Phase dynamics of convective rolls

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The equation of motion for the slow time dependence of convective rolls due to long-wavelength inhomogeneities is shown to have a singular dependence on the wave vector of the disturbance. Consequences for the skew varicose instability and the wave-number selection principle in textures of curved rolls suggested by Pomeau and Manneville are discussed.

I. INTRODUCTION

In many nonequilibrium systems driven by a spatially homogeneous forcing there is a transition from a uniform solution to one varying periodically in space. The Rayleigh-Benard instability is a canonical example: A transition occurs from the conducting state, with zero fluid velocities and a vertical temperature gradient independent of horizontal coordinates, to the convective roll state, with these quantities varying periodically with the horizontal direction. In the idealized limit of a laterally infinite system the overall position of the rolls and their orientation are free variables—a uniform displacement or rotation leads to an equivalent solution. An interesting motion of the system above onset then involves the slow change of these variables driven by slow spatial inhomogeneities. In this motion the local roll structure is only slightly perturbed, and the dynamics may be described in terms of slowly varying functions describing the position and orientation of the local roll structure. Such motion may well be involved in the low-frequency turbulence observed by Ahlers and Behringer[1] in large aspect ratio (ratio of horizontal extent to fluid depth) cylindrical cells. It has also been directly observed in a large aspect ratio rectangular cell a finite distance above threshold by Gollub and Steinman.[2] In this latter case the motion of defects in the roll pattern is a prominent feature.

Since the symmetries imply that homogeneous translations and rotations lead to no time dependence, it seems natural to assume a gradient expansion for the dynamics of slow spatial inhomogeneities. Such an approach was suggested by Pomeau and Manneville[3] who introduced the general idea of a phase variable to describe this motion (although the phase variable was already an explicit feature of the complex amplitude equation introduced by Newell and Whitehead[4] and Segel[5] to describe convection close to onset) and suggested the notion of "phase diffusion." In fact, we will show that the gradient expansion for the phase dynamics is singular, with, for example, the dynamics of a sinusoidal disturbance $\sin(\vec{K}\cdot\vec{r})$ about a parallel roll state depending on the ratio K_x/K_y (with the x direction normal to the rolls) as K_x and K_y approach zero.

There are two important consequences of this analysis. The first is the modification of the wave-number selection principle in axisymmetric convection due to Pomeau and Manneville.[6] If we first assume that the gradient expansion of the phase dynamics is smooth, then it is readily seen that at large distances from the center of curvature in axisymmetric convection the phase dynamics driven by the roll curvature cannot be balanced by other terms. A static solution is then obtained only if the basic wave number of the rolls is such as to make the transverse diffusion constant $D_\perp$, and hence the driving, equal to zero. Since this criterion is also that for marginal stability against transverse fluctuations, Pomeau and Manneville suggest the fascinating result of wave-number selection at a marginal stability point. In fact, the gradient expansion is *not* smooth, and although a wave-number selection principle remains, it is no longer at the marginal instability.

The second result is an understanding of the skew varicose instability from the long-wavelength phase dynamic equation. The skew varicose instability is a long-wavelength instability (i.e., it occurs first for the disturbance wave number $K\rightarrow 0$ and so should be derivable from the phase dynamics) with a finite value of K_x/K_y. It is very important in limiting the band of stable stationary convection close to threshold in low Prandtl number fluids. A smooth gradient expansion leads to no such instability, and understanding the singular nature of the expansion is crucial in obtaining a simple description of this instability.

II. DISCUSSION

The phase variable describing the slow modulation is introduced as follows. Consider a time-independent solution $\vec{v}(kx,z)$ periodic in the x direction with period $2\pi k^{-1}$. In general, $\vec{v}$ is a vector field of all the hydrodynamic variables. Translational invariance implies that $\vec{v}(kx+\phi,z)$, with ϕ an arbitrary constant, is an equally good solution. If ϕ is varying slowly with the horizontal coordinates then $\vec{v}(kx+\phi(x,y),z)$ is close to being a solution, and the time dependence of $\vec{v}$, and hence ϕ itself, may be expected to be correspondingly slow. For ϕ small, corresponding to small deviations from a straight roll structure, Pomeau and Manneville[3] suggest an equation of the form

$$\dot{\phi}=D_{\parallel}\partial_x^2\phi+D_{\perp}\partial_y^2\phi+O(\partial^4\phi)\,, \tag{1}$$

where $D_{\parallel}$ and $D_{\perp}$ are diffusion constants.

The dynamics of the Rayleigh-Benard instability may be calculated just above onset using the amplitude equation, which is derived as a systematic expansion in $\epsilon=(R-R_c)/R_c$, with R the Rayleigh number and R_c its critical value. Using the lowest-order amplitude equation of Newell and Whitehead and Segel, Eq. (1) may indeed be derived with

$$\begin{aligned}\tau_0 D_{\parallel}&=\left[\frac{1-3Q^2}{1-Q^2}\right]\xi_0^2+O(\epsilon^{1/2})\,,\\ \tau_0 D_{\perp}&=(q/k_0)\xi_0^2+O(\epsilon)\,,\end{aligned} \tag{2}$$

where

$$Q=\epsilon^{-1/2}\xi_0 q=\epsilon^{-1/2}\xi_0(k-k_0) \tag{3}$$

with k_0 the wave number minimizing R_c, ξ_0 is the convenient length scale for horizontal coordinates equal to $0.385d$ for rigid upper and lower boundaries with separation d, and τ_0, equal to $19.65\sigma(\sigma+0.5117)^{-1}$ vertical thermal diffusion times, sets the time scale.

An important result easily derived from the phase diffusion equation is the long-wavelength instability of the convective pattern, signalled by a negative diffusion constant. Equations (1)–(3) reproduce the known results close to onset for the longitudinal (Eckhaus) instability at $Q^2=\frac{1}{3}$, and transverse (zig-zag) instability at $Q=0$. Stable convection close to onset is then limited to the range of wave numbers

$$0<k-k_0<\frac{1}{\sqrt{3}}\epsilon^{1/2}\xi_0^{-1}\,, \tag{4}$$

with $O(\epsilon)$ corrections.

In a subsequent paper Pomeau and Manneville[6] suggest the fascinating result that for axisymmetric convection there is a unique roll wave number sufficiently far away from the center of curvature, that is also determined by the marginal stability condition for Eq. (1), $D_{\perp}(Q)=0$ [i.e., $Q=0$ for the lowest-order calculation given by Eq. (2)]. The reader is referred to the original paper for the very elegant derivation leading to this result. The essential ingredient is to compare an r^{-1} expansion, with r the distance from the center, to the gradient expansion leading to Eq. (1). The only assumption made is the smoothness of these expansions; no assumption of proximity to onset is needed. Their result may, however, be simply illustrated from Eq. (1) in this limit. If instead of a solution periodic in x, a basic solution periodic in r (at large r) is assumed, the changes to Eq. (1) are given by

$$\begin{aligned}\partial_x^2\phi&\to\partial_r^2\phi=\frac{dq}{dr}\,,\\ \partial_y^2\phi&\to k_0/r\,,\end{aligned} \tag{5}$$

where $q(r)$ is the local wave-number change from k_0, assumed small. Stationarity then requires

$$\frac{dq}{dr}+\frac{q}{r}=0\,, \tag{6}$$

where the lowest-order results [Eq. (2)] have been used. This leads to

$$q=c/r\xrightarrow{r\to\infty}0\,, \tag{7}$$

with c an integration constant: A unique wave number is found, for sufficiently large r, at the zig-zag marginal instability. This result has also been derived by Zippelius[7] directly from the axisymmetric amplitude equation of Brown and Stewartson.[8] The result may be seen to follow from the fact that at large distances a nonzero value of the curvature term $D_{\perp}(Q)k_0r^{-1}$ forcing the circular rolls to grow or shrink cannot be balanced. Note, however, that the balancing is determined by the constant c, not fixed by this argument. Presumably if the center of curvature is part of the flow field $c=O(1)$. For axisymmetric flow outside a cylinder c will be determined by boundary effects at this cylinder.

Although Pomeau and Manneville derived their result for axisymmetric convection, an analysis of the lowest-order amplitude equation generalized to arbitrary orientations (see, for example, Cross[9]) suggests a similar result for any set of curved rolls. It is conceivable that in the textures characteristically observed in convection in finite cells, involving curved rolls with defects, this mechanism may provide a wave-number selection principle in regions of relatively undistorted rolls far away from the defects. In this case the result assumes an importance far greater than that suggested by the somewhat idealized situation of axisymmetric convection.

A crucial assumption in the Pomeau-Manneville analysis is the gradient expansion [Eq. (1)]. In fact, we will show that this is not, in general, correct for the Rayleigh-Benard convection, and that Eq. (1) must be modified to

$$\dot{\phi} = D_{\parallel}\partial_x^2\phi + D_{\perp}\partial_y^2\phi + U \ , \tag{8}$$

where U is a drift term arising from a horizontal fluid velocity, in turn, driven by phase gradients. The important and rather surprising point is that if U is eliminated in favor of the phase variable, the gradient expansion is, in fact, singular. For plane-wave variation $\phi \propto \sin(\vec{K}\cdot\vec{r})$ the drift U will depend on K_x/K_y as $K \to 0$.

We will demonstrate the singular nature of the gradient expansion using the formal expansion scheme of Newell and Whitehead and Segel, in which the single small parameter ϵ is used to scale both the amplitudes of the hydrodynamic variables and the rate of spatial variation of the basic roll pattern and its perturbations. Calculations are then done to a chosen order in ϵ. Ultimately, for particular quantitative calculations (e.g., of stability boundaries) it may well be better to introduce various independent expansion parameters. For the qualitative conclusion sought here, the single straightforward expansion scheme provides a simpler derivation.

For the physically artificial but formally instructive case of free-slip boundary conditions at the upper and lower plate, the modified phase equation (8) follows directly from the new amplitude equation of Siggia and Zippelius[10] and is implicit in their stability analysis, which leads to results quite different from those implied by Eq. (1). The additional singular drift term arises from the vertical vorticity with uniform z dependence. Free-slip boundaries apply no restoring force on this flow in the long-wavelength limit, and the response to the driving by the nonlinearities in the Navier-Stokes equation becomes singular. The long-wavelength vertical vorticity is an additional "dangerous mode" in the system (i.e., mode of zero eigenvalue). It is not, then, too surprising that the equations become smooth in a gradient expansion only when written as coupled equations for ϕ and U. In an expansion above onset the corrections to the phase motion, in fact, appear at the same order in ϵ as the lowest-order terms [Eq. (2)], and completely change the stability analysis at this order.[10]

On the other hand, rigid top and bottom boundaries lead to a finite $O(1)$ decay rate for vertical vorticity even in the long-wavelength limit, there are no extra dangerous modes at onset in addition to the convection modes, and a gradient expansion seems more likely to be valid. In fact, we will see that a singular drift term arises here too, not from a *zero* eigenvalue for the horizontal velocity at long wave vectors $\vec{q}_\perp$ as in the free case, but because the eigenvalue and eigenvector depend in a singular way on the limit $q_\perp \to 0$, with the longitudinal and transverse modes giving different limits. The gradient expansion may be seen to break down because of the effectively long-range forces introduced by the incompressibility of the fluid on the time scale involved in Eq. (8). In an expansion above onset the singular correction U appears only at the next order in $\epsilon^{1/2}$ beyond the Newell-Whitehead-Segel-type analysis. In fact, at this order we suggest an equation of the form [Eq. (8)] with

$$\begin{aligned} \tau_0 D_{\parallel} &= \left[\frac{1-3Q^2}{1-Q^2}\right]\xi_0^2 - \epsilon^{1/2}Qa(\sigma,Q) + O(\epsilon) \ , \\ \tau_0 D_{\perp} &= (q/k_0)\xi_0^2 + \epsilon b(\sigma,Q) + O(\epsilon^{3/2}) \ , \end{aligned} \tag{9}$$

and

$$\begin{aligned} \tau_0(\partial_x^2+\partial_y^2)U = \partial_x^2[&\epsilon^{1/2}Q\alpha(\sigma,Q)\partial_x^2\phi \\ &-\epsilon\beta(\sigma,Q)\partial_y^2\phi] \ , \end{aligned} \tag{10}$$

where a, b, α, and β are, in general, nonzero for $Q \to 0$. We will evaluate α and β explicitly. The functions a,b have not been calculated directly for the rigid case, but their properties may be inferred from the free-slip calculations and previous numerical work. The important quantity $b(\sigma,0)$ has recently been obtained by Manneville and Piquemal.[11]

Since the correction U arises from the nonlinearities in the Navier-Stokes equations, it is clear that the relative importance of this term will be greatest at small Prandtl numbers, and we will emphasize this limit. At large Prandtl numbers where the thermal nonlinearities dominate, the singular corrections will become unimportant, and the wave-number selection principle of Pomeau and Manneville,[6] should be essentially correct.

III. DERIVATION

It is convenient to eliminate the pressure P and the horizontal velocity $\vec{u}$ from the linear part of the Boussinesq equations to give two equations for the vertical velocity w and T the deviation of the temperature from the linear conducting profile:

$$(-\sigma^{-1}\partial_t+\nabla^2)\nabla^2 w+\nabla_\perp^2 T=\sigma^{-1}\{\nabla_\perp^2[(\vec{u}\cdot\vec{\nabla}_\perp+w\partial_z)w]-\partial_z\nabla_\perp[(\vec{u}\cdot\vec{\nabla}_\perp+w\partial_z)\vec{u}]\}\ , \tag{11}$$

$$(-\partial_t+\nabla^2)T+Rw=(\vec{u}\cdot\vec{\nabla}_\perp+w\partial_z)T\ . \tag{12}$$

The equations have been reduced to dimensionless form in the conventional way,[12,13] and σ is the Prandtl number.

The expansion scheme[4,5] is to use $\epsilon=(R-R_c)/R_c$ to scale slow spatial and temporal derivatives of the basic convection pattern of rolls parallel to the y direction according to

$$X=\epsilon^{1/2}x\ ,\quad Y=\epsilon^{1/4}y\ ,\quad \tau=\epsilon t\ . \tag{13}$$

The different scaling of x and y coordinates is suggested by the rotational invariance of the onset problem. For the results of our problem such a scaling is not obviously beneficial, but we will continue to use it for ease of comparison with previous work. The envelope function $\bar{A}_0(X,Y,\tau)$ is defined by the equations

$$\begin{aligned} w&=\epsilon^{1/2}(\bar{A}_0e^{ik_0x}+\text{c.c.})w_0(z)+O(\epsilon)\ ,\\ T&=\epsilon^{1/2}(\bar{A}_0e^{ik_0x}+\text{c.c.})T_0(z)+O(\epsilon)\end{aligned} \tag{14}$$

with $[w_0(z),T_0(z)]\exp(ik_0x)$ the critical solution satisfying the time-independent, linearized equations at onset, and k_0 the critical wave number.

At each order in $\epsilon^{1/2}$ an equation will be obtained in the form

$$L_0(w,T)=(f_1,f_2)\ , \tag{15}$$

where L_0 is the linear operator at onset:

$$L_0=\begin{bmatrix}-\nabla^4 & -\nabla_\perp^2\\ R_c & \nabla^2\end{bmatrix} \tag{16}$$

with all derivatives here with respect to the fast variables x, y, and f_1,f_2 are functions of $\bar{A}_0$ and its (slow) derivatives known from previous orders. Equation (15) has the solution

$$(w,T)=L_0^{-1}(f_1,f_2)\ , \tag{17}$$

providing (f_1,f_2) satisfies the solvability condition of zero component along the eigenvector of L_0 with zero eigenvalue (i.e., the critical solution). The solvability condition leads to the amplitude equation at each order. The lowest nontrivial order leads to the Newell-Whitehead-Segel result

$$\tau_0\dot{A}_0=\epsilon A_0+\xi_0^2\left[\partial_x-\frac{i}{2k_0}\partial_y^2\right]^2A_0-g\,|A_0|^2A_0 \tag{18}$$

with τ_0, ξ_0, and g Prandtl number dependent constants tabulated, for example, in Ref. 12, and $A_0=\epsilon^{1/2}\bar{A}_0$.

Once (w,T) at each order is calculated from Eq. (17) the horizontal velocity must be found. It is given at each order by

$$\sigma\nabla^2\vec{u}=\vec{\nabla}_\perp P+(\vec{u}\cdot\vec{\nabla}_\perp+w\partial_z)\vec{u}+\partial_t\vec{u}\ , \tag{19}$$

where the right-hand side involves terms from previous orders, and the pressure P is given by integrating the equation for the vertical velocity:

$$\partial_z P=\sigma\nabla^2 w+T-(\vec{u}\cdot\vec{\nabla}_\perp+w\partial_z)w-\partial_t w \tag{20}$$

with all terms on the right-hand side known. In integrating Eq. (20) a complementary function $P_0(x,y)$ must be introduced. This function is ultimately determined using the continuity equation. The correct treatment of the equation for $\vec{u}$ introduces the singular gradient dependence into the phase equation. Notice that for driving terms on the right-hand side of Eq. (19) with fast x dependence, such as $\exp(ik_0x)$ the equation may be directly inverted. It is the possibility of driving terms varying horizontally with only the slow coordinates X,Y that leads to the singular results. At this point the formal treatment for the free and rigid boundaries diverge.

For free boundary conditions a constant horizontal velocity field is a natural solution to the operator ∇^2 in Eq. (19) together with the boundary conditions $\partial_z\vec{u}(\pm\frac{1}{2})=0$. Consequently the response $\vec{u}(x,y)$ to a slowly varying driving is anomalously large. The potential part of this driving is canceled by the pressure $P_0(x,y)$, leaving an anomalously large vertical vorticity response $\Omega=(\vec{\nabla}\times\vec{u})_z$ given by

$$(-\sigma^{-1}\partial_t+\nabla_\perp^2)\Omega=\sigma^{-1}\{\vec{\nabla}\times[(\vec{u}\cdot\vec{\nabla}_\perp+w\partial_z)\vec{u}]\}_z\ , \tag{21}$$

where the slowly varying component independent of z of the driving is taken. The driving term on the right-hand side is first nonzero at $O(\epsilon^{3/2})$, and leads to the results of Siggia and Zippelius.[10] [The form of the driving is easily obtained from Eq. (25) below.] Calculating $\vec{u}$ from Ω involves inverting a slow horizontal Laplacian $\partial_X^2+\partial_Y^2$ and leads to the singular gradient expansion discussed before. For the analysis of the consequences of this result, see the original work of Ref. 10.

Rigid boundary conditions $\vec{u}(\pm\frac{1}{2})=0$ permit no natural solutions to the operator in Eq. (19) at long wavelengths. The equation may be integrated to give rapidly varying terms, which present no problem, together with the slow drift velocity $\vec{u}_D$:

$$\vec{u}_D=\sigma^{-1}\int^z dz'\int^{z'} dz''[\vec{\nabla}_\perp P+(\vec{u}\cdot\vec{\nabla}_\perp+w\partial_z)\vec{u}+\partial_t\vec{u}]\,|_{\text{slow}}\,, \tag{22}$$

where constants of integration are uniquely determined by $\vec{u}(\pm\frac{1}{2})=0$. At first sight there is no singular gradient dependence. However, the complementary function $P_0(x,y)$ is as yet undetermined, and leads to the contribution

$$\vec{u}_D=\sigma^{-1}\vec{\nabla}_\perp P_0\tfrac{1}{2}(z^2-\tfrac{1}{4})+\vec{f}\ , \tag{23}$$

where $\vec{f}$ is the remainder of the right-hand side of Eq. (22), including the particular integral of Eq. (20). (It is convenient to choose this to give no contribution to the flux integrated over the depth of the fluid, although any other choice merely leads to a redefinition of P_0 and does not, of course, affect the results.) In fact, P_0 is determined by the integrated continuity equation:

$$\int_{-1/2}^{1/2}dz\ \vec{\nabla}_\perp\cdot\vec{u}_D=0=-\tfrac{1}{12}\sigma^{-1}\nabla_\perp^2 P_0+\vec{\nabla}_\perp\cdot\int_{-1/2}^{1/2}\vec{f}\,dz\ . \tag{24}$$

Solution for P_0 involves inversion of the slow horizontal Laplacian $\partial_X^2+\partial_Y^2$, and, providing $\langle\vec{f}\rangle=\int\vec{f}\,dz$ is not purely potential, introduces a contribution to $\vec{u}_D$ that is again singular in the gradient expansion. The explicit calculation of $\vec{f}$ is a straightforward extension of Refs. 3 and 4. We find the lowest-order contributions (leading to $\vec{\nabla}_\perp\cdot\vec{u}_D$ of order ϵ^2):

$$f_x^{(3/2)}=2\sigma^{-1}\left[A_0^*\left[\partial_x-\frac{i}{2k_0}\partial_y^2\right]A_0+\text{c.c.}\right]h(z)\ , \tag{25}$$

$$f_y^{(7/4)}=\sigma^{-1}k_0^{-1}\left[\partial_y A_0^*\left[\partial_x-\frac{i}{2k_0}\partial_y^2\right]A_0-iA_0^*\left[\partial_x-\frac{i}{2k_0}\partial_y^2\right]\partial_y A_0+\text{c.c.}\right]h(z)$$

with

$$h(z)=\int^z dz'\int^{z'}dz''\left[w_0\frac{\partial}{\partial z''}\frac{\partial\phi_0}{\partial q_0^2}-\frac{\partial w_0}{\partial q_0^2}\frac{\partial\phi_0}{\partial z''}\right] \tag{26}$$

and the constants of integration are fixed by $h(\pm\frac{1}{2})=0$. Other contributions to $\vec{f}$ that are explicitly potential and so do not lead to any singular behavior have been ignored: They are assumed incorporated into the parameters a,b. Finally, when substituted back into f_1,f_2 of Eq. (15) we find the lowest-order singular contribution to be added to the amplitude equation (18) proportional to

$$[(\partial_x P_0)iq_0A_0+(\partial_y P_0)\partial_y A_0]$$

with P_0 given by Eq. (24).

A number of remarks may make this result more transparent. The form of the result is an additional "convection" of the envelope function by an effective transverse velocity proportional to $\vec{\nabla}_\perp P_0(x,y)$, with the proportionality constant depending in a complicated way on the shape of the z dependence of the velocity $\vec{u}_D$ and of the critical solutions w_0,T_0. The singular part of $\vec{u}_D$ may be thought of as the subtraction of a singular potential contribution (with z dependence $z^2-\frac{1}{4}$), from a slowly varying drift $\vec{f}$ that has a complicated z dependence but has a nonsingular gradient expansion at the order in ϵ considered. Although the full z dependences must be retained to arrive at the correct numerical prefactors, a qualitative understanding is given by considering the convection of the zeroth-order solutions by the z-integrated horizontal drift:

$$\langle\vec{u}_D\rangle=\int_{-1/2}^{1/2}\vec{u}_D dz\ . \tag{27}$$

It is easy to see that the potential part of $\langle\vec{u}_D\rangle$ is canceled by the pressure P_0 leaving a singular vorticity contribution to the average drift

$$\langle\vec{u}_D\rangle=\vec{\nabla}\times\zeta\hat{z} \tag{28}$$

with

$$\nabla_{\perp}^{2}\zeta = -(\vec{\nabla}\times\vec{f})_z \ . \tag{29}$$

With the approximation of considering only the integrated drift, the role of the long-wavelength vertical vorticity is emphasized as in the free case analyzed by Siggia and Zippelius.[10] Their phenomenological extension to the rigid case is thus seen to be qualitatively correct, at least as far as the singular part of the expression is concerned, although it seems clear that there are other, nonsingular contributions of formally the same order, even at small Prandtl number.

Notice that the singular drift velocity has the particular z dependence of Eq. (23). An alternative scheme for calculating the horizontal velocity is to calculate it as the sum of a potential term, given via the continuity equation, and a vorticity term, for which a rather simple equation, Eq. (21), exists. The solution for both contributions involve the inversion of a slow horizontal Laplacian, and at first sight would lead to a singular expression for *all z* eigenfunctions. This is, in fact, misleading, since it is readily shown that the sum of these contributions is smooth, except for the particular component with dependence $z^2-\frac{1}{4}$.

For the purposes of calculating the equation for small phase deviations from a straight roll pattern the expression for $\vec{f}$ simplifies to

$$\vec{f}=\sigma^{-1}|A_0|^2[2\partial_x(\delta|A_0|/|A_0|) + k_0^{-1}\partial_y^2\phi, k_0^{-1}\partial_x\partial_y\phi]h(z) \ , \tag{30}$$

where we have retained only those terms leading to quadratic derivatives of the phase. The perturbation of the magnitude is given by

$$\frac{\delta|A_0|}{|A_0|} = -\frac{\xi_0^2 q}{\epsilon-\xi_0^2 q^2}(\partial_x\delta\phi)[1+O(\epsilon^{1/2})] \ . \tag{31}$$

Thus we see there are two contributions to f_x tending to drive a drift velocity convecting the phase: A term in $\partial_x^2\phi$ also proportional to the deviation q of the wave number from the critical value; and a term in $\partial_y^2\phi$ the curvature of the rolls. This drift velocity plays an important role in the instabilities of straight rolls near onset, as will be discussed below.

Finally, for U we find the expression (10) with

$$\alpha(\sigma,Q)=\xi_0\gamma(\sigma) \ , \tag{32}$$

$$\beta(\sigma,Q)=k_0^{-1}(1-Q^2)\gamma(\sigma) \ , \tag{33}$$

where

$$\gamma(\sigma)=0.45\sigma^{-2}(1+2.90\sigma)g^{-1} \ . \tag{34}$$

IV. ANALYSIS OF RESULTS

In this section we consider the consequences of adding the singular drift term U to the phase diffusion equation. There are two effects of interest. The first is the modification of the long-wavelength instabilities that limit the band of stable stationary convection solutions. Although the singular nature of the expansion is crucial only to the skew varicose instability, a better understanding is given by first studying the influence of the horizontal drift on the Eckhaus and zig-zag instabilities. Since the case of free-slip boundaries has been completely discussed by Siggia and Zippelius[10] we will restrict our attention to rigid boundaries. Secondly, the wave-number selection principle in axisymmetric convection suggested by Pomeau and Manneville,[6] will be discussed.

A. Long-wavelength instabilities

The Eckhaus instability arises if phase variation in the x direction only is assumed. In this case there is an effective parallel diffusion constant:

$$\tau_0 D_{\parallel} = \left[\frac{1-3Q^2}{1-Q^2}\right]\xi_0^2 - \epsilon^{1/2}Q[a(\sigma,Q)-\alpha(\sigma,Q)]+O(\epsilon) \ , \tag{35}$$

where a and α are nonzero for $Q\to 0$. The role of the drift term is easily understood. The nonlinearities in the horizontal velocity equation tend to drive a longitudinal potential flow towards the region of increased phase gradient for $Q>0$, according to Eqs. (23) and (25). This is a destabilizing effect, and contributes to the coefficient $a(\sigma,Q)$ in Eq. (35). However, the integrated component $\langle\vec{u}_D\rangle$ is canceled by the flow driven by the pressure P_0 [the $\alpha(\sigma,Q)$ term in Eq. (35)] leaving only the net effect $\sim a-\alpha$.

The quantity a has not yet been evaluated for rigid boundaries. Since, however, neither the vertical vorticity nor the average drift is involved in this purely longitudinal motion, it is instructive to evaluate $D_{\parallel}$ from the work on free-slip boundaries: The result for rigid boundaries will be qualitatively similar. In the small Prandtl number limit we find

$$\tau_0 \xi_0^{-2} D_{\parallel}(Q) = \left[\frac{1-3Q^2}{1-Q^2}\right] - \frac{5\sigma^{-2}}{8\sqrt{3}} \epsilon^{1/2} Q \quad (36)$$

with corrections of relative order $\epsilon\sigma^{-1}$. The resulting contour for the stability boundary $D_{\parallel}=0$ for $\sigma \to 0$ is shown in Fig. 1. Note that the Eckhaus instability is dramatically enhanced for $Q>0$ and for small Prandtl numbers. The Prandtl number dependence of this result needs further discussion. The strong Prandtl number dependence of $D_{\parallel}(Q)$ for free-slip boundaries evident in Fig. 1 arises from the *independence* of $|A_0|^2$ on σ in this limit. This is the phenomenon of inertial convection[14]: The $O(\sigma^{-2})$ nonlinearities in the Navier-Stokes equation are zero for the linear onset solution, and cause no saturation, leaving only the $O(1)$ thermal nonlinearities. On the other hand, for rigid boundaries it can be shown that there is no singular dependence for $\sigma \to 0$, and the expression for $D_{\parallel}$ above cannot strictly apply. However, even in this case the Prandtl number dependence of $A_0^2 \sim g^{-1}$, with

$$g \simeq 0.70 - 0.005\sigma^{-1} + 0.008\sigma^{-2}$$

is weak (i.e., thermal nonlinearities remain the dominant saturation mechanism) except at very small Prandtl numbers $\sigma \lesssim 0.1$. We may therefore expect the parameters a, α, $a-\alpha$, etc., depending on $\sigma^{-2}|A_0|^2$ at low Prandtl numbers to increase to large values as σ decreases, eventually saturating for $\sigma \lesssim 0.1$. Figure 1 should then remain qualitatively correct for rigid boundaries and small Prandtl numbers.

The zig-zag instability corresponds to phase variation in the y direction. Now the roll curvature tends to drive a drift velocity $\vec{u}_D$ that convects the rolls back towards the straight configuration. This drift is divergence free, and the singular correction due to P_0 is not involved. The drift thus is a stabilizing influence that will be large at small Prandtl numbers as argued above. The instability occurs for

$$q = -\epsilon \xi_0^2 k_0 b(\sigma,0) + O(\epsilon^{3/2}) . \quad (37)$$

Recently $b(\sigma,0)$ has been calculated by Manneville and Piquemal.[11] These authors do indeed find a

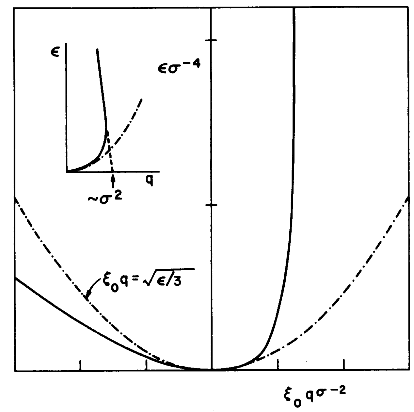

FIG. 1. Eckhaus instability boundary for free-slip boundaries and small Prandtl numbers σ. Insert shows the qualitative behavior for small but finite Prandtl numbers. Slope of the boundary at large ϵ depends on terms not calculated, but should be rather insensitive to small values of the Prandtl number. Shape of the boundaries are argued to be qualitatively similar for both the Eckhaus and skew varicose instabilities for rigid boundaries, although the numerical factors will be different, and the scalings with σ^{-1} will break down for too small values of $\sigma \lesssim 0.1$.

stabilizing effect at small Prandtl numbers (b large), also consistent with the numerical work of Clever and Busse.[15]

Finally, if we allow variation in both the x and y directions, the skew varicose instability results. Near onset this instability is readily understood in terms of a modified Eckhaus instability. As we observed there, for $Q>0$ the slowly varying horizontal drift acts as a destabilizing influence, but the integrated component $\vec{u}_D$ is canceled by the effect of the slowly varying pressure P_0. If, however, phase variation in the y direction is allowed in addition, this pressure is relieved, and the destabilizing integrated drift may now flow. Thus, for $Q>0$, the Eckhaus instability may be preempted by the skew varicose instability with phase variation wave vector $\vec{K}$ neither purely in the x or y direction. This result may be illustrated by calculating the instability boundary near threshold. For a general orientation of $\vec{K}$, we may define effective diffusion constants which near onset are

$$\begin{aligned} \tau_0 D_{\parallel}^{\mathrm{eff}} &= \left[\frac{1-3Q^2}{1-Q^2}\right] \xi_0^2 - \epsilon^{1/2} Q[a(\sigma,Q) - \alpha(\sigma,Q)\cos^2\phi] + O(\epsilon) , \\ \tau_0 D_{\perp}^{\mathrm{eff}} &= (q/k_0)\xi_0^2 + \epsilon[b(\sigma,Q) - \beta(\sigma,Q)\cos^2\phi] + O(\epsilon^{3/2}) , \end{aligned} \quad (38)$$

where ϕ is the angle between $\vec{K}$ and the roll normal, and the terms in $\cos^2\phi$ arise from the singular drift term. The long-wavelength instability criterion is then $\lambda < 0$, with

$$\lambda/K^2=D_{\parallel}^{\text{eff}}\cos^2\phi+D_{\perp}^{\text{eff}}\sin^2\phi \ . \tag{39}$$

Neglecting the $O(\epsilon)$ terms in Eq. (38) leads to the result

$$\cos\phi=(\xi_0/k_0\alpha)^{1/4}\sim\sigma^{1/2} \ ,$$

where we assume $\alpha\sim\sigma^{-2}$ for not too small Prandtl numbers (cf. above). The stability boundary is then given by

$$\left[\frac{1-3Q^2}{1-Q^2}\right]\xi_0^2-\epsilon^{1/2}Q[a-\alpha+(\sqrt{\alpha}-\sqrt{\xi_0/q_0})^2]=0 \ . \tag{40}$$

Thus for $\alpha>\xi_0/k_0$ the skew varicose instability preempts the Eckhaus instability. The form of Eq. (40) is as in Fig. 1, but with rescaled axes to incorporate the change in the second factor. Equation (40) only applies very close to threshold where the details of the behavior of the skew varicose instability boundary have not been investigated numerically. Nevertheless the trends suggested in Fig. 1 are evident in the numerical results,[16] particularly the approach of the boundary to $q=0$ for small Prandtl numbers (again saturating for very small Prandtl numbers at a finite value). Clearly for a complete understanding of the skew varicose instability, terms of higher order in ϵ in the expansion about threshold should be calculated. The present calculation does, however, demonstrate the crucial role of the singular drift in producing the instability.

B. Wave-vector selection in axisymmetric convection

In an asixymmetric configuration the curvature of the rolls leads to a nonlinear driving $\vec{f}$ in the horizontal velocity equation (30) giving a radial drift velocity. The continuity equation implies that a pressure $P_0(r)$ must build up to cancel the z-integrated component of this flow. Contrast this with the analysis of the zig-zag instability, where again roll curvature tends to drive a drift normal to the rolls, but which in that case is divergence free, inducing no counteracting pressure. Thus the analysis of Pomeau and Manneville[6] follows through except that the wave-number selection criterion, which involves the singular pressure-induced drift term, is no longer identical to the zig-zag instability criterion, which does not. In fact, it is clear that the radial drift driven by the component $f_r=\sigma^{-1}|A_0|^2r^{-1}$ of Eq. (30) must be canceled by the pressure term. This leads to a unique wave number at large distances in axisymmetric convection given by [cf. Eq. (37)]

$$q=-\epsilon\xi_0^2k_0[b(\sigma,0)-\tfrac{1}{2}\beta(\sigma,0)] \ , \tag{41}$$

a wave number within the band of solutions stable towards the zig-zag instability. Recently Manneville and Piquemal[17] have independently reached similar conclusions by direct calculation: The difference between the results [Eqs. (41) and (37)] given by the parameter β of Eqs. (33) and (34) agrees with their explicit calculation.

A similar result holds for the free-slip case. Here Pomeau and Manneville[6] explicitly calculated the wave-number selection criterion and found

$$k-k_0=O(\epsilon^{3/2}) \ . \tag{42}$$

As might be expected, this agrees with the old calculations of the zig-zag instability boundary,[18] which neglected the important vertical vorticity corrections. Siggia and Zippelius[10] showed that including these corrections removed the long-wavelength zig-zag instability boundary close to threshold altogether. Symmetry, however, requires no vertical vorticity in axisymmetric convection, so that the calculation leading to Eq. (42) remains correct.

Thus in each case, the wave-number selection criterion for axisymmetric convection is found to operate, but the singular drift terms move the selected wave number to a value within the band stable against the zig-zag instability. A similar result will apply for convection rolls of closed contours with a more general shape, although because of the long-range nature of the drift correction the exact wave-number selection may well depend on the details of the curvature of the rolls.

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