cubic code has property that if the cluster is neutral, E is supported in the minimal box that encloses the cluster. This is because the erasure process [26] is possible as long as it does not encounter a defect.

By the duality between Z - and X -type operator, we assume E is consisted entirely of Z errors. Since in the cubic code there is only one X -type of stabilizer generators, the positions of defects completely determines the X -type syndrome. We shall use double-letter indices to represent single-site Pauli operators; for example, $IZ \equiv I \otimes Z$, $, $, etc. Let the qubits have integer coordinates, and the defects have half-integer coordinates.$$

Elementary syndromes are depicted in Fig. 4. We fix a box B that encloses all the defects in the neutral cluster. We will sweep the defects to bottom-left-back corner. Since each defect is a Z_2 charge, they will disappear in the end. The algorithm begins with the top-right foremost vertex of B on the dual lattice. If there is a defect at $(x + \frac{1}{2}, y + \frac{1}{2}, z + \frac{1}{2})$, we apply ZI at (x, y, z) to eliminate it. This might create another defects as there are four defects in the elementary syndrome. Important is that the potentially new defects are all contained in the box B we started with. Continuing with ZI we push all the defects in the foremost plane of B to the left vertical line and the bottom horizontal line. During this process, we record where ZI has been applied.

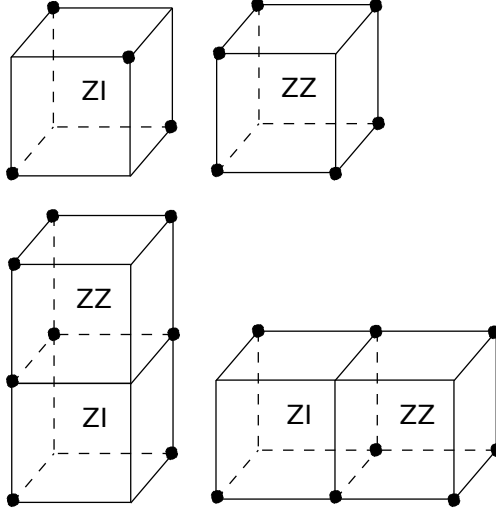


FIG. 4. Elementary syndromes created by Z errors. The vertices which are on the dual lattice, represent the defects created by the error at the center. The elementary syndrome by ZI is used to push the defects to the bottom and to the left. The syndromes by errors of weight three is used to push the defects to the bottom-left corner.

For the defects on the vertical line at the left or on the horizontal line at the bottom, we use the operator of weight 3 to further move the defects to the bottom-left corner. See Fig. 4. This will in general create more defects behind, all of which are still contained in B . Thus, we have moved all the defects on the foremost plane to the bottom-left corner except for the three sites t, u, v :

$$\begin{array}{ccc} t & o & \\ | & & \\ u & - & v \end{array} \quad (28)$$

That is, if E' is the recorded operator during the sweeping process, the syndrome $S(EE') \subseteq B$ has potential defects only at t, u, v on the foremost plane.

Let o be at $(x_o + \frac{1}{2}, y_o + \frac{1}{2}, z_o + \frac{1}{2})$. By considering the multiplication by suitable stabilizer generators Q^Z , we can assume that EE' is the identity on the plane $x = x_o$, except (x_o, y_o, z_o) . Since there is no defect at o , the operator at (x_o, y_o, z_o) has to commute with XX ; it is either II or ZZ . Applying ZZ if necessary, the operator at (x_o, y_o, z_o) will become II , and the defects at t, u, v will disappear. In this way, we have successfully pushed all the defects to the next-to-furthest plane. We emphasize that the box B still envelops all the defects, and further B can be shrunk in one direction.

Due to the threefold symmetry of the cubic code, one can carry out this broom algorithm along any of three directions. We will have at last a box B of volume 1 that encloses all defects. The defects in the cluster must be from one of the three elementary syndrome cubes created either by

ZI , IZ , or ZZ , which are easily eliminated. It is clear that in time $O(V)$ the error operator has been computed up to stabilizer, where V is the initial volume of the minimal enclosing box of the cluster.

Note that a similar sweeping algorithm is applicable to the toric code family for any dimensions.

VII. CORRECTABILITY OF ERRORS WITH A LOGARITHMIC ENERGY BARRIER

Let P be an unknown Pauli error. Suppose we are promised that P has a sufficiently small energy barrier, namely, $\Delta(P) \leq c \log L$, for some constant c that will be chosen later. In this section we prove that any such error P will be corrected by the RG decoder for any topological stabilizer code satisfying the no-strings rule. For completeness, we define the no-strings rule formally.

A. No-strings rule

Let P be any Pauli operator supported on a cube with linear size at most L_{tqo} , and let $S = S(P)$ be its syndrome. Suppose S can be partitioned into a pair of disjoint clusters, $S = C_1 \cup C_2$, such that $d(C_a) \leq r$ and $d(C_1, C_2) \geq \alpha r$ for some $r \geq 1$ and $\alpha \geq 1$. We call such P a *string segment* with *aspect ratio* α . Intuitively, a string segment with a large aspect ratio, $\alpha \gg 1$, is an operator capable of creating a pair of well-separated clusters of defects from the vacuum. We say P is a *trivial* string segment if the clusters of defects C_1, C_2 are neutral (note that the full syndrome $S = C_1 \cup C_2$ is always neutral due to our assumptions on P). Throughout this section we only consider topological stabilizer codes following Definition 1. Let $\alpha \geq 1$ be any constant.

Definition 3. A family of topological stabilizer codes obeys the *no-strings rule* with a constant α if any logical string segment with the aspect ratio greater than α is trivial.

The 3D Cubic Code obeys no-string rule with $\alpha = 5$ under the current ℓ_∞ -metric convention [26].

B. Errors with a logarithmic energy barrier

In order to assess the capability of the RG decoder rigorously, we will need some of the results from Ref. [7]. Assume throughout this section that a family of topological stabilizer codes $\{C_L\}_L$ with $L_{tqo} \geq L^\gamma$ obey the no-strings rule with some constant α . Define

$$\xi(p) = (10\alpha)^p$$

for notational convenience.

Definition 4 ([7]). A syndrome S is called *sparse at level p* if S can be partitioned into disjoint clusters, $S = C_1 \cup \dots \cup C_n$, such that

$$d(C_a) \leq \xi(p) \quad \text{and} \quad d(C_a \cup C_b) > \xi(p+1)$$

for all $a \neq b$. Otherwise, S is called *dense at level p* .

(The term ‘dense’ should not be confused with the ‘dense subset’ of a topological space, with which any nonempty open set intersects; the two notions are even conceptually different. Here, the denseness is just the negation of sparseness.)

Proposition 5 (Lemma 1 of [7]). *If S is dense at all levels $q = 0, \dots, p$ then S contains at least $p + 2$ defects.*

Let $\Gamma = (P_0, P_1, \dots, P_t)$ be an error path implementing P with the energy cost $m(\Gamma) = m$, see Section V A. Here $P_0 = I$, $P_t = P$, while $E_j \equiv P_j P_{j-1}$ are single-qubit Pauli operators for all j . A sequence of syndromes $S(j) \equiv S(P_j)$, $j = 0, \dots, t$ is called a *level-0 syndrome history*. Note that $S(0) = 0$ and $S(t) = S$. For any integer $p \geq 1$ define inductively a *level- p syndrome history* as a subsequence of $\{S(j)\}_{j=0, \dots, t}$ including only $S(0)$, $S(t)$ and all syndromes $S(j)$, $0 < j < t$ which are dense at all levels $q = 0, \dots, p - 1$. Proposition 5 implies that there is a level $p_{max} \leq m - 1$ such that the level- p_{max} syndrome history contains only the initial empty syndrome $S(0) = 0$ and the final syndrome $S(t) = S$. The key technical result is the following.

Proposition 6 (Lemma 2 of [7]). *Let S' and S'' be any consecutive pair of syndromes in the level- p syndrome history. Let E be the product of all errors E_j that occurred in the error path Γ between S' and S'' . If $4m(2 + \xi(p)) < L_{tq_0}$ then there exists a stabilizer $G \in \mathcal{G}$ such that $E \cdot G$ is supported on the $\xi(p)$ -neighborhood of $S' \cup S''$.*

This is useful to characterize a low energy barrier error.

Lemma 2. *Let P be any Pauli error, $S = S(P)$ be its syndrome, and $m = \Delta(P)$ be its energy barrier. Suppose*

$$8m(10\alpha)^m < L_{tq_0}. \tag{29}$$

Then, there exists a stabilizer $G \in \mathcal{G}$ such that $P \cdot G$ has support on the $\xi(m)$ -neighborhood of S . Any R -connected component of S is neutral for $2\xi(m) < R \leq 4\xi(m)$.

Proof. Let us apply Proposition 6 to the level p_{max} , the smallest integer such that the syndrome history at that level has only the initial and final syndrome. Since $p_{max} < m$, the condition $4m(2 + \xi(p_{max})) < L_{tqo}$ is satisfied. We have $S' = 0$, $S'' = S$, and $E = P$. Hence, there exists a stabilizer $G \in \mathcal{G}$ such that $P \cdot G$ is supported on the $\xi(p_{max})$ -neighborhood of S . It proves the first statement of the lemma.

Let $r = \xi(m) = (10\alpha)^m$. Choose any R such that $2r < R \leq 4r$ and let C_a be any R -connected component of S . Since C_a contains at most m defects, the diameter of C_a is at most mR . Restricting $P \cdot G$ on the r -neighborhood of C_a , we obtain a Pauli operator P_a supported on a cube of linear size at most $mR + 2r \leq 4rm + 2r < L_{tqo}$ by assumption. Furthermore, the support of P_a is separated from $(P_a)^{-1}(P \cdot G)$ by distance at least $R - 2r > 0$. Hence, P_a creates the cluster C_a from the vacuum. Therefore, C_a is neutral. \square

We wish to have a well-separated cluster decomposition.

Lemma 3. *Let S be any cluster of $m > 0$ defects. For any integer $\mu \geq 1$, there exists a nonnegative integer $p < m$ and a decomposition $S = C_1 \cup \dots \cup C_n$ such that $d(C_a) \leq 4^p \mu$ and $d(C_a, C_b) > 2 \cdot 4^p \mu$ for all $a \neq b$.*

Proof. Let us say that a partition of S into clusters is μ -good if it satisfies the statement. By grouping all defects occupying the same elementary cube into a cluster, one obtains a partition $S = C_1 \cup \dots \cup C_g$. Obviously, $g \leq m$, and $d(C_a) \leq \mu$. If this partition is not 0-good, then $g \geq 2$ and there is a pair, say, C_1, C_2 such that $d(C_1, C_2) \leq 2\mu$. Merging C_1 and C_2 into a single cluster C'_2 , one obtains a partition $S = C'_2 \cup C_3 \cup \dots \cup C_g$ where $d(C'_2) \leq 4\mu$. If this partition is not 1-good, then $g \geq 3$ and one can repeat the merging again. After at most $g - 1$ iterations, one arrives at a good partition. \square

Note that the minimal enclosing boxes of distinct cluster do not overlap, since

$$d(b(C_a), b(C_b)) > 2 \cdot 4^p \mu - 4^p \mu - 4^p \mu = 0.$$

The following is the promised property of the RG decoder.

Lemma 4. *Let P be any Pauli error with energy barrier $m = \Delta(P)$. Suppose*

$$(80\alpha)^m < L_{tqo}.$$

Then calling the RG decoder on the syndrome $S(P)$ returns a correcting operator P_{ec} such that PP_{ec} is a stabilizer.

Thus, the RG decoder corrects P if $\Delta(P) < \frac{\gamma}{\log(80\alpha)} \log L$.

Proof. Let $S = S(P)$ be the syndrome. Let p be the integer such that $2\xi(m) < 2^p \leq 4\xi(m)$. Setting $\mu = 2^p$ in Lemma 3, S is decomposed into $S = C_1 \cup \dots \cup C_n$ such that $d(C_a) \leq 2^{p'}$ and $d(C_a, C_b) > 2^{p'+1}$ for all $a \neq b$, where p' is an integer such that $p \leq p' < 2m + p$. By Lemma 2, each C_a is neutral for being a disjoint union of neutral 2^p -connected components. The RG subroutines $EC(s)$ with $s = 0, 1, \dots, p-1$, can only annihilate some neutral 2^s -connected components of C_a , which does not alter the neutrality of C_a . Therefore, the RG decoder from level-0 to p will annihilate each cluster C_a , and hence S at last.

We need to show that $P \cdot P_{ec}$ is a stabilizer, where P_{ec} is the returned correcting operator. Let B_a be the $(10\alpha)^m$ -neighborhood of $b(C_a)$. Our assumptions imply that B_a has diameter smaller than L_{tqo} and distinct B_a 's do not intersect. By construction, the operators P_{ec} and $P \cdot G$ have support in the union $B_1 \cup \dots \cup B_n$. Therefore, $P \cdot G \cdot P_{ec} = Q_1 \cdots Q_n$, where Q_a has support on B_a and has trivial syndrome. Topological order condition implies that Q_a are stabilizers, so is the product. \square

As was mentioned earlier, the full hierarchy of the RG decoder is not necessary to correct the error with the low energy barrier. A single level- p error correction with p proportional to $\log L_{tqo}$, will be sufficient. We nevertheless include the hierarchy since in practice it corrects errors with slightly higher (although only by a constant factor) energy barrier at a marginal slowdown of the decoder. If one wishes to apply the decoder against random errors, the hierarchy becomes necessary. We discuss the character of the RG decoder against random errors in Appendix.

C. Further specialization

A closer analysis reveals a simplification of TestNeutral defined in Section VIC for the 3D Cubic Code. We defined TestNeutral to return the identity operator if a cluster turns out to be charged. The modified TestNeutral' just applies the broom algorithm and returns recorded operator in any case. It gives the *same* characteristic as stated in the Lemma 4. $EC'(p)$ using TestNeutral' will transform a charged cluster C_a to a *different* cluster C'_a , but C'_a is still contained in $b(C_a)$. Due to Lemma 3, $b(C_a)$ do not overlap at a high level p , and $EC'(p)$ will eliminate neutral clusters at last. This specialized version of RG decoder is used in our numerical simulation of the next section.

VIII. NUMERICAL SIMULATION

We test our theoretical bound by numerical simulations. The interaction of the memory system with a thermal bath is simulated by Metropolis evolution. As we wish to observe low temperature behavior we adopt continuous time algorithm by Bortz, Kalos, and Lebowitz (BKL) [6]. A pseudo-random number generation package `RngStream` by L'Ecuyer [35] was used. As before, the coupling constant in the Hamiltonian is set to $J = 1/2$ so a single defect has energy 1. Although the 3D Cubic Code is inherently quantum, it is relevant to consider only X -type errors (bit flip) in the simulation, thanks to the duality of the X - and Z -type stabilizer generators of the 3D Cubic Code. The simulation thus is purely classical. The errors are represented by a binary array of length $2L^3$, and the corresponding syndrome by a binary array of length L^3 .

The memory time is measured to be the first time when the memory becomes unreliable. There are two cases the memory is unreliable: either the broom algorithm fails to remove all the defects so we have to reinitialize the memory, or a nontrivial logical error is occurred. It is thus necessary in our simulation to keep track of the error operator during the time evolution. In fact, most of the time, it was the broom algorithm's failure that made the memory unreliable. Nontrivial logical errors occurred only for very small system sizes $L = 5, 7$.

It is too costly to decode the system every time it is updated. Alternatively, we have performed a trial decoding every fixed time interval

$$T_{ec} = \frac{e^{4\beta}}{100}$$

where β is the inverse temperature. Although the time evolution of the BKL algorithm is stochastic, a single BKL update typically advances time much smaller than T_{ec} . So it makes sense to decode the system every T_{ec} . The exponential factor appears naturally because BKL algorithm advances time exponentially faster as β increases. It is to be emphasized that we do not alter the system by the trial decodings (a copy of the actual syndrome has been created for each trial decoding).

The system sizes L^3 for the simulation are chosen such that the code space dimension is exactly 2, for which the complete list of logical operators is known. If the linear size L is ≤ 200 , this is the case when L is not a multiple of 2, 15, or 63 [26]. For these system sizes, to check whether a logical operator is nontrivial is to compute the commutation relation with the known nontrivial logical operators.

The measured memory time for a given L and β is observed to follow an exponential distribution; a memory system is corrupted with a certain probability given time interval. Specifically, the

probability that the measured memory time is t is proportional to $e^{-t/\tau}$. Thus the memory time should be presented as the characteristic time of the exponential distribution. We choose the estimator for the characteristic time to be the sample average $\bar{T} = \frac{1}{n} \sum_i^n T_i$. The deviation of the estimator will follow a normal distribution for large number n of samples. We calculated the confidence interval to be the standard deviation of the samples divided by \sqrt{n} . For each L , 400 samples when $\beta \leq 5.0$ and 100 samples when $\beta > 5.0$ were simulated. The result is summarized in Fig. 2,3.

Figure 3 clearly supports $\log T_{mem} = \frac{1}{4}c\beta^2 + \dots$. Figure 2 demonstrates the power law for small system size:

$$T_{mem} \propto L^{2.93\beta - 10.5}$$

We wish to relate some details of the model with the numerical coefficients. The Rigorous analysis of the previous section, gives a relatively small coefficient c of the energy barrier for correctable errors by our RG decoder. However, we expect that the coefficient of β in the exponent is the same as the constant c that appear in the energy barrier

$$E = c \log_2 R$$

to create an isolated defect separated from the other by a distance R . This is based on an intuition that the output P' of the decoder would have roughly the same support as the real error P for the most of the time, provided that the error has energy barrier less than $\Delta = c \log_2 L_{tqo}$. Thus, an error of energy barrier less than Δ would be corrected by the decoder. Our empirical formula supports this intuition. It suggests that $c = 2.93 \log 2 = 2.03 \sim 2$.

Indeed, we can illustrate explicitly an error path that separates a single defect from the rest by distance 2^p during which only $2p + 4$ defects are needed. This is an improvement in the upper bound on the energy barrier for separating a charged set of defects, for which it was $c \leq 4$ in [7].

Consider an error of weight 2 that creates 4 defects as shown in the top of Fig. 5. We call it the *level-0 hook*. The bottom sequence depicts a process to create a configuration shown at the bottom-left, which we call *level-1 hook*. One sees that level-1 hook is similar with ratio 2 to level-0, and is obtained from level-0 with extra 2 defects. One defines level- p hooks hierarchically. We claim that a level- p hook can be constructed from the vacuum using $2p + 4$ defects. The proof is by induction. The case $p = 1$ is treated in the diagrams. Suppose we can construct level- p hook using $2p + 4$ defects. Consider the 2^{nd} , 4^{th} , 6^{th} , and 8^{th} steps in Fig. 5. They can be viewed as a minuscule version of level- p steps that construct a level- $(p + 1)$ hook from the level- p hooks. It

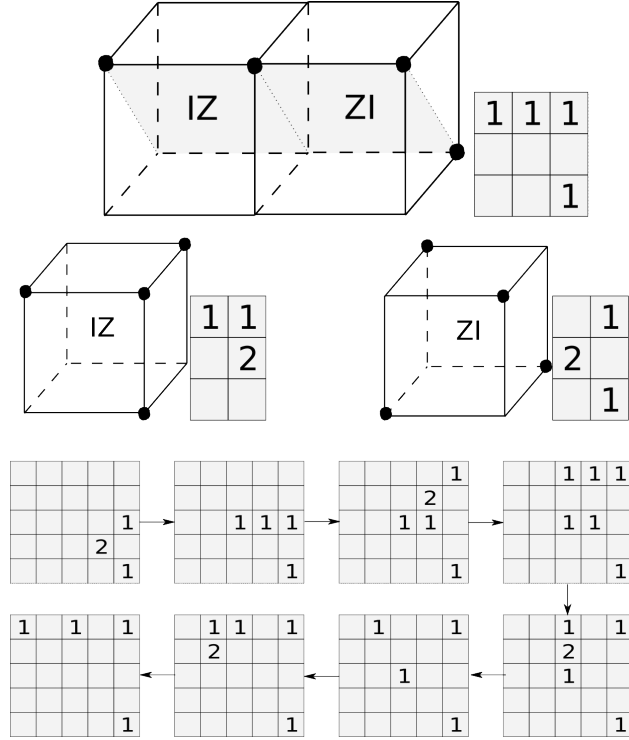


FIG. 5. Construction of a hook of level 2 from the vacuum. The grid diagram represents the position and the number of defects in the $(x = z)$ -plane. For each transition, an operator of weight 1 is applied. The total number of defects never exceeds 6. From a level-0 hook (the second diagram in the sequence), a level-1 hook (the last in the sequence) is constructed using extra 2 defects.

requires at most $2p + 4 + 2$ defects to perform the level- p step; this completes the induction. It may not be obvious whether a high level hook corresponds to a nontrivial logical operator, but such a large hook is bad enough to make our decoder fail.

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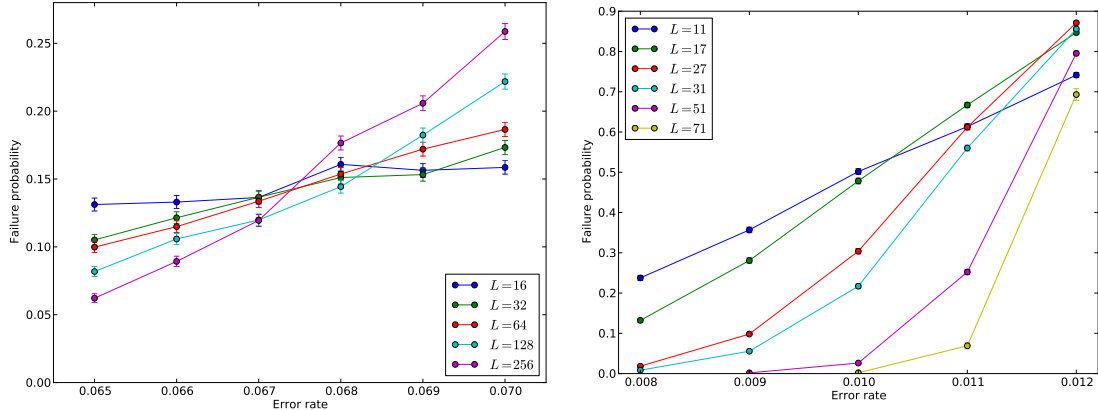


FIG. 6. The thresholds of 2D toric code (left) and 3D cubic code (right) under independent random bit-flip errors using our RG decoder, are measured to be $p_c(2\text{D toric}) = 6.7(1)\%$ and $p_c(3\text{D cubic}) \gtrsim 1.1\%$.

Appendix A: Benchmark of the decoder

Given a decoder, a family of quantum codes indexed by code length (system size) is said to have an *error threshold* p_c if the probability for decoder to fail approaches zero in the limit of large code length provided the random error rate p is less than p_c . We tested our decoder with respect to random uncorrelated bit-flip errors on the well-studied 2D toric code. The error threshold is measured to be $6.7(1)\%$. See Fig. 6. It is roughly two-thirds of the best known value 10.3% based on the perfect matching algorithm [18, 28], or 9% based on a renormalization group decoder of similar nature to ours [19]. This is remarkable for our decoder's simplicity and applicability. The 3D cubic code has threshold $\gtrsim 1.1\%$ under independent bit-flip errors.

Appendix B: Threshold theorem for topological stabilizer codes

In this section we prove that any topological stabilizer code can tolerate stochastic local errors with a small constant rate assuming that the error correction is performed using the RG decoder. We assume without loss of generality that each stabilizer generator is supported on a unit cube. Each site of the lattice may contain finitely many qubits. A generator at a cube c may act only on qubits of c . We shall assume that errors at different sites are independent and identically distributed. More precisely, let $E(P)$ be the set of sites at which a Pauli error P acts non-trivially. We shall assume that

$$\Pr[E(P) = E] = (1 - \epsilon)^{V - |E|} \epsilon^{|E|} \quad (\text{B1})$$

where $0 \leq \epsilon \leq 1$ is the error rate and $V = L^D$ is the total number of sites (the volume of the lattice). For example, the depolarizing noise in which every qubit experiences X, Y, Z errors with the probability $p/3$ each, satisfies Eq. (B1) with the error rate $\epsilon = 1 - (1 - p)^q$, where q is the number of qubits per site.

Theorem 2. *Suppose a family of stabilizer codes has topological order, see Definition 1. Then, there exists a constant threshold $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$ the RG decoder corrects random independent errors with rate ϵ with the failure probability at most $e^{-\Omega(L^\eta)}$ for some constant $\eta > 0$.*

In the rest of this section we prove the theorem. Our proof borrows some techniques from [21, 23, 28], specifically Section 5.1 of [23]. For reader's convenience, we briefly summarize the RG decoding algorithm below (see Section VI for details).

Recall that we use ℓ_∞ -metric. A cube of linear size r thus has diameter r . Let $S = S(P)$ be the syndrome of a Pauli error P considered as a set of defects (violated stabilizers). A subset $M \subseteq S$ is called r -connected iff M cannot be partitioned into a pair of disjoint proper subsets separated by distance more than r . A maximum r -connected subset of S is called an r -connected component of S . A cluster of defects $M \subseteq S$ is called *neutral* if it can be created from the vacuum by a Pauli operator P supported in a cube of linear size L_{tq_0} . The smallest rectangular box enclosing a cluster M will be denoted $b(M)$. Any neutral cluster M can be created by a Pauli operator supported on the 1-neighborhood of $b(M)$, see Definition 1.

The *level- p error correction* $EC(p)$ on a syndrome S is the following subroutine. (i) find all neutral 2^p -connected components M of S , (ii) for each M found at step 1, calculate and apply a Pauli operator P supported on the 1-neighborhood of $b(M)$ that annihilates M , and update the syndrome accordingly. Calling the full RG decoder on a syndrome S involves the following steps: (i) run $EC(0), EC(1), \dots, EC(\lfloor \log_2 L_{tq_0} \rfloor)$, (ii) if the resulting syndrome S is empty, return the accumulated Pauli operator applied by the subroutines $EC(p)$. Otherwise, declare a failure.

Below we shall use the term ‘error’ both for the error operator P and for the subset of sites E acted on by P , whenever the meaning is clear from the context. Let us choose an integer $Q \gg 1$ and define a class of errors which are properly corrected by the RG decoder, see Lemma 5 below. We will see later that this class of errors actually includes all errors which are likely to appear for small enough error rate.

Definition 5. Let E be a fixed error. A site $u \in E$ is called a *level-0 chunk*. A non-empty subset of E is called a *level- n chunk* ($n \geq 1$) if it is a disjoint union of two level- $(n - 1)$ chunks and its

diameter is at most $Q^n/2$.

The term ‘chunk’ is chosen in order to avoid confusion with ‘cluster’, which is used for a set of defects. Note that a level- n chunk contains exactly 2^n sites. Given an error E , let E_n be the union of all level- n chunks of E . If $u \in E_{n+1}$, then by definition u is an element of a level- $(n+1)$ chunk. Since a level- $(n+1)$ chunk is a union of two level- n chunks, u is contained in a level- n chunk. Hence, $u \in E_n$, and the sequence E_n form a descending chain

$$E = E_0 \supseteq E_1 \supseteq \cdots \supseteq E_m,$$

where m is the smallest integer such that $E_{m+1} = \emptyset$. Let $F_i = E_i \setminus E_{i+1}$, so $E = F_0 \cup F_1 \cup \cdots \cup F_m$ is expressed as a disjoint union, which we call the *chunk decomposition* of E .

Proposition 7. *Let $Q \geq 6$ and M be any Q^n -connected component of F_n . Then M has diameter $\leq Q^n$ and is separated from $E_n \setminus M$ by distance $> \frac{1}{3}Q^{n+1}$.*

Proof. We claim that for any pair of sites $u \in F_n = E_n \setminus E_{n+1}$ and $v \in E_n$ we have $d(u, v) \leq Q^n$ or $d(u, v) > \frac{1}{3}Q^{n+1}$. Suppose on the contrary to the claim, that there is a pair $u \in F_n$ and $v \in E_n$ such that $Q^n < d(u, v) \leq Q^{n+1}/3$. Let $C_u \ni u$ and $C_v \ni v$ be level- n chunks that contains u and v , respectively. Since the diameters of C_u, v are $\leq Q^n/2$ and $d(u, v) > Q^n$, we deduce that C_u and C_v are disjoint. On the other hand,

$$d(C_u \cup C_v) \leq d(u, v) + d(C_u) + d(C_v) \leq Q^{n+1}/2$$

since $Q \geq 6$. Thus, $C_u \cup C_v$ is a level- $(n+1)$ chunk that contains u which shows that $u \in E_{n+1}$. It contradicts to our assumption that $u \in F_n = E_n \setminus E_{n+1}$. \square

Note that in the chunk decomposition a Q^n -connected component P of E_n may not be separated from the rest $E \setminus P$ by distance $> Q^n$.

Lemma 5. *Let $Q \geq 10$. If the length m of the chunk decomposition of an error E satisfies $Q^{m+1} < L_{tqo}$, then E is corrected by the RG decoder.*

Proof. Consider any fixed error P supported on a set of sites E . Let $E = F_0 \cup F_1 \cup \cdots \cup F_m$ be the chunk decomposition of E , and let $F_{j,\alpha}$ be the Q^j -connected components of F_j . Also, let $B_{j,\alpha}$ be the 1-neighborhood of the smallest box enclosing the syndrome created by the restriction of P onto $F_{j,\alpha}$. Proposition 7 implies that

$$d(B_{j,\alpha}) \leq Q^j + 2 \quad \text{and} \quad d(B_{j,\alpha}, B_{k,\beta}) > \frac{1}{3}Q^{1+\min(j,k)} - 2. \quad (\text{B2})$$

Let $P_{ec}^{(p)}$ be the accumulated correcting operator returned by the levels $0, \dots, p$ of the RG decoder. Let us use induction in p to prove the following statement.

1. The operator $P_{ec}^{(p)}$ has support on the union of the boxes $B_{j,\alpha}$.
2. The operators $P_{ec}^{(p)}$ and P have the same restriction on $B_{j,\alpha}$ modulo stabilizers for any j such that $2^p \geq Q^j + 2$.

The base of induction is $p = 0$. Using Eq. (B2) we conclude that any 1-connected component of the syndrome $S(P)$ is fully contained inside some box $B_{j,\alpha}$. It proves that $P_{ec}^{(0)}$ has support on the union of the boxes $B_{j,\alpha}$. The second statement is trivial for $p = 0$.

Suppose we have proved the above statement for some p . Then the operator $P \cdot P_{ec}^{(p)}$ has support only inside boxes $B_{j,\alpha}$ such that $2^p < Q^j + 1$ (modulo stabilizers). It follows that any 2^{p+1} -connected component of the syndrome caused by $P \cdot P_{ec}^{(p)}$ is contained in some box $B_{j,\alpha}$ with $2^p < Q^j + 1$. Note that the RG decoder never adds new defects; we just need to check that 2^{p+1} -connected components do not cross the boundaries between the boxes $B_{j,\alpha}$ with $2^p < Q^j + 1$. This follows from Eq. (B2). Hence $P_{ec}^{(p+1)}$ has support in the union of $B_{j,\alpha}$. Furthermore, if $2^p < Q^j + 1 \leq 2^{p+1}$, the cluster of defects created by $P \cdot P_{ec}^{(p)}$ inside $B_{j,\alpha}$ forms a single 2^{p+1} -connected component of the syndrome examined by $EC(p+1)$. This cluster is neutral since we assumed $Q^{m+1} < L_{tqo}$. Hence $P_{ec}^{(p+1)}$ will annihilate this cluster. The annihilation operator is equivalent to the restriction of $P \cdot P_{ec}^{(p)}$ onto $B_{j,\alpha}$ modulo stabilizers, since the linear size of $B_{j,\alpha}$ is smaller than L_{tqo} . It proves the induction hypothesis for the level $p+1$. \square

The preceding lemma says that errors by which the RG decoder could be confused are those from very high level chunks. What is the probability of the occurrence of such a high level chunk if the error is random according to Eq.(B1)? Since our probability distribution of errors depend only on the number of sites in E , this question is completely percolation-theoretic.

Let us review some terminology from the percolation theory[24]. An event is a collection of configurations. In our setting, a configuration is a subset of the lattice. Hence, we have a partial order in the configuration space by the set-theoretic inclusion. An event \mathcal{E} is said to be *increasing* if $E \in \mathcal{E}, E \subseteq E'$ implies $E' \in \mathcal{E}$. For example, the event defined by the criterion that there exists an error at $(0,0)$, is increasing. The *disjoint occurrence* $\mathcal{A} \circ \mathcal{B}$ of the events \mathcal{A} and \mathcal{B} is defined as the collection of configurations E such that $E = E_a \cup E_b$ is a disjoint union of $E_a \in \mathcal{A}$ and $E_b \in \mathcal{B}$. To illustrate the distinction between $\mathcal{A} \circ \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B}$, consider two events defined as $\mathcal{A} =$ “there are errors at $(0,0)$ and at $(1,0)$ ”, and $\mathcal{B} =$ “there are errors at $(0,0)$ and at $(0,1)$ ”. The

intersection $\mathcal{A} \cap \mathcal{B}$ contains a configuration $\{(0, 0), (1, 0), (0, 1)\}$, but the disjoint occurrence $\mathcal{A} \circ \mathcal{B}$ does not. A useful inequality by van den Berg and Kesten (BK) reads [24, 45]

$$\Pr[\mathcal{A} \circ \mathcal{B}] \leq \Pr[\mathcal{A}] \cdot \Pr[\mathcal{B}] \quad (\text{B3})$$

provided the events \mathcal{A} and \mathcal{B} are increasing.

Proof. (of Theorem 2) Consider a D -dimensional lattice and a random error E defined by Eq. (B1). Let B_n be a fixed cubic box of linear size Q^n and B_n^+ be the box of linear size $3Q^n$ centered at B_n . Define the following probabilities:

$$\begin{aligned} p_n &= \Pr[B_n \text{ has a non-zero overlap with a level-}n \text{ chunk of } E] \\ \tilde{p}_n &= \Pr[B_n^+ \text{ contains a level-}n \text{ chunk of } E] \\ q_n &= \Pr[B_n^+ \text{ contains 2 disjoint level-}(n-1) \text{ chunks of } E] \\ r_n &= \Pr[B_n^+ \text{ contains a level-}(n-1) \text{ chunk of } E] \end{aligned}$$

Note that all these probabilities do not depend on the choice of the box B_n due to translation-invariance. Since a level-0 chunk is just a single site of E , we have $p_0 = \epsilon$. We begin by noting that

$$p_n \leq \tilde{p}_n \leq q_n.$$

Here we used the fact that any level- n chunk has diameter at most $Q^n/2$ and that any level- n chunk consists of a disjoint pair of level- $(n-1)$ chunks. Let us fix the box B_n^+ and let \mathcal{Q}_n be the event that B_n^+ contains a disjoint pair of level- $(n-1)$ chunks of E . Let \mathcal{R}_n be the event that B_n^+ contains a level- $(n-1)$ chunk of E . Then $\mathcal{Q}_n = \mathcal{R}_n \circ \mathcal{R}_n$. It is clear that \mathcal{Q}_n and \mathcal{R}_n are increasing events. Applying the van den Berg and Kesten inequality we arrive at

$$q_n \leq r_n^2.$$

Finally, since B_n^+ is a disjoint union of $(3Q)^D$ boxes of linear size Q^{n-1} , the union bound yields

$$r_n \leq (3Q)^D p_{n-1}.$$

Combining the above inequalities we get $p_n \leq (3Q)^{2D} p_{n-1}^2$, and hence

$$p_n \leq (3Q)^{-2D} ((3Q)^{2D} \epsilon)^{2^n}.$$

The probability p_n is doubly exponentially small in n whenever $\epsilon < (3Q)^{-2D}$. If there exists at least one level- n chunk, there is always a box of linear size Q^n that overlaps with it. Hence, on

the finite system of linear size L , the probability of the occurrence of a level- m chunk is bounded above by $L^D p_m$. Employing Lemma 5, we conclude that the RG decoder fails with probability at most $p_{fail} = L^D p_m$ for any m such that $Q^{m+1} < L_{tqo}$. Since we assumed that $L_{tqo} \geq L^\gamma$, one can choose $m \approx \gamma \log L / \log Q$. In this case $p_{fail} = \exp(-\Omega(L^\eta))$ for $\eta \approx \gamma / \log Q$. We have proved our theorem with $\epsilon_0 = (3Q)^{-2D}$ where $Q = 10$. \square

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- [1] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher. Non-abelian statistics and topological quantum information processing in 1D wire networks. *Nature Physics*, 7:412–417, 2011.
 - [2] R. Alicki, M. Fannes, and M. Horodecki. On thermalization in Kitaev’s 2D model. *J. Phys. A: Math. Theor.*, 42:065303, 2009.
 - [3] R. Alicki, M. Horodecki, P. Horodecki, and R. Horodecki. On thermal stability of topological qubit in Kitaev’s 4D model. *Open Syst. Inf. Dyn.*, 17:1, 2010.
 - [4] T. Asano and H. Tanaka. In-place algorithm for connected components labeling. *JPRR*, 1:10–22, 2010.
 - [5] D. Bacon. Operator quantum error correcting subsystems for self-correcting quantum memories. *Phys. Rev. A*, 73:012340, 2006.
 - [6] A. B. Bortz, M. H. Kalos, and J. L. Lebowitz. A new algorithm for Monte Carlo simulation of Ising spin systems. *J. Comp. Phys.*, 17(1):10 – 18, 1975.
 - [7] S. Bravyi and J. Haah. On the energy landscape of 3D spin hamiltonians with topological order. *Phys. Rev. Lett.*, 107:150504, 2011.
 - [8] S. Bravyi, M. Hastings, and S. Michalakis. Topological quantum order: stability under local perturbations. *J. Math. Phys.*, 51:093512, 2010.
 - [9] S. Bravyi and M. B. Hastings. A short proof of stability of topological order under local perturbations. *Comm. Math. Phys.*, 307:609–627, 2011.
 - [10] S. Bravyi and R. Köenig. Disorder-assisted error correction in Majorana chains. 2011, arXiv:1108.3845.
 - [11] S. Bravyi, B. Leemhuis, and B. M. Terhal. Topological order in an exactly solvable 3D spin model. *Ann. Phys.*, 326(4):839 – 866, 2011.
 - [12] S. Bravyi and B. M. Terhal. A no-go theorem for a two-dimensional self-correcting quantum memory based on stabilizer codes. *New J. Phys.*, 11:043029, 2009.
 - [13] C. Castelnovo and C. Chamon. Entanglement and topological entropy of the toric code at finite temperature. *Phys. Rev. B*, 76(18):184442, 2007.
 - [14] C. Chamon. Quantum glassiness. *Phys. Rev. Lett.*, 94:040402, 2005.
 - [15] S. Chesi, D. Loss, S. Bravyi, and B. M. Terhal. Thermodynamic stability criteria for a quantum memory based on stabilizer and subsystem codes. *New J. Phys.*, 12:025013, 2009.
 - [16] S. Chesi, B. Röthlisberger, and D. Loss. Self-correcting quantum memory in a thermal environment.

- Phys. Rev. A*, 82(2):022305, 2010.
- [17] E. B. Davies. Markovian master equations. *Comm. Math. Phys.*, 39:91–110, 1974.
- [18] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill. Topological quantum memory. *J. Math. Phys.*, 43:4452–4505., 2002.
- [19] G. Duclos-Cianci and D. Poulin. Fast decoders for topological quantum codes. *Phys. Rev. Lett.*, 104:050504, 2009.
- [20] L. Fu and C. L. Kane. Superconducting proximity effect and Majorana fermions at the surface of a topological insulator. *Phys. Rev. Lett.*, 100:096407, 2008.
- [21] P. Gács. Reliable computation with cellular automata. *J. Comp. and System Sciences*, 32(1):15 – 78, 1986.
- [22] D. Gottesman. A theory of fault-tolerant quantum computation. *Phys.Rev. A*, 57:127, 1998.
- [23] L. Gray. A reader’s guide to Gács’s “positive rates” paper. *J. Stat. Phys.*, 103:1–44, 2001.
- [24] G. Grimmett. *Percolation, 2ed.* Springer, 1999.
- [25] J. Haah. in preparation.
- [26] J. Haah. Local stabilizer codes in three dimensions without string logical operators. *Phys. Rev. A*, 83(4):042330, 2011.
- [27] A. Hamma, C. Castelnovo, and C. Chamon. Toric-boson model: Toward a topological quantum memory at finite temperature. *Phys. Rev. B*, 79(24):245122, 2009.
- [28] J. Harrington. *Analysis of quantum error-correcting codes: symplectic lattice codes and toric codes.* PhD thesis, California Institute of Technology, 2004.
- [29] M. B. Hastings. Topological order at non-zero temperature. *Phys. Rev. Lett*, 107:210501, 2011.
- [30] L. Hormozi, N. E. Bonesteel, and S. H. Simon. Topological quantum computing with Read-Rezayi states. *Phys. Rev. Lett.*, 103:160501, 2009.
- [31] S. Iblisdir, D. Pérez-García, M. Aguado, and J. Pachos. Thermal states of anyonic systems. *Nuclear Physics B*, 829(3):401 – 424, 2010, 0812.4975.
- [32] A. Kay and R. Colbeck. Quantum self-correcting stabilizer codes. 2008, arXiv:0810.3557.
- [33] B.R. Kiran, K.R. Ramakrishnan, Y.S. Kumar, and K.P. Anoop. An improved connected component labeling by recursive label propagation. In *Communications (NCC), 2011 National Conference on*, pages 1 –5, 2011.
- [34] A. Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Ann. Phys.*, 303:2–30, 2003.
- [35] P. L’Ecuyer, R. Simard, E. J. Chen, and W. D. Kelton. An object-oriented random-number package with many long streams and substreams. *Operations Research*, 50(6):1073–1075, 2002.
- [36] S. Michalakis and J. Pytel. Stability of frustration-free hamiltonians. 2011, arXiv:1109.1588.
- [37] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma. Non-abelian anyons and topological quantum computation. *Rev. Mod. Phys.*, 80:1083, 2008.
- [38] Z. Nussinov and G. Ortiz. Autocorrelations and thermal fragility of anyonic loops in topologically quantum ordered systems. *Phys. Rev. B*, 77(6):064302, 2008.

- [39] Y. Oreg, G. Refael, and F. von Oppen. Helical liquids and Majorana bound states in quantum wires. *Phys. Rev. Lett.*, 105:177002, 2010.
- [40] F. Pastawski, L. Clemente, and J. I. Cirac. Quantum memories based on engineered dissipation. *Phys. Rev. A*, 83:012304, 2011.
- [41] J. D. Sau, R. M. Lutchyn, S. Tewari, and S. Das Sarma. Generic new platform for topological quantum computation using semiconductor heterostructures. *Phys. Rev. Lett.*, 104:040502, 2010.
- [42] H. Spohn. An algebraic condition for the approach to equilibrium of an open n -level system. *Lett. Math. Phys.*, 2:33–38, 1977.
- [43] T. D. Stanescu, R. M. Lutchyn, and S. Das Sarma. Majorana fermions in semiconductor nanowires. *Phys. Rev. B*, 84:144522, 2011.
- [44] C. Stark, A. Imamoglu, and R. Renner. Localization of toric code defects. *Phys. Rev. Lett.*, 107:030504, 2011.
- [45] J. van den Berg and H. Kesten. Inequalities with applications to percolation and reliability. *J. Applied Probability*, 22:556–569, 1985.
- [46] J. R. Wootton and J. K. Pachos. Bringing order through disorder: Localisation of errors in topological quantum memories. *Phys. Rev. Lett.*, 107:030503, 2011.
- [47] B. Yoshida. Feasibility of self-correcting quantum memory and thermal stability of topological order. *Ann. Phys.*, 326:2566, 2011.