# Strings in $AdS_3$ and the SL(2,R) WZW model. III. Correlation functions

Juan Maldacena
*Jefferson Physical Laboratory, Harvard University, Cambridge, Massachusetts 02138*
*and Institute for Advanced Study, Princeton, New Jersey 08540*

Hirosi Ooguri
*California Institute of Technology 452-48, Pasadena, California 91125*
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We consider correlation functions for string theory on $AdS_3$. We analyze their singularities and we provide a physical interpretation for them. We explain which worldsheet correlation functions have a sensible physical interpretation in terms of the boundary theory. We consider the operator product expansion of the four-point function and we find that it factorizes only if a certain condition is obeyed. We explain that this is the correct physical result. We compute correlation functions involving spectral flowed operators and we derive a constraint on the amount of winding violation.

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## I. INTRODUCTION

This is the third installment of our series of papers on the SL(2,R) Wess-Zumino-Witten (WZW) model and its relation to string theory in $AdS_3$, three-dimensional anti–de Sitter space. In the first two papers [1,2], we determined the structure of the Hilbert space of the WZW model, computed the spectrum of physical states of the string theory, and studied the one-loop amplitude. In this paper, we will discuss the correlation functions of the model.

The SL(2,R) WZW model has many important applications in string theory and related subjects. It has close connections to the Liouville theory of two-dimensional gravity (for a review, see, for example, [3]) and three-dimensional Einstein gravity [4]. It is used to describe string theory in two-dimensional black-hole geometries [5]. Its quotients are an important ingredient in understanding string theory in the background of Neveu-Schwarz (NS) 5-branes [6], and they capture aspects of strings propagating near singularities of Calabi-Yau spaces [7–9]. One can also construct a black-hole geometry in three dimensions by taking a quotient of the SL(2,R) group manifold [10]. Moreover, sigma models with noncompact target spaces such as SL(2,R) have various applications to condensed-matter physics [11]. For these reasons, the model has been studied extensively for more than a decade.[1] Recently the model has becomes particularly important in connection with the AdS conformal field theory (CFT) correspondence [12,13] since it describes the worldsheet of a string propagating in $AdS_3$ with a background NS-NS $B$ field. According to the correspondence, type IIB superstring theory on $AdS_3 \times S^3 \times M_4$ is dual to the supersymmetric nonlinear sigma model in two dimensions whose target space is the moduli space of Yang-Mills instantons on $M_4$ [13,14]. Here $M_4$ is a four-dimensional Ricci flat Kähler manifold, which can be either a torus $T^4$ or a $K3$ surface. So far this has been the only case in which we have been able to explore the correspondence beyond the supergravity approximation with complete control over the worldsheet theory.

Besides the AdS/CFT correspondence, understanding string theory in $AdS_3$ is interesting since $AdS_3$ is the simplest example of a curved spacetime where the metric component $g_{00}$ is nontrivial. Using this model, one can discuss various questions which involve the concept of time in string theory. This will give us important lessons on how to deal with string theory in geometries which involve time in more complicated ways. In this connection, there had been a long-standing puzzle, first raised in [15,16], about whether the no-ghost theorem holds for string in $AdS_3$. The proof of the no-ghost theorem in this case is more involved than in Minkowski space since the time variable in $AdS_3$ does not decouple from the rest of the degrees of freedom on the worldsheet. The task was further complicated by the fact that $AdS_3$ is a noncompact space and the worldsheet CFT is not rational. Thus it was difficult to decipher the spectrum of the worldsheet theory.

This problem was solved in [1,2]. In [1], we proposed the spectrum of the WZW model and gave a complete proof of the no-ghost theorem base on the proposed spectrum. This proposal itself was verified in [2] by exact computation of the one-loop free energy for a string on $AdS_3 \times \mathcal{M}$, where $\mathcal{M}$ is a compact space represented by a unitary conformal field theory on the worldsheet. Although the one-loop free energy receives contributions only from physical states of the string theory, we can deduce the full spectrum of the SL(2,R) WZW model from the dependence of the partition function on the spectrum of the internal CFT representing $\mathcal{M}$, which can be arbitrary as far as it has the appropriate central charge. Thus the result of [2] can be regarded as a string theory proof of the full spectrum proposed in [1].

The spectrum of the SL(2,R) WZW model established in [1,2] is as follows. Since the model has the symmetry generated by the SL(2,R)×SL(2,R) current algebra, the Hilbert space $\mathcal{H}$ is decomposed into its representations as

[1] For a list of historical references, see the bibliography in [1].

 

$$\mathcal{H}=\oplus_{w=-\infty}^{\infty}\left[\left(\int_{1/2}^{(k-1)/2}dj\,\mathcal{D}_j^w\otimes\mathcal{D}_j^w\right)\oplus\left(\int_{(1/2)+i\mathbf{R}}dj\int_0^1 d\alpha\, C_{j,\alpha}^w\otimes C_{j,\alpha}^w\right)\right]. \tag{1.1}$$

Here $\mathcal{D}_j^w$ is an irreducible representation of the SL$(2,R)$ current algebra generated from the highest weight state $|j;w\rangle$ defined by

$$\begin{aligned} &J^+_{n+2}|j;w\rangle=0,\quad J^-_{n-w-1}|j;w\rangle=0,\\ &J^3_n|j;w\rangle=0\quad (n=1,2,...)\\ &J^3_0|j;w\rangle=\left(j+\frac{k}{2}w\right)|j;w\rangle,\\ &\left[-\left(J_0^3-\frac{k}{2}w\right)^2+\frac{1}{2}(J^+_wJ^-_{-w}+J^-_{-w}J^+_w)\right]|j;w\rangle\\ &\qquad=-j(j-1)|j;w\rangle, \end{aligned} \tag{1.2}$$

and $\mathcal{C}_j^w$ is generated from the state $|j,\alpha;w\rangle$ obeying

$$\begin{aligned} &J^+_{n\pm w}|j,\alpha;w\rangle=0,\quad J^3_n|j,\alpha;w\rangle=0\ \ (n=1,2,...)\\ &J^3_0|j,\alpha;w\rangle=\left(\alpha+\frac{k}{2}w\right)|j,\alpha;w\rangle,\\ &\left[-\left(J_0^3-\frac{k}{2}w\right)^2+\frac{1}{2}(J^+_wJ^-_{-w}+J^-_{-w}J^+_w)\right]|j,\alpha;w\rangle\\ &\qquad=-j(j-1)|j,\alpha;w\rangle. \end{aligned} \tag{1.3}$$

The representations with $w=0$ are conventional ones, where $|j;0\rangle$ and $|j,\alpha;0\rangle$ are annihilated by the positive frequency modes of the currents $J_n^{\pm,3}$ $(n\geqslant 1)$. These representations $\mathcal{D}_j^0$ and $\mathcal{C}_{j,\alpha}^0$ are called the discrete and continuous representations, respectively.[2] The representations with $w\neq 0$ are related to the ones with $w=0$ by the spectral flow automorphism of the current algebra, $J_n^{3,\pm}\to\tilde{J}_n^{3,\pm}$, defined by

$$\begin{aligned} &\tilde{J}_n^{\pm}=J^{+}_{n\pm w},\\ &\tilde{J}_n^3=J_n^3-\frac{k}{2}w\,\delta_{n,0}. \end{aligned} \tag{1.4}$$

In the standard WZW model, based on a compact Lie group, spectral flow does not generate new types of representations; it simply maps a conventional representation into another, where the highest weight state of one representation turns into a current algebra descendant of another. In the case of SL$(2,R)$, representations with different amounts of $w$ are not equivalent.

In [1], it was shown that Eq. (1.1) leads to the physical spectrum of string in AdS$_3$ without ghosts and that the spectrum agrees with various aspects of the dual two-dimensional CFT (CFT$_2$) on the boundary of the target space. In particular, it is shown that the spectral flow images of the continuous representations lead to physical states with continuous energy spectrum of the form

$$J_0^3=\frac{k}{4}w+\frac{1}{w}\left(\frac{s^2+\frac{1}{4}}{k-2}+N+h-1\right), \tag{1.5}$$

where $s$ is a continuous parameter for the states, $N$ is the amount of the current algebra excitations before we take the spectral flow, and $h$ is the conformal weight of the state in the internal CFT representing the compact directions in the target space. These states are called "long strings" with winding number $w$, and their continuous spectrum is related to the presence of noncompact directions in the target space of the dual CFT$_2$ [17,18]. The continuous parameter $s$ is identified with the momentum in the noncompact directions. The continuous representations with $w=0$ give no physical states except for the tachyon, which is projected out in superstring. On the other hand, the discrete representations and their spectral flow images give the so-called "short strings," whose physical spectra are discrete.

In this paper, we compute amplitudes of these physical states of the string theory and interpret them as correlation functions of the dual CFT$_2$. We show that the string theory amplitudes satisfy various properties expected for correlation functions of the dual CFT$_2$.

The dual CFT$_2$ is unitary with a Hamiltonian of positive-definite spectrum, and the density of states grows much slower than the exponential of the energy.[3] Therefore, one should be able to analytically continue the time variable of CFT$_2$ to Euclidean time. Correspondingly, the AdS$_3$ geometry can be analytically continued to the three-dimensional hyperbolic space $H_3$, whose boundary is $S^2$. The worldsheet of the string on $H_3$ is described by the SL$(2,C)$/SU$(2)$ coset model. We would like to stress that the SL$(2,R)$ WZW model and the SL$(2,C)$/SU$(2)$ coset model are quite distinct even though their actions are formally related by analytical continuations of field variables. For example, the Hilbert spaces of the two models are completely different since all the states in the Hilbert space (1.1) of the SL$(2,R)$ WZW model, except for the continuous representations with

[2] We call $\mathcal{D}_j^0$ a discrete representation even though the spectrum of $j$ in the Hilbert space (1.1) of the WZW model is continuous. It would have been discrete if the target space were the single cover of the SL$(2,R)$ group manifold. In order to avoid closed timelike curves, we take the target space to be the universal cover of SL$(2,R)$, in which case the spectrum of $j$ is continuous. We still call these representations discrete since their $J_0^3$ eigenvalues are related to the values of the Casimir operator, $-j(j-1)$, while the $J_0^3$ eigenvalue for continuous representations is not related to their values of the Casimir operator.

[3] The Cardy formula states that the density of states of conformal field theory on a unit circle grows as $\exp(2\pi\sqrt{cE/6})$, where $E$ is the energy and $c$ is the central charge of the theory.

$w=0$, correspond to non-normalizable states in the $SL(2,C)/SU(2)$ model. This means that all the physical states in string theory, except for the tachyon, are represented by non-normalizable states in the $SL(2,C)/SU(2)$ model. It is in the context of string theory computations of physical observables that one can establish connections between the two worldsheet models. We will discuss in detail how this connection works when we use string theory to compute correlation functions of the dual $CFT_2$ on the boundary of the target space.

Correlation functions of the $SL(2,C)/SU(2)$ model have been derived in [19–21] for operators corresponding to normalizable states and some non-normalizable states simply related to them by analytic continuation. Although the correlation functions for normalizable states are completely normal, those for non-normalizable states contain singularities of various kinds. Thus we need to understand the origins of these singularities and learn how to deal with them.

For clarity, we separate our discussion into two parts. First we will discuss the origins of these singularities purely from the point of view of the worldsheet theory. We will show how functional integrals of the $SL(2,C)/SU(2)$ model generate these singularities. We find that some of these singularities can be understood in the point-particle limit while others come from large "worldsheet instantons."

After explaining all the singularities from the worldsheet point of view, we turn to string theory computations and interpret these singularities from the point of view of the target spacetime physics. Some of the singularities are interpreted as due to operator mixings, and others originate from the existence of the noncompact directions in the target space of the dual $CFT_2$. In addition to the singularities in the worldsheet correlation functions, the integral over the moduli space of string worldsheets can generate additional singularities of a stringy nature. In Minkowski space, singularities are all at boundaries of moduli spaces (e.g., when two vertex operators collide with each other or when the worldsheet degenerates) and divergences coming from them are interpreted as due to the propagation of intermediate physical states. For strings in $AdS_3$, we find that amplitudes can have singularities in the middle of moduli space. We have already encountered such phenomena in a one-loop free-energy computation in [2], and they are attributed to the existence of the long string states in the physical spectrum. We will find related singularities in our computation of four-point correlation functions.

By taking into account these singularities on the worldsheet moduli space, we prove the factorization of four-point correlation functions in the target space. We show that the four-point correlation function, obtained by integrating over the moduli space of the worldsheet, is expressed as a sum of products of three-point functions summed over possible intermediate physical states. The structure of the factorization agrees with the physical Hilbert space of a string given in [1,2]. We also check that normalization factors for intermediate states come out precisely as expected. The resulting factorization formula shows a partial conservation of the total "winding number" $w$ of a string.[4] We will explain its origin from the worldsheet $SL(2,R)$ current algebra symmetry and the structure of the two- and three-point functions. In the course of this, we will clarify various issues about the analytic continuation relating the $SL(2,C)/SU(2)$ model and the $SL(2,R)$ model.

We find that, in certain situations, the four-point functions fail to factorize into a sum of products of three-point functions with physical intermediate states. We show that this failure of the factorization happens exactly when it is expected from the point of view of the boundary $CFT_2$. Namely, the four-point functions factorize only when they should.

This paper is organized as follows. In Sec. II, we review correlation functions of the $SL(2,C)/SU(2)$ coset model derived in [19,20] and explain the worldsheet origin of their singularities. In Sec. III, we turn to the string theory computation and discuss the target space interpretation of the singularities in two- and three-point correlation functions. In Sec. IV, we give a detailed treatment of four-point correlation functions. On the worldsheet, a four-point function of the $SL(2,C)/SU(2)$ model is expressed as an integral over solutions to the Knizhnik-Zamolodchikov equation [21]. We integrate the amplitude over the worldsheet moduli, which in this case is the cross ratio of the four points on $S^2$, and obtain the target space four-point correlation function. We examine factorization properties of the resulting correlation function. We explain when it factorizes and why it sometimes fails to factorize. In Sec. V, we compute two- and three-point functions of states with nonzero winding numbers. We also explain the origin of the constraint on the winding number violation. In Sec. VI, we use the result of Sec. V to show that the factorization of the four-point function works with precisely the correct coefficients.

In Appendix A, we derive the target space two-point function of a short string with $w=0$. The normalization of the target space two-point function is different from that of the worldsheet two-point function. The target space normalization is precisely the one that shows up in the factorization of the four-point amplitudes. In Appendix B, we derive some properties of conformal blocks of four-point functions. In Appendix C, we derive a formula for integrals of hypergeometric functions used in Sec. IV. In Appendix D, we derive a constraint on winding number violation from the $SL(2,R)$ current algebra symmetry of the theory. In Appendix E, we introduce another definition of the spectral flowed operator, working directly in the coordinate basis (rather than in the momentum basis) on the boundary of the target space. We compute two- and three-point functions containing the spectral flowed operators using this definition.

Some aspects of correlation functions of the $SL(2,C)/SU(2)$ model have also been discussed in [22–27].

[4]As explained in [1], $w$ is in general a label of the type of representation and is not the actual winding number of the string, although, for some states, it could coincide with the winding number of the string in the angular direction of $AdS_3$.

## II. GENERAL REMARKS ABOUT THE SL(2,C)/SU(2) MODEL

In this section, we study properties of the sigma model whose target space is Euclidean $AdS_3$ or three-dimensional hyperbolic space, which is denoted by $H_3$. This sigma model is a building block for the construction of string theory in $H_3\times\mathcal{M}$, where $\mathcal{M}$ is an internal space represented by some unitary conformal field theory. It is also used to compute string amplitudes for the Lorentzian signature $AdS_3$. A precise prescription for computing string amplitudes will be given in Sec. III. Before discussing the string-theory interpretation, let us clarify some properties of the sigma model itself.

The hyperbolic space $H_3$ can be realized as a right-coset space $SL(2,C)/SU(2)$ [28]. Accordingly, the conformal field theory with the target space $H_3$ and a nonzero NS-NS two-form field $B_{\mu\nu}$ can be constructed as a coset of the $SL(2,C)$ WZW model by the right action of SU(2). The action of the $SL(2,C)/SU(2)$ model can be expressed in terms of the Poincaré coordinates $(\phi,\gamma,\bar{\gamma})$ and the global coordinates $(\rho,\theta,\varphi)$ of $H_3$ as

$$S=\frac{k}{\pi}\int d^2z(\partial\phi\bar{\partial}\phi+e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma)$$

$$=\frac{k}{\pi}\int d^2z[\partial\rho\bar{\partial}\rho+\sinh^2\rho(\partial\theta\bar{\partial}\theta+\sin^2\theta\partial\varphi\bar{\partial}\varphi)+i(\tfrac{1}{2}\sinh 2\rho-\rho)\times\sin\theta(\partial\theta\bar{\partial}\varphi-\bar{\partial}\theta\partial\varphi)]. \quad (2.1)$$

We are considering the Euclidean worldsheet with $\partial=\partial_z$, etc. Near the boundary, $\rho\to\infty$, the action becomes

$$S\sim\frac{k}{\pi}\int d^2z[\partial\rho\bar{\partial}\rho+\tfrac{1}{4}e^{2\rho}(\partial\theta-i\sin\theta\partial\varphi)\times(\bar{\partial}\theta+i\sin\theta\bar{\partial}\varphi)-i\rho\sin\theta(\partial\theta\bar{\partial}\varphi-\bar{\partial}\theta\partial\varphi)]. \quad (2.2)$$

Because of the second term on the right-hand side, contributions from large values of $\rho$ are suppressed in the functional integral as $\sim\exp(-\alpha e^{2\rho})$; the coefficient $\alpha$ is positive semidefinite, and it vanishes only when $(\theta,\varphi)$ is a holomorphic map from the worldsheet to $S^2$ obeying

$$\bar{\partial}\theta+i\sin\theta\bar{\partial}\varphi=0. \quad (2.3)$$

Even for $\alpha=0$, if the map is nontrivial, the last term in Eq. (2.2) may grow linearly in $\rho$. For constant $\rho$ and $(\theta,\varphi)$ obeying Eq. (2.3), the action goes as $S\sim 2kn\rho$, where $n$ is the number of times the worldsheet wraps the $S^2$.

The action on the Euclidean worldsheet is real-valued since the $B$ field is pure imaginary.[5] The action is positive definite, and all normalizable operators have positive conformal weights. Thus one expects Euclidean functional integrals to behave reasonably well in this model. The only novelty is the fact that the target space $H_3$ of this sigma model is noncompact, but it is just as in the case of a free scalar field taking values in **R**, which is also noncompact.

The interpretation of this model on a Lorentzian worldsheet is more subtle. Because of the $B$ field, the action (2.1) is not invariant under reflection of the Euclidean time, and it becomes complex-valued after analytically continuing to the Lorentzian worldsheet. Thus the Hilbert space of the $SL(2,C)/SU(2)$ model on the Lorentzian worldsheet may not have a positive-definite inner product; in fact, an action of the $SL(2,C)$ current $J^3_{-n}$ generates negative norm states. As we mentioned in the above paragraph, the model on the Euclidean worldsheet appears to be completely normal, except that it does not have an analytic continuation to a normal field theory on a Lorentzian worldsheet.[6]

What is the space of states of this conformal field theory? In the semiclassical approximation, which is valid when $k$ in the action (2.2) is large, states are given by normalizable functions on the target space. More precisely, since the target space $H_3$ is noncompact, we allow functions to be continuum-normalizable. Because of the $SL(2,C)$ isometry of $H_3$, the space of continuum-normalizable functions is decomposed into a sum of irreducible unitary representations of $SL(2,C)$. The representations are parameterized by $j=\frac{1}{2}+is$ with $s$ being a real number, and the Casimir operator of each representation is given by $-j(j-1)$. The Casimir operator is proportional to the eigenvalue of the Laplacian on $H_3$. Corresponding to each of these states, there is an operator in the $SL(2,C)/SU(2)$ model, which is also called normalizable. They can be conveniently written as [30,19]

$$\Phi_j(x,\bar{x};z,\bar{z})=\frac{1-2j}{\pi}(e^{-\phi}+|\gamma-x|^2e^{\phi})^{-2j}. \quad (2.4)$$

The labels $x,\bar{x}$ are introduced to keep track of the $SL(2,C)$ quantum numbers [31].[7] The $SL(2,C)$ currents act on it as

$$J^a(z)\Phi_j(x,\bar{x};w,\bar{w})\sim\frac{D^a}{z-w}\Phi_j(x,\bar{x};w,\bar{w}),\quad a=\pm,3, \quad (2.5)$$

[5]This is so that the $B$ field becomes real-valued after analytically continuing the target space to the Lorentzian signature $AdS_3$.

[6]This is somewhat of a reflection of the situation of the $SL(2,R)$ WZW model. The $SL(2,R)$ model makes sense in the Lorentzian worldsheet as discussed in [29,1]. However, we cannot analytically continue to the Euclidean worldsheet since the Hamiltonian of the model is not positive definite. Note that here we are talking about analytically continuing the worldsheet without analytically continuing the spacetime.

[7]In the string theory interpretation discussed in Sec. III, $(x,\bar{x})$ is identified as the location of the operator in the dual CFT on $S^2$ on the boundary of $H_3$ [32].

where $D^a$ are differential operators with respect to $x$ defined by

$$D^+ = \frac{\partial}{\partial x}, \quad D^3 = x\frac{\partial}{\partial x} + j, \quad D^- = x^2\frac{\partial}{\partial x} + 2jx. \tag{2.6}$$

By using this and the Sugawara construction of the energy-momentum tensor,

$$T(z) = \frac{1}{k-2}[J^+(z)J^-(z) - J^3(z)J^3(z)], \tag{2.7}$$

we find the precise expression for the conformal weights of these operators as

$$\Delta(j) = -\frac{j(j-1)}{k-2} = \frac{s^2+\frac{1}{4}}{k-2}. \tag{2.8}$$

Operators with $j=\frac{1}{2}+is$ have positive conformal weight, as we expect for normalizable operators in a well-defined theory with Euclidean target space. It was shown in [28] that states corresponding to these operators and their current algebra descendants make the complete Hilbert space of the $SL(2,C)/SU(2)$ model.

The vacuum state of the $SL(2,C)/SU(2)$ model is not normalizable. This again is not unfamiliar; the vacuum state for the free scalar field on $\mathbf{R}$ is also non-normalizable since its norm is proportional to $\mathrm{vol}(\mathbf{R})=\infty$. In this case, we do not consider the vacuum in isolation. The vacuum state always appears with an operator, such as in $e^{ipX(0)}|0\rangle$. Similarly, on $H_3$, the vacuum state $|0\rangle$ is not normalizable, but we can consider a state given by operators of the form (2.4) with $j=\frac{1}{2}+is$ acting on it.[8]

The two- and three-point functions of operators like Eq. (2.4) were computed in [19–21]. The two-point function has the form

$$\langle \Phi_j(x_1,\overline{x_1};z_1,\overline{z_1})\Phi_{j'}(x_2,\overline{x_2};z_2,\overline{z_2})\rangle = \frac{1}{|z_{12}|^{4\Delta(j)}}\left[\delta^2(x_1-x_2)\,\delta(j+j'-1) + \frac{B(j)}{|x_{12}|^{4j}}\,\delta(j-j')\right]. \tag{2.9}$$

The coefficient $B(j)$ is given by

$$B(j) = \frac{k-2}{\pi}\frac{\nu^{1-2j}}{\gamma\left(\frac{2j-1}{k-2}\right)}, \tag{2.10}$$

where

$$\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{2.11}$$

The choice of the constant $\nu$ will not affect the discussion in the rest of this paper. In [21], it is set to be

$$\nu = \pi\frac{\Gamma\left(1-\frac{1}{k-2}\right)}{\Gamma\left(1+\frac{1}{k-2}\right)}, \tag{2.12}$$

by requiring a certain consistency between the two- and three-point functions.

The three-point function is expressed as

$$\langle \Phi_{j_1}(x_1,\overline{x_1};z_1,\overline{z_1})\Phi_{j_2}(x_2,\overline{x_2};z_2,\overline{z_2})\Phi_{j_3}(x_3,\overline{x_3};z_3,\overline{z_3})\rangle$$

$$= C(j_1,j_2,j_3)\frac{1}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}|z_{31}|^{2(\Delta_3+\Delta_1-\Delta_2)}}\frac{1}{|x_{12}|^{2(j_1+j_2-j_3)}|x_{23}|^{2(j_2+j_3-j_1)}|x_{31}|^{2(j_3+j_1-j_2)}}. \tag{2.13}$$

The $z$ and $x$ dependence is determined by $SL(2,C)$ invariance of the worldsheet and the target space. The coefficient $C(j_1,j_2,j_3)$ is given by

$$C(j_1,j_2,j_3) = -\frac{G(1-j_1-j_2-j_3)G(j_3-j_1-j_2)G(j_2-j_3-j_1)G(j_1-j_2-j_3)}{2\pi^2\nu^{j_1+j_2+j_3-1}\gamma\left(\frac{k-1}{k-2}\right)G(-1)G(1-2j_1)G(1-2j_2)G(1-2j_3)}, \tag{2.14}$$

where

[8]In the flat space case, the vacuum $|0\rangle$ can be regarded as the $p\to 0$ limit of $e^{ipX(0)}|0\rangle$, and therefore it is a part of the continuum-normalizable states. Such an interpretation is not possible in the case of $H_3$ since there is a gap of $1/[4(k-2)]$ between the conformal weight (2.8) of the normalizable states and that of the vacuum.

$$G(j)=(k-2)^{[j(k-1-j)]/[2(k-2)]}\Gamma_2(-j|1,k-2)\times\Gamma_2(k-1+j|1,k-2) \quad (2.15)$$

and $\Gamma_2(x|1,\omega)$ is the Barnes double gamma function defined by[9]

$$\log(\Gamma_2(x|1,\omega))=\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left[\sum_{n,m=0}^{\infty}(x+n+m\omega)^{-\epsilon}-\sum_{\substack{n,m=0\\(n,m)\neq(0,0)}}(n+m\omega)^{-\epsilon}\right]. \quad (2.16)$$

This shows that $\Gamma_2$ has poles at $x=-n-m\omega$ with $n,m=0,1,2,\ldots$. The function $G(j)$ defined by Eq. (2.15) then has poles at

$$j=n+m(k-2),\quad -(n+1)-(m+1)(k-2)\quad (n,m=0,1,2,\ldots). \quad (2.17)$$

These will play an important role in the following discussion.

Another important fact about $G(j)$ is that it obeys the functional relations

$$G(j+1)=\gamma\left(-\frac{j+1}{k-2}\right)G(j),$$
$$G(j-k+2)=\frac{1}{(k-2)^{2j+1}}\gamma(j+1)G(j). \quad (2.18)$$

For example, one can use the first of these relations to show that

$$\lim_{\epsilon\to 0}\frac{G(j_1-j_2+\epsilon)G(j_2-j_1+\epsilon)}{G(-1)G(1-2\epsilon)}=(k-2)\gamma\left(\frac{k-1}{k-2}\right)\lim_{\epsilon\to 0}\frac{2\epsilon}{(j_1-j_2)^2-\epsilon^2}=-2\pi(k-2)\gamma\left(\frac{k-1}{k-2}\right)\delta(s_1-s_2), \quad (2.19)$$

when $j_1=\frac{1}{2}+is_1$ and $j_2=\frac{1}{2}+is_2$. From this, it follows that

$$C(j_1,j_2,0)=B(j_1)\delta(j_1-j_2), \quad (2.20)$$

verifying that the three-point function including the identity operator $\Phi_{j=0}$ is in fact equal to the two-point function. Similarly, by using the second of Eqs. (2.18), we can show

$$C\left(j_1,j_2,\frac{k}{2}\right)=(k\text{-dependent coefficient})\times\delta(j_1+j_2-k/2). \quad (2.21)$$

Unlike the case of Eq. (2.20), the proportionality factor depends only on $k$ and not on $j_1$, $j_2$. This identity is used in later sections when we evaluate correlation functions involving spectral flowed states.

These two- and three-point functions are perfectly well behaved and finite for normalizable operators with $j=\frac{1}{2}+is$. Similarly, one expects the four-point function of such states to be given by summing over intermediate normalizable states [21,33].[10] The four-point function will be discussed in detail in Sec. IV. These properties are familiar and happen in all conformal field theories. The noncompactness of the target space does not pose a problem; we deal with it as in the case of a free noncompact scalar field.

### A. Analytic continuation and singularities

Life would be relatively simple if all we were interested in were operators like Eq. (2.4) with $j=\frac{1}{2}+is$.

The complications in our case show up because the operators we are going to be interested in are non-normalizable operators [35,3]. This is also familiar in standard flat space computations in string theory. There, we are interested in vertex operators which go as $e^{p_0X^0_{\rm Euclid}}$, where $p_0$ is the energy carried by the operator and is real, and $X^0_{\rm Euclid}$ is the scalar field representing the Euclidean time coordinate. It is sometimes said that we compute amplitudes in Euclidean signature space (with pure imaginary $p_0$) and then we analytically continue the results in $p_0$. This analytic continuation is possible if correlation functions with non-normalizable operators of the form $e^{p_0X^0_{\rm Euclid}}$ make sense in the model with Euclidean target space. There might be singularities for complex values of $p_0$, but we should be able to go around them to arrive at real values of $p_0$. Original correlators with pure imaginary $p_0$ are well defined in the Euclidean theory and never infinite since these operators correspond to normalizable states of the theory. When we analytically continue to real (or complex) $p_0$, there can be singularities where the amplitudes diverge. In flat space string theory, these singularities arise when we integrate over the positions of the operators on the worldsheet. The integrated four-point function can become singular as a function of the momenta. The interpretation of these singularities is of course well known in flat target spacetime; they correspond

[9]The sums over $n$, $m$ in the right-hand side are defined by analytic regularization. Namely, the sums are defined for $\mathrm{Re}(\epsilon)>2$, where they are convergent, and the result is analytically continued to $\epsilon\to 0$.

[10]Recently, it was shown in [34] that the four-point function of the $SL(2,C)/SU(2)$ model has the same form as that of the five-point function of the Liouville model where the cross ratio of four $x_i$'s in the $SL(2,C)/SU(2)$ model is related to the location of the fifth vertex operator in the Liouville model. This in particular shows that the four-point function obeys the crossing symmetry, the monodromy invariance, and so on, assuming that Liouville correlation functions also satisfy these properties. The monodromy invariance of the four-point function is proven explicitly in Sec. IV B.

to poles in the $S$ matrix and they are due to the propagation of an intermediate on-shell state. The lesson from the flat space case is that we should be able to interpret any singularity that appears in the physical computation of string amplitudes. Part of the definition of the physical theory is the choice of operators we consider. In the fact space case, $p_0$ has to be pure imaginary in order for the vertex operator $e^{p_0 X^0_{\rm Euclid}}$ to be normalizable. These are the operators that are most natural (i.e., normalizable) from the point of view of the Euclidean worldsheet theory. On the other hand, for applications to string theory, we need to consider the case in which $p_0$ is real as these are the ones that correspond to physical states in target space.

In our case, we can define non-normalizable operators by taking $j$ away from the line $j=\frac{1}{2}+is$. In the string theory application, we will be interested in the case in which $j$ is real. One can define correlation functions of these operators by analytically continuing the well-defined expressions that were found for $j=\frac{1}{2}+is$. In fact, the expressions for complex $j$ were derived in [19] by using special properties of operators at particular real values of $j$, so analyticity in $j$ was an input to the calculation. A feature of this analytic continuation is that correlation functions that were perfectly finite and well behaved can develop singularities for particular values of $j$. In the following subsections, we will explain the origin of these singularities in the $SL(2,C)/SU(2)$ model. We will also explain that there are other non-normalizable operators that are necessary for the string theory application which are *not* obtained by analytic continuation in $j$ of Eq. (2.4). In Sec. V, we will discuss how to compute correlation functions of these operators.

### B. Singularities in two-point functions

The first thing we need to understand is how the operators (2.4) with real $j$ are defined. It seems that all we need to do is to insert the vertex operators (2.4) in the path integral. As usual, we need to remove short-distance singularities in the worldsheet theory when we insert these operators. This is the standard renormalization procedure we need to use to define vertex operators. In this case, however, we also need to be careful with singularities on the worldsheet theory that arise due to the fact that the sigma model is noncompact. The vertex operator $\Phi_j(z,x)$ defined by Eq. (2.4) has the property that, depending on whether $\gamma(z)=x$ or $\neq x$, it behaves as $\Phi_j \sim e^{2j\phi}$ or $\sim e^{-2j\phi}$ for large $\phi$. For $\mathrm{Re}(j)<\frac{1}{2}$, we see that, once we take into account the measure factor $e^{2\phi}$, the two-point function will have a divergence. This divergence comes from the region where $\gamma \neq x$ and $\phi \to \infty$, and therefore it is not localized near $x$ in target space; it is spread all over the $x$ space. On the other hand, if $\mathrm{Re}(j)>\frac{1}{2}$, this divergence is localized at $\gamma = x$. This distinction between these two cases will be very important for the string theory application discussed in the next section. From the worldsheet point of view, operators of the form $\mathrm{Re}(j)\neq\frac{1}{2}$ are not normalizable. Analytic continuation is defining these operators in some way. We also get a divergence in the two-point function coming from the delta function $\delta(j-j')$ in Eq. (2.9). This comes from the volume of the subgroup of target space global $SL(2,C)$ transformations that leave two points fixed (the two points $x_1$ and $x_2$ where the operators are inserted).

The analytically continued expression (2.10) has other divergences. It has poles at

$$j=\frac{n}{2}(k-2)+\frac{1}{2}, \quad n=1,2,\ldots . \tag{2.22}$$

Let us understand these poles when $k$ is large. Before we continue, let us note that we know the exact expression (2.9), and there is no need to reevaluate it approximately. The purpose of this exercise is to understand the origin of these singularities. This will help us to interpret them in the context of string theory later. It may also be useful in analyzing similar singularities in situations in which we do not know exact answers.

Let us start with the $n=1$ case. Since $j\sim k$ and the semiclassical limit corresponds to $k\to\infty$, these poles can be thought of as arising from nonperturbative effects on the worldsheet. The nonperturbative effect we have in mind is due to a worldsheet instanton. The target space has a boundary that is an $S^2$, and our worldsheet instanton approaches it while wrapping on this $S^2$ once. These are sometime called "long strings" [36], which are related to the long strings in the spectrum of the $SL(2,R)$ WZW model. To evaluate effects of the instanton, it is useful to use global coordinates in $H_3$. As we discussed earlier, the worldsheet action (2.2) grows exponentially large toward the boundary $\rho\to\infty$ unless the worldsheet obeys the holomorphicity condition (2.3). For a holomorphic worldsheet, the action grows linearly as $S \sim 2k\rho$ for large $\rho$. The effect is of the order $e^{-2k\rho}$, which is indeed nonperturbative if we identify $k\sim 1/g^2$, where $g$ is the coupling constant on the worldsheet. These worldsheet instanton effects are similar to the ones which appear in the computation of the Yukawa coupling of the type II string compactification, where the instantons wrap topologically nontrivial 2-cycles in a Calabi-Yau threefold (a complex three-dimensional manifold). In our case, however, the $S^2$ is contractible in $H_3$. In fact, the instanton action $\sim 2k\rho$ is not a topological invariant, but it depends on the size $\rho$ of the worldsheet. Thus the instanton configuration is not topologically stable, and it is continuously connected to the vacuum.[11] Without additional effects, the factor $e^{-2k\rho}$ tends to suppress large instantons.

This observation can be used to explain the poles in the two-point function at $2j\sim k$ in the following way. As we noted, depending on whether $\gamma(z)=x$ or $\gamma(z)\neq x$, the vertex operator behaves as $\Phi_j(z,x)\sim e^{2j\phi}$ or $\sim e^{-2j\phi}$ for large $\phi$. On the worldsheet $S^2$ with the two vertex operators inserted, one can always find a holomorphic map such that $\gamma(z_i) = x_i$ $(i=1,2)$. In fact, there is a one-complex parameter family of instantons, generated by dilatation and rotation which

[11]In several respects, these instantons are similar to instantons in ordinary Yang-Mills theory in four dimensions. In this latter case, their action depends logarithmically on the size of the instanton (analogous to $e^{-\rho_0}$ in our case) and if we are in a given theta vacuum, the instanton can dissolve into the vacuum.

keep fixed the two points, and the integral over the family is responsible for the delta function $\delta(j-j')$ in the two-point function. On such instantons, the vertex operator is evaluated as $\Phi_j(z_i,x_i)=e^{2j\phi}$ in Poincaré coordinates. In the global coordinates, it behaves as $\Phi_j\sim e^{2j\rho}$ for large $\rho$. Therefore, in the two-point function, the integral over the zero mode $\rho_0$ of the instanton size is of the form

$$\int d\rho_0 e^{-2k\rho_0}e^{2j\rho_0}e^{2j\rho_0}, \tag{2.23}$$

where the first factor is the instanton action and the last two factors come from the vertex operator insertions. We see that the integral (2.23) converges at large $\rho_0$ only for $j<k/2$ [the exact answer (2.10) is finite only for $j<(k-1)/2$].[12] Thus the instanton effect explains the origin of the singularity as due to the noncompact direction in field space which can be explored with finite cost in the action. Since this divergence is coming from the large $\rho$ region, it does not matter that the instanton is not topologically stable in the full space of the worldsheet fields. What is important is that the large $\rho$ region gives a dominant contribution to the functional integral. We can therefore say that this divergence is an IR effect in the target space. It is interesting that the divergence is not localized on the worldsheet and therefore cannot be considered as an UV effect there. The standard lore about the correspondence between IR effects in the target space and UV effects on the worldsheet does not hold in this case.

Thus we have shown that there is a divergence for $\mathrm{Re}(j)\geqslant(k-1)/2$ due to large worldsheet instantons. In the analytic regularization, the divergence is converted into a pole at $j=(k-1)/2$. Of course, the formula (2.9) is precisely the result of such analytic continuation. These poles were also discussed in [24] in the context of the SL(2,$R$)/U(1) coset model using a dual description [20]. Similarly, by considering an instanton which wraps $n$ times the $S^2$, we can explain the pole at $2j\sim nk$ in the two-point function.

### C. Singularities in the three-point function

The three-point function (2.13) has various poles which come from the poles in $G(j)$ [Eq. (2.17)]. One finds that $C(j_1,j_2,j_3)$ has poles at

$$j=n+m(k-2),\quad -(n+1)-(m+1)(k-2)$$

$$(n,m=0,1,2,...),$$

where

$$j=1-j_1-j_2-j_3,\quad j_3-j_1-j_2,$$
$$j_2-j_3-j_1,\ \text{or}\ j_1-j_2-j_3. \tag{2.24}$$

Our first task is to understand the origin of these singularities from the point of view of the SL(2,$C$)/SU(2) sigma model on the worldsheet.

Let us first consider the poles at

$$j_3-j_1-j_2=n\quad (n=0,1,2,...). \tag{2.25}$$

Here we use the standard large $k$ approximation treating $\phi$ and $\gamma$, $\bar{\gamma}$ as constant on the worldsheet (this is the point-particle approximation). The vertex operator (2.4) goes like $e^{2j\phi}$ at $\gamma=x$ and it decays like $e^{-2j\phi}$ for $\gamma\neq x$. When $j_3>j_1+j_2$, a divergence in the three-point amplitude arises from the integral region where $\gamma=x_3$ (and therefore $\gamma\neq x_1,x_2$) so that $\Phi_{j_3}(x_3)\sim e^{2j_3\phi}$ and $\Phi_{j_1},\Phi_{j_2}\sim e^{-2j_1\phi},e^{-2j_2\phi}$. The integral over $\phi$ then takes the form

$$\int d\phi\, e^{2(j_3-j_1-j_2)\phi}, \tag{2.26}$$

where the measure factor $e^{2\phi}$ is canceled by the integral over $\gamma$, $\bar{\gamma}$. The amplitude is divergent for $j_3\geqslant j_1+j_2$, and analytic regularization gives a pole at $j_3=j_1+j_2$. This explains the pole with $n=0$ in Eq. (2.25). To reproduce the other poles with $n=1,2,...$, we just have to expand $\Phi_{j_3}(z_3,x_3)$ in powers of $|\gamma(z_3)-x_3|^2$ and repeat the above exercise. Thus we have interpreted the poles (2.25) in the exact expression (2.14) from the point of view of the worldsheet theory. There are also poles when $(j_2-j_3-j_1)$ and $(j_1-j_2-j_3)$ are non-negative integers and they are explained in a similar way. In Sec. III, we will discuss how these divergences are dealt with in string theory. We will see that these are very analogous to poles in the $S$ matrix in the flat space computation.

The other poles in Eq. (2.24) can be explained by the worldsheet instanton effects. Since one can always find a holomorphic map from the worldsheet to the target space such that $\gamma(z_i)=x_i$ $(i=1,2,3)$, the worldsheet instanton can grow large whenever $\mathrm{Re}(j_1+j_2+j_3)$ exceeds $\sim k$. This explains the first pole in Eq. (2.24) with $(n,m)=(0,1)$. As in the case of the two-point function, this divergence is nonlocal in target space. The remaining poles in Eq. (2.24) can be interpreted in similar ways.

### D. Singularities in four-point functions

Let us now move on to the four-point function. By worldsheet conformal invariance and target space isometries, it depends nontrivially only on the cross ratios of $z_i$'s and $x_i$'s $(i=1,...,4)$,

$$z=\frac{(z_1-z_2)(z_4-z_3)}{(z_1-z_3)(z_4-z_2)},\quad x=\frac{(x_1-x_2)(x_4-x_3)}{(x_1-x_3)(x_4-x_2)}. \tag{2.27}$$

For special values of $j_i$, the labels of the four operators, the dependence of the four-point function of $z$ and $x$ can be de-

[12] In principle, we expect the computation in (2.23) to give us only the leading order in $k$ behavior. By being a bit more careful about the integral over quadratic fluctuations, we can see that the amplitude can be better approximated as $\int d\rho_0 e^{2\rho_0}e^{-2(k-2)\rho_0}e^{2(j-1)\rho_0}e^{2(j-1)\rho_0}$, where the first factor comes from the measure of the $\rho_0$ integral, the shift in $k$ comes from the determinants, and the shift in $j$ comes from the integral over $\gamma,\bar{\gamma}$. This gives the exact bound $j<(k-1)/2$.

termined by differential equations. These values of $j_i$ are outside the range which leads to physical operators in the string theory.

For generic values of $j_i$, one very useful piece of information is that it obeys the Knizhnik-Zamolodchikov (KZ) equation, which follows from the Sugawara construction of the stress tensor (2.7). The idea is to compute $\langle T(w)\Phi_{j_1}(z_1,x_1)\cdots\Phi_{j_4}(z_4,x_4)\rangle$ in two different ways. One is to convert $T(z)$ into derivatives with respect to $z_i$'s using the conformal Ward identity. Another is to use Eq. (2.7) to express $T(z)$ in terms of the currents $J^a$ and to turn them into differential operators on $x$ by the $\mathrm{SL}(2,C)$ Ward identities (2.5). Combining these two expressions together and going over to the cross ratios (2.27), one finds [21] that the four-point function $\mathcal{F}_{\mathrm{SL}(2)}=\langle\Phi_{j_1}\cdots\Phi_{j_4}\rangle$ obeys

$$\frac{\partial}{\partial z}\mathcal{F}_{\mathrm{SL}(2)}=\frac{1}{k-2}\left(\frac{P}{z}+\frac{Q}{z-1}\right)\mathcal{F}_{\mathrm{SL}(2)}, \tag{2.28}$$

where $P$ and $Q$ are differential operators with respect to $x$ defined by

$$\begin{aligned} P&=x^2(x-1)\frac{\partial^2}{\partial x^2}+[(-\kappa+1)x^2-2j_1\\ &\quad -2j_2x(1-x)]\frac{\partial}{\partial x}-2\kappa j_2x-2j_1j_2,\\ Q&=-(1-x)^2x\frac{\partial^2}{\partial x^2}+[(\kappa-1)(1-x)^2\\ &\quad +2j_3(1-x)+2j_2x(1-x)]\frac{\partial}{\partial x}\\ &\quad -2\kappa j_2(1-x)-2j_2j_3, \end{aligned} \tag{2.29}$$

with

$$\kappa=j_4-j_1-j_2-j_3. \tag{2.30}$$

Because of the factor $z^{-1}$ and $(z-1)^{-1}$ on the right-hand side of the KZ equation (2.28), the amplitude $\mathcal{F}_{\mathrm{SL}(2)}(z,x)$ has singularities at $z=0$, 1, and $\infty$. Such singularities are familiar in conformal field theory and appear when locations of two operators coincide on the worldsheet. This leads to the operator product expansion, which will be discussed extensively in Sec. IV.

Quite unexpectedly, the equation also implies a singularity at $z=x$. This is because the coefficients in front of $\partial^2/\partial x^2$ in $P$ and $Q$ cancel each other out at $z=x$. Substituting the ansatz $\mathcal{F}_{\mathrm{SL}(2)}\sim(z-x)^{\delta}$ into Eq. (2.28) and solving the equation to the leading order in $(z-x)$, the exponent $\delta$ is determined as

$$\delta=0 \quad \text{or} \quad k-j_1-j_2-j_3-j_4. \tag{2.31}$$

The solution with $\delta=0$ is regular at $z=x$. However, as we will see in Sec. IV, monodromy invariance of the amplitude $\mathcal{F}_{\mathrm{SL}(2)}$ around $z=0,1,\infty$ as well as around $z=x$ requires that we include the other solution with $\delta=k-j_1-j_2-j_3-j_4$. Therefore, $\mathcal{F}_{\mathrm{SL}(2)}$ has to have a singularity of the form

$$\mathcal{F}_{\mathrm{SL}(2)}|z-x|^{2(k-j_1-j_2-j_3-j_4)}. \tag{2.32}$$

Here we combined holomorphic and antiholomorphic parts so that the amplitude is monodromy-invariant around $z=x$.

The presence of the singularity at $z=x$ is very surprising from the point of view of the worldsheet theory since this is a point in the middle of moduli space. In a standard conformal field theory, amplitudes become singular only at boundaries of moduli spaces. A very closely related divergence appears in the one-loop diagram [2].[13] The interpretation of this singularity is again associated with instanton effects. In the case of the four-point function, worldsheet instantons can grow large if and only if $z=x$ since there has to be a holomorphic map from the worldsheet to the boundary $S^2$ of the target space such that $\gamma(z_i)=x_i$ $(i=1,\ldots,4)$. Such a map exists only when the worldsheet modulus $z$ coincides with the target space modulus $x$. The instanton approximation also explains the value of $\delta$ in the following way. If $z$ is not equal to $x$ but close to it, there is a harmonic map $(\theta,\phi)$ for which

$$\int (\partial\theta-i\sin\theta\partial\phi)(\bar{\partial}\theta+i\sin\bar{\theta}\partial\phi)\sim|z-x|^2. \tag{2.33}$$

We can then insert this into Eq. (2.2) to estimate the action for large $\rho$ as

$$S\sim 2k\rho_0+\alpha e^{2\rho_0}|z-x|^2 \tag{2.34}$$

for some positive constant $\alpha$. Here we only show the dependence on the zero mode $\rho_0$ of $\rho$. The functional integral for the four-point function is then approximated as

$$\begin{aligned}\int d\rho_0 e^{2\rho_0}e^{-2(k-2)\rho_0+\alpha|x-z|^2e^{2\rho_0}}e^{2\Sigma_i(j_i-1)\rho_0}\\ \sim|z-x|^{2(k-j_1-j_2-j_3-j_4)},\end{aligned} \tag{2.35}$$

reproducing the singularity (2.32). This is related to the remark in [18] that the dynamics of long strings is approximated by the Liouville theory; here, $|x-z|^2$ plays the role of the cosmological constant. By a simple extension of this argument, we expect that $n$-point amplitudes have singularities when the worldsheet moduli coincide with the target space moduli. For $n>4$, there can also be singularities when a subset of the worldsheet moduli coincides with a subset of

[13] In [2], we considered the finite-temperature situation in which we periodically identify the target space Euclidean time, and computed a partition function on a worldsheet torus. We found that, in addition to the divergence at the boundary of the worldsheet moduli space $\tau\to i\infty$, there are singularities when $\tau$ is related to the periodicity of the target space Euclidean time. These singularities are interpreted as due to worldsheet instantons from the worldsheet torus to the finite-temperature target space (i.e., the Euclidean black hole in $\mathrm{AdS}_3$).

the target space moduli. In this case, only the corresponding part of the worldsheet grows large.

### E. Correlation functions of spectral flowed states

So far, we have discussed some general properties of (analytically continued) correlation functions of the operators (2.4) in the $SL(2,C)/SU(2)$ model, and we have explained the origin of various singularities in the correlation functions. It turns out that there are other non-normalizable operators we will need to consider for the string theory application.

The operators $\Phi_j$ and their descendents by the $SL(2,C)$ current algebra are not the only operators we will be interested in. The current generators $J^a(z)$ act on $\Phi_j$ as Eq. (2.5), which means that $\Phi_j$ and their analytic continuations also obey the conditions

$$J_n^{\pm}|\Phi\rangle=0, \quad J_n^3|\Phi\rangle=0 \quad (n=1,2,...). \tag{2.36}$$

These lead to the conventional representations of the current algebra. In WZW models based on compact Lie groups, these are all the operators we need to consider; other operators are just current algebra descendents of these. In the $SL(2,R)$ WZW model, there are other states one needs to take into account. These are states in spectral flowed representations of the types described in Eqs. (1.2) and (1.3). Correspondingly, there are non-normalizable operators in the $SL(2,C)/SU(2)$ model that are different from the ones obtained by analytic continuation of $\Phi_j$. In fact, by taking worldsheet operator product expansion OPE of operators of the form (2.4), which obey Eq. (2.36), we can produce operators which are not in the conventional representations obeying Eq. (2.5). For example, we shall see in detail in Sec. V B that we can construct an operator which generates spectral flow from the operator in Eq. (2.4) with $j=k/2$; the spectral flowed representations are generated by the worldsheet OPE's with this operator.

In the remainder of this section, we will argue from a semiclassical point of view that these are natural operators to consider. In particular, we will build operators that are non-normalizable, but such that their "non-normalizability" is concentrated at a point $x$ on target space.

To formulate the problem, let us consider a vertex operator $\Psi_j(z_0,x_0)$ defined so that it imposes the boundary condition

$$\begin{aligned}\phi(z)&\sim-\frac{j}{k}\log|z-z_0|^2,\\ \gamma(z)&\sim x_0+o(|z-z_0|^{2j/k}).\end{aligned} \tag{2.37}$$

The reason that the subleading term in the second line of Eqs. (2.37) has to be smaller than $|z-z_0|^{2j/k}$ will become clear below. We will also show that, when $\frac{1}{2}<\mathrm{Re}(j)<(k-1)/2$, the operator $\Psi_j$ coincides to the operator $\Phi_j$. What happens when $j$ is outside of this range? Let us express $j$ as $j=\tilde{j}+(k/2)w$ with $\frac{1}{2}<\mathrm{Re}(\tilde{j})<(k-1)/2$. The semiclassical analysis that follows shows that the operator $\Psi_j$ defined by Eqs. (2.37) is identified as $\Phi_j^w$, which is defined by acting the $w$ amount of the spectral flow on $\Phi_{\tilde{j}}^-$. In the semiclassical approximation, the spin $\tilde{j}$ will actually be found to be in the range $0<\mathrm{Re}(\tilde{j})<k/2$. In the exact computation, this becomes $\frac{1}{2}<\mathrm{Re}(\tilde{j})<(k-1)/2$.

To explain this, let us consider the two-point function of the vertex operators $\Psi_j$ at $(z,x)=(0,0)$ and $(\infty,\infty)$. We consider the case in which $j$ is real. It was shown in [32] that a general solution to the classical equation of motion for Eq. (2.1) is given by

$$\begin{aligned}\phi&=\rho(z)+\overline{\rho(\bar z)}+\log[1+b(z)\overline{b(\bar z)}],\\ \gamma&=a(z)+\frac{e^{-2\rho(z)}\overline{b(\bar z)}}{1+b(z)\overline{b(\bar z)}},\\ \bar\gamma&=\overline{a(\bar z)}+\frac{e^{-2\overline{\rho(\bar z)}}b(z)}{1+b(z)\overline{b(\bar z)}},\end{aligned} \tag{2.38}$$

for some holomorphic functions $\rho,a,b$ of $z$. The simplest solution obeying the boundary conditions (2.37) is

$$\begin{aligned}\phi&=-\frac{j}{k}\log|z|^2,\\ \gamma&=0.\end{aligned} \tag{2.39}$$

This solution corresponds to $\rho=-(j/k)\log z$ and $a=b=0$ in Eqs. (2.38). This clearly satisfies the boundary conditions at $z=0$. To see that it also obeys the boundary conditions at $z=\infty$, we use the inversion of Poincaré coordinates as

$$\begin{aligned}e^{\phi'}&=e^{-\phi}(1+e^{2\phi}|\gamma|^2),\\ \gamma'&=-\frac{e^{2\phi}\bar\gamma}{1+e^{2\phi}|\gamma|^2},\\ \bar\gamma'&=-\frac{e^{2\phi}\gamma}{1+e^{2\phi}|\gamma|^2}.\end{aligned} \tag{2.40}$$

Note that, at $\phi\to\infty$, this corresponds to the inversion $\gamma'=-1/\gamma$ of the complex coordinates on $S^2$. We then find

$$\begin{aligned}\phi'&=-\frac{j}{k}\log|z'|^2,\\ \gamma'&=0,\end{aligned} \tag{2.41}$$

where $z'$ is the worldsheet coordinate appropriate near $z=\infty$,

$$z'=-\frac{1}{z}. \tag{2.42}$$

Thus the solution (2.39) obeys the boundary conditions both at $z=0$ and $\infty$. This solution describes a cylindrical worldsheet of zero radius, connecting $x=0$ and $\infty$.

Now let us examine what type of perturbations are allowed to this solution. The simplest ones are of the form,

$$\phi = -\frac{j}{k}\log|z|^2. \qquad \gamma = \epsilon z^n \tag{2.43}$$

for small $\varepsilon$. We claim that this deformation corresponds to the action of the current algebra generator $J_n^+$ on the solution (2.39). To see this, we note that the point $g$ in the coset $SL(2,C)/SU(2)$ is parametrized by the coordinates $(\phi,\gamma,\bar{\gamma})$ as

$$g = \begin{pmatrix} e^{-\phi} + \gamma\bar{\gamma}e^{\phi} & e^{\phi}\gamma \\ e^{\phi}\bar{\gamma} & e^{\phi} \end{pmatrix}, \tag{2.44}$$

and the action of $J_n^+$ is given by

$$J_n^+ : g \to g + \begin{pmatrix} 0 & \epsilon z^n \\ 0 & 0 \end{pmatrix} g. \tag{2.45}$$

One can easily verify that (2.45) indeed maps Eqs. (2.39)–(2.43).

One should ask whether this perturbation is normalizable or not. The norm of worldsheet fluctuations is defined using the target space metric as[14]

$$\|(\delta\phi, \delta\gamma, \delta\bar{\gamma})\|^2 = \int \frac{d^2 z}{|z|^2}(\delta\phi^2 + e^{2\phi}\delta\gamma\delta\bar{\gamma}). \tag{2.46}$$

Therefore, the perturbation (2.43) is normalizable (at small $z$) if

$$n = w+1, w+2, w+3, \ldots, \tag{2.47}$$

and non-normalizable if

$$n = w, w-1, w-2, \ldots . \tag{2.48}$$

Normalizable perturbations should be integrated out when we perform the functional integral over the worldsheet and therefore do not change the boundary conditions. This explains why we require that the subleading term in the second line of Eqs. (2.37) has to be smaller than $|z-z_0|^{2j/k}$ since any perturbation equal to or greater than that term is non-normalizable. Non-normalizable perturbations change boundary conditions and correspond to inserting different operators on the worldsheet. Since these perturbations correspond to the action of $J_n^+$ on the worldsheet as in (2.45), one can say that the vertex operator $\Psi_j$ is annihilated by $J_n^+$ which generates normalizable perturbations, i.e.,

$$J_n^+ \Psi_j = 0, \quad n = w+1, w+2, w+3, \ldots . \tag{2.49}$$

One can repeat this analysis for the action of $J_n^-$. This gives a perturbed solution of the form

$$\phi = -\frac{j}{k}\log|z|^2, \qquad \gamma = \epsilon |z|^{4j/k}\bar{z}^n. \tag{2.50}$$

A similar analysis shows that this perturbation is normalizable[15] for

$$n = -w, -w+1, -w+2, \ldots \tag{2.51}$$

and is non-normalizable for

$$n = -w-1, -w, -w+1, \ldots . \tag{2.52}$$

This means $\Psi_j$ is annihilated by $J_n^-$ as

$$J_n^- \Psi_j = 0, \quad n = -w, -w+1, -w+2, \ldots . \tag{2.53}$$

Combining Eqs. (2.49) and (2.53), we find that $\Psi_j$ corresponds to the highest weight state of a discrete representation with $w$ amount of spectral flow. By evaluating $J^3$ for the solution (2.39), one finds that it carries the $J^3$ charge $j$. According to the rule of the spectral flow (1.4), this means that the Casimir operator of the representation before the spectral flow is given by $-\tilde{j}(\tilde{j}-1)$, where $\tilde{j} = j - (k/2)w$.

Something special must happen when $2j/k$ is an integer since the amount $w$ of spectral flow jumps there. What happens is that the solution (2.43) with $n=w$ coincides with the solution (2.50) with $n=-w$ and both are non-normalizable. This means that we have a new type of state, not annihilated by $J_{-w}^-$ and $J_w^+$. It is in the continuous representation with $w$ amount of spectral flow. The fact that the two solutions coincide means that there is a new solution. In fact, when $2j/k = w$, there is a new solution,

$$\phi = -\frac{w}{2}\log|z|^2, \qquad \gamma = \epsilon z^w \log|z|^2. \tag{2.54}$$

One can think of $\epsilon$ as the radial momentum carried by the long string. This is a Euclidean version of the phenomenon discussed in Sec. 3 of [1] in the context of string theory in the Lorentzian $AdS_3$.

Here we have explained how to define the vertex operators $U_j(z,x)$ for the spectral flowed representations. In Sec. V, we will give exact expressions for correlation functions of these operators.

## III. SPACETIME INTERPRETATION OF THE SINGULARITIES IN TWO- AND THREE-POINT FUNCTIONS

In the previous section, we have discussed properties of non-normalizable operators in the $SL(2,C)/SU(2)$ model in general. In this section, we will discuss which subset of those

[14] Here the worldsheet metric is set to $|z|^{-2}dz\, d\bar{z}$, which is appropriate when the worldsheet is an infinite cylinder, since we will use this computation to identify the state corresponding to the vertex operator $\Psi_j$.

[15] Here we assume $2j/k$ is not an integer. See the discussion below.

operators we will consider as physical operators. The physical theory we have in mind is string theory on $H_3 \times \mathcal{M}$, where $\mathcal{M}$ is a compact target space represented by some standard unitary CFT. We will interpret singularities in the amplitudes discussed in the previous section from the point of view of this string theory. According to the AdS/CFT conjecture, the string theory is dual to a boundary conformal field theory (BCFT) on $S^2$ [13]. The observables of BCFT are local normalizable operators on the boundary of the target space. In string perturbation theory, they are represented on the worldsheet by products of non-normalizable operators in the $\mathrm{SL}(2,C)/\mathrm{SU}(2)$ theory times normalizable operators in the unitary CFT for $\mathcal{M}$.[16] The same is true in flat space computations where normalizable plane waves in the target space theory are represented by non-normalizable operators of the form $e^{p_L^0 X_E^0}$ times normalizable operators in the internal CFT in the Euclidean worldsheet theory. (In this discussion we have neglected the tachyon which could be both normalizable in the Euclidean worldsheet theory and physical in the string theory; it is projected out in superstring.) Notice that in the $\mathrm{AdS}_3$ case the Euclidean worldsheet computations are directly related to the Euclidean BCFT computations. We will concentrate on the interpretation of the string theory as a Euclidean field theory. The rotation to Lorentzian target space then should be the standard rotation of the BCFT to Lorentzian signature.

### A. Two-point functions

Our first task will be to pick a set of non-normalizable operators in the $\mathrm{SL}(2,C)/\mathrm{SU}(2)$ model which we will use to construct physical observables. The BCFT is a unitary CFT and it makes sense to analytically continue the target space to $\mathrm{AdS}_3$ with a Lorentzian signature metric. By the standard state-operator correspondence, a normalizable operator of the BCFT corresponds to a normalizable state in the BCFT in the Lorentzian signature space. In the regime where perturbative string theory is applicable, these states correspond to single-particle states and multiparticle states of string theory on Lorentzian $\mathrm{AdS}_3 \times \mathcal{M}$. The worldsheet theory of the string on the Lorentzian $\mathrm{AdS}_3$ is the $\mathrm{SL}(2,R)$ WZW model. The spectrum of the WZW model was proposed in [1] based on a semiclassical analysis, and the proposal was verified by an exact computation of one-loop free energy in [2]. The spectrum of the WZW model is decomposed into a sum of irreducible representations of the $\mathrm{SL}(2,R)\times \mathrm{SL}(2,R)$ current algebra. As shown in Eq. (1.1), it contains the discrete representations $\mathcal{D}_j^0 \otimes \mathcal{D}_j^0$ with $\frac{1}{2}<j<(k-1)/2$ and their spectral flow images corresponding to short strings, and the continuous representations $\mathcal{C}_{j,\alpha}^0 \otimes \mathcal{C}_{j,\alpha}^0$ with $j=\frac{1}{2}+is$ for real $s$ and their spectral flow images corresponding to long strings.

Going back to the $\mathrm{SL}(2,C)/\mathrm{SU}(2)$ model, these states correspond to the operators with

$$j=\tfrac{1}{2}+is \tag{3.1}$$

[16] More precisely, these are what "single-particle" operators correspond to [13].

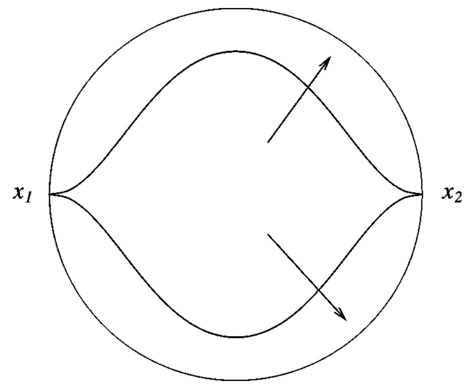


FIG. 1. If $\mathrm{Re}(j)>(k-1)/2$, the worldsheet for the two-point function grows uniformly on $S^2$ toward the boundary.

or

$$\frac{1}{2}<j<\frac{k-1}{2} \tag{3.2}$$

and all their spectral flow images. Though operators with $j=\frac{1}{2}+is$ are normalizable in the worldsheet theory, their spectral flow images are not. After imposing the physical state conditions, the only states with $j=\frac{1}{2}+is$ and $w=0$ are tachyons. Neglecting the tachyons, we see that all the operators of interest are non-normalizable on the worldsheet theory.

Though we just argued for the conditions (3.1) and (3.2) on the basis of the Lorentzian theory, we can make a similar argument purely in the Euclidean theory. The operators on the worldsheet that can correspond to good spacetime BCFT operators are those non-normalizable operators for which the divergences are localized at the point $x$ which we want to interpret as the point where the BCFT operator is inserted. In other words, the "non-normalizability" of the worldsheet vertex operator should be concentrated around $\gamma\sim x$ in target space. Indeed, we saw in Sec. II that if $j$ is outside the range (3.2), there are divergences on the worldsheet theory that are not localized on the boundary $S^2$. For $j<\frac{1}{2}$, these can be interpreted in the usual point-particle limit, while for $j>(k-1)/2$ the divergences came from worldsheet instantons. Let us clarify the target space implication of the latter. Instead of the analytic regularization, one may choose to compute the two-point function by using an explicit target space cutoff regularization by limiting the functional integral to be over $\rho<\rho_0$ for some large value of $\rho_0$. From the discussion in Sec. II B, we expect that, if $j$ is in the range (3.2), the worldsheet never grows large for generic $\gamma$ and all cutoff dependence is localized near $\gamma\sim x_i$. On the other hand, if $j$ exceeds the upper bound, the amplitude depends on $\rho_0$ since the worldsheet can grow larger than $\rho_0$. So the large $\rho_0$ dominates the functional integral and the two-point function is divergent. The divergence is not localized in target space around the points $x_i$, but it is spread all over target space, as shown in Fig. 1. Thus the two-point function of the operator $\Phi_j$ in the Euclidean theory makes sense as a local operator in $x,\bar{x}$ only in the region (3.2). One can nevertheless define the worldsheet operators $\Phi_j$ outside the range (3.2), via analytic continuation. In this definition, one is implicitly subtracting counterterms that are not localized in $x$. From the point of view of the worldsheet theory, there seems to be nothing wrong with this. In fact, operators outside (3.2) are very useful for computing correlation functions on the worldsheet [19–21]. However, worldsheet operators outside (3.2) cannot be identified with local operators in the BCFT. In fact, our analysis in Sec. II E shows that, if one tries to exceed the upper bound in the Euclidean worldsheet theory, one is naturally led to operators in spectral flowed representations.

The coefficient $B(j)$ in the worldsheet two-point function (2.9) given by Eq. (2.10) is well defined and positive for $j$ belonging to the range (3.2). In the string theory computation, we need to divide the amplitude by the volume of the conformal group $V_{\rm conf}$ which keeps the two points fixed. It cancels the divergence coming from evaluating the delta function $\delta(j-j')$ in Eq. (2.9) at $j=j'$, leaving a finite answer, as explained in [29].[17] The cancellation of the two divergent factors requires some care since it may leave some finite $j$-dependent factor. In Sec. V, we will given a heuristic argument to say that the target space two-point function comes with an extra factor of $(2j-1)$ as

$$\langle \Phi_j(x_1)\Phi_j(x_2)\rangle_{\rm target} = \frac{1}{V_{\rm conf}}\langle \Phi_j(x_1;z_1=0) \times \Phi_j(x_2;z_2=0)\rangle_{\rm worldsheet} = \frac{(2j-1)B(j)}{|x_{12}|^{4j}}. \tag{3.3}$$

A more rigorous derivation of the extra factor $(2j-1)$ is given in Appendix A, where we show that this is required by the consistency with the target space Ward identities. The target space two-point function (3.3) is also well behaved in the physical range (3.2).

We can also compute target space two-point functions for any spectral flowed states; this will be done explicitly in Sec. V. We will find that they are all regular and have positive-definite two-point functions in the region (3.2). The extra factor $(2j-1)$ mentioned in the above paragraph is generalized to $|2j-1+(k-2)w|$ when $w\neq 0$.

As shown in [1], the spectral flowed continuous states ($j=\frac{1}{2}+is$) correspond to operators in the BCFT which have continuous dimensions. We conclude from this that the BCFT has a noncompact target space (at least it is noncompact in the leading order in string perturbation theory). The nature of this noncompactness was discussed in [18] in the case of $AdS_3\times S^3\times M_4$, where $M_4=K3$ or $T^4$. In these cases, BCFT is the supersymmetric sigma model whose target space is the moduli space of the Yang-Mills instantons on $M_4$. The noncompact directions are related to the limits where instantons become small. The relation between the existence of the continuous spectrum in CFT and the noncompact directions in its target space is familiar in the case of a free noncompact scalar. We would like to stress that there is nothing particularly nonlocal about the sigma model with a continuous spectrum. The operators corresponding to these states are local on the space where the BCFT is defined. This is for the same reason that an operator like $e^{ikX}$ is local on the worldsheet of the free scalar field $X(z,\overline{z})$. In our case, these operators are the spectral flowed versions of $j=\frac{1}{2}+is$. Their target space two-point function will be computed in Sec. V and is given by

$$\langle \hat{\Phi}^{jw}_{J\bar{J}}(x_1)\hat{\Phi}^{j'w}_{J\bar{J}}(x_2)\rangle_{\rm target} \sim \left[\delta(s+s')+\delta(s-s')\,\frac{\pi B(j)}{\gamma(2j)}\,\frac{\Gamma\left(j-\frac{k}{2}w+J\right)}{\Gamma\left(1-j-\frac{k}{2}w+J\right)}\,\frac{\Gamma\left(j+\frac{k}{2}w-\bar{J}\right)}{\Gamma\left(1-j+\frac{k}{2}w-\bar{J}\right)}\right]\frac{1}{x_{12}^{2J}\bar{x}_{12}^{2\bar{J}}}. \tag{3.4}$$

Here $j=\frac{1}{2}+is$, $j'=\frac{1}{2}+is'$, the spacetime conformal weight of the operator $J$ is given by

$$J=\frac{k}{4}w+\frac{1}{w}\left(\frac{s^2+\frac{1}{4}}{k-2}+h-1\right), \tag{3.5}$$

and $h$ is the conformal weight of the vertex operator for the internal CFT, whose two-point function we assumed to be unit-normalized in Eq. (3.4). Equation (3.5) comes from the $L_0=1$ condition. Unlike the case of short strings, the two-point function of long strings does not receive the extra factor of $|2j-1+(k-2)w|$ when we transform the worldsheet computation into the target space computation. Note that the term multiplying the second $\delta$ function in Eq. (3.4) is a pure phase as

[17]The target space two-point function receives contribution from the internal CFT. Since this part is diagonal in the conformal weight, the physical state condition for the short string implies that we need to set $j=j'$ to have a nonzero two-point function in the target space.

$$e^{i\delta(s)} \equiv \frac{\pi B(j)}{\gamma(2j)} \frac{\Gamma\left(j-\frac{k}{2}w+J\right)}{\Gamma\left(1-j-\frac{k}{2}w+J\right)} \frac{\Gamma\left(j+\frac{k}{2}w-\bar{J}\right)}{\Gamma\left(1-j+\frac{k}{2}w-\bar{J}\right)}$$

$$= \nu^{-2is} \frac{\Gamma\left(-\frac{2is}{k-2}\right)}{\Gamma\left(+\frac{2is}{k-2}\right)} \frac{\Gamma(-2is)}{\Gamma(+2is)} \frac{\Gamma\left(\frac{1}{2}+is-\frac{k}{2}w+J\right)}{\left(\frac{1}{2}-is-\frac{k}{2}w+J\right)}$$

$$\times \frac{\Gamma\left(\frac{1}{1}+is+\frac{k}{2}w-\bar{J}\right)}{\Gamma\left(\frac{1}{2}-is+\frac{k}{2}w+\bar{J}\right)}. \qquad (3.6)$$

This is the phase shift that occurs when a long string comes from the boundary and back, which in terms of the BCFT is a small instanton becoming large and small again.

In summary, the singularities in the two-point function are outside of the range (3.2) of our choice of operators. Now we can ask whether this choice removes all singularities in all *n*-point functions. The answer is *no*. We will see, however, that the singularities can be interpreted physically and we will give a prescription for how to deal with them. In other words, all singularities that appear are interpretable in the BCFT.

### B. Three- and four-point functions

The three-point function has poles at $j_3 = j_1 + j_2 + n$ and their permutations in $j_1$, $j_2$, $j_3$. These poles are standard and easy to understand. They appear in all $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ examples [37,38]. These poles are due to mixing with two-particle states. The string perturbation expansion in AdS corresponds to a $1/N$ expansion in the boundary theory. To leading order in $1/N$ the operators are single particles and multiparticle states in AdS. When we compute $1/N$ corrections, these operators can mix. The mixing is generically small, of order $1/N$, but if two operators have the same conformal weight at leading order in $1/N$, then the mixing can be of order 1, since we are doing degenerate perturbation theory. If $j_3 = j_1 + j_2 + n$, then we have two operators with the same conformal weight, namely $O_{j_3}$ and: $\partial_{12}^n O_{j_1} O_{j_2}$:, where the $O_{j_1}$ are single-particle operators and the derivatives act on both operators in such a way that the result is a primary operator under $\mathrm{SL}(2,R) \times \mathrm{SL}(2,R)$ symmetry at large *N*. These two operators can mix in the subleading order in $1/N$, and the divergence in the three-point function is canceled if we take into account this mixing effect.

It is instructive to look at the semiclassical description of this divergence. Suppose $j_i$ are large, then correlation functions can be computed by considering a particle of masses proportional to $j_i$ with trajectories that intersect the boundary at the points where the operators are inserted [39]. If $j_3 < j_1 + j_2$ (and the same holds for other permutations of 123), the dominant contribution is given in Fig. 2(a). On the other

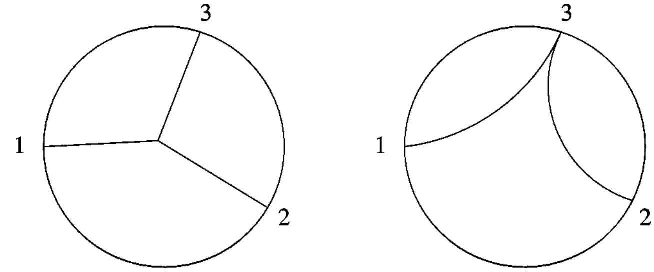


FIG. 2. Here we see the change in behavior of the semiclassical geodesics when we go from the case of $j_3 < j_1 + j_2$ in (a) to the case $j_3 > j_1 + j_2$ in (b).

hand, if $j_3 > j_1 + j_2$, we cannot find a configuration where the interaction point is in the interior; the interaction point moves to the boundary as shown in Fig. 2(b). In the semiclassical approximation, $n > 0$ becomes a continuous variable. If we quantize the fields, we see that *n* is an integer. This divergence is eliminated by a redefinition of the operator $O_{j_3}$ which mixes the single-particle operator with the two-particle operator. That a local redefinition of the operator can cancel the divergence is related to the fact that the divergence is coming from the region close to the point on the boundary where $O_{j_3}$ is inserted.

The three-point function has also a divergence at $\Sigma_i j_i = k$. This divergence appears even if all $j_i$'s are within the range (3.2). From the point of view of the worldsheet theory, this divergence is due to instanton corrections as we saw in Sec. II. This means that the divergence appears because the worldsheet can be very close to the boundary of AdS with no cost in action; see Fig. 3.

One might think that this is a nonlocal effect in the BCFT. In order to remove it, it seems that we need counterterms which are spread all over the $S^2$ where the BCFT is defined. We would like to propose a different interpretation. The BCFT is local and this divergence is simply due to the noncompactness of the BCFT target space. In other words, we do not remove the divergence. The origin of this divergence, which we will explain below, suggests that only three-point functions with $\Sigma j_i < k$ make sense in the BCFT.

In order to clarify this point, let us consider a quantum-mechanical example which has a phenomenon very analogous to what we are dealing with. Suppose that we have the

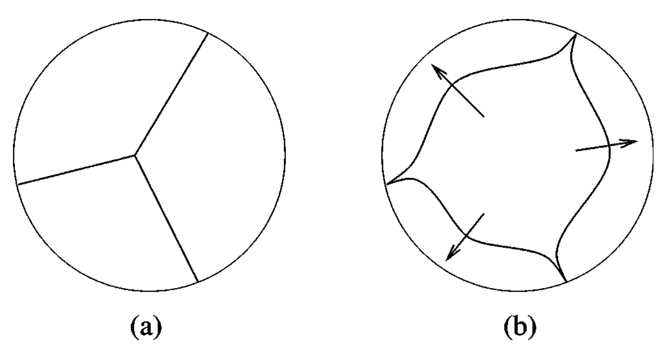


FIG. 3. Change in behavior of the classical worldsheet when $\Sigma j_i < k$ in (a) to the case where $\Sigma j_i > k$ in (b). In (b), the worldsheet is driven to the boundary of AdS.

quantum mechanics of a particle in a potential well, where the potential asymptotes to zero at infinity and it is negative at the origin, so that the system has a normalizable ground-state wave function $\psi(x)$ which for large $x$ decays as $\psi(x) \sim e^{-\kappa x/2}$. In this system, we can consider operators of the form $O_\lambda = e^{\lambda x}$. The expectation value on the ground state of the product of two of these operators is well defined as long as $\lambda < \kappa/2$. If we insert several operators and we try to compute $\langle \psi | O_{\lambda_1}(t_1) \cdots O_{\lambda_n}(t_n) | \psi \rangle$, we will find that we can only do the computation if $\Sigma \lambda_i < \kappa$. In other words, there seems to be a nonlocal constraint (in time) on the operators whose correlators we can compute. The theory is perfectly local, and the divergence is just an IR effect in the target space coming from the noncompactness of the target space. It is a well-known fact that there are operators in quantum mechanics that have a domain and a range, and some operators can take a state out of the Hilbert space.[18] In this quantum-mechanical model, there are other operators, of the form $e^{ikx}$, for example, which are perfectly well defined for any real value of $k$.

Our BCFT is very similar to this quantum-mechanical example. It has a normalizable ground state, and the vacuum expectation value of discrete states with $\Sigma j_i > k$ is not defined. There are other operators, the ones in the spectral flowed continuous representations, which we can consider. These operators are analogous to $e^{ikx}$ in the quantum-mechanical model. Correlation functions of these are well defined without any additional constraint. Notice that the target space BCFT has a normalizable ground state, despite having a noncompact target space since there is a gap between the ground-state energy and the threshold where the continuum starts due to the noncompactness.

Based on these observations, we claim that correlation functions of discrete states are only well defined if $\Sigma j_i < k$. The expression (2.13) can be defined for $\Sigma j_i > k$ by analytic continuation, but it does not make physical sense as it does not represent a well-defined computation in the BCFT. In order to define it, we need to add counterterms that are spread over $S^2$ in target space.

For the four-point function, the singularity at $z = x$ (2.32) implies, after integrating over $z$, that there is a divergence in the four-point function if $\Sigma_i j_i = k+1$.[19] So a four-point function makes sense only for $\Sigma j_i < k+1$. It might be possible to extend the four-point function to $\Sigma j_i > k+1$ by analytic continuation, but it does not have any immediate physical interpretation.

Note that we are not saying that there is a bound on the spacetime conformal weight of the operators we add. By using spectral flowed operators, we can compute correlation functions of operators whose conformal weights are as high as we like. These spectral flowed operators were defined precisely to avoid the divergences associated to long strings.

In order to stress once again that these divergences have nothing to do with nonlocal behavior of the BCFT, let us consider an example with $N=4$ super Yang-Mills (SYM) theory in $d=4$ where this feature appears. Consider $N=4$ SYM theory on $T^2 \times S^1 \times (\mathrm{time})$ with antiperiodic boundary conditions for the fermions on $S^1$ and periodic on $T^2$. The supergravity solution describing the ground state of this theory was described in [40]. It is the near-extremal black three brane doubly Wick rotated. It is a nonsingular geometry with topology $T^2 \times D^2$, where $D^2$ is a disk whose boundary is the $S^1$ (we concentrate on the geometry of the radial direction and the three spatial dimensions of the brane). This theory has finite-energy excitations which correspond to placing a D3 brane at some radial position and winding on $T^2 \times S^1$. These are analogous to the long strings described above. They lead to divergences in computations of certain correlation functions, in a very similar fashion to how long strings lead to divergences in the $AdS_3$ case. These divergences come from the fact that there is a Coulomb branch that we can explore with finite cost in energy.

Finally let us note that, both in the $AdS_3$ case and in the $N=4$ SYM example we have given above, we can remove the noncompact direction in field space by deforming the Lagrangian of the theory. In the $AdS_3$ case we can add some Ramond-Ramond (RR) fields, which in the BCFT has the effect of making the target space compact. In the $N=4$ example, we can add mass terms for all scalar fields.

In $AdS_3$ with RR backgrounds, the continuum states become discrete and we can compute the correlation functions of any number of operators. If we take the limit of RR fields going to zero, we will find that states with high conformal weight with $j > (k-1)/2$ will lead to operators in the SL(2)/SU(2) model which are spectral flowed. Similarly, we expect that if we compute a three-point function for three discrete states with $\Sigma j_i < k$, the result will go over smoothly to Eq. (2.13) as we take the RR fields to zero. On the other hand, there is no reason why the correlation function of states with $\Sigma j_i > k$ should go over smoothly to Eq. (2.13) when we remove the RR fields; in fact, we expect that the correlation function diverges in the limit.

## IV. FOUR-POINT FUNCTION

In this section, we compute four-point functions in target space by performing the integration over the moduli space of the string worldsheet. A four-point amplitude depends nontrivially on the cross ratio $x$ of the four points on the boundary of $AdS_3$ where the operators $\mathcal{O}_1, \ldots, \mathcal{O}_4$ are inserted. In other words, we can use conformal invariance to fix the operators as

$$\mathcal{F}_{\mathrm{target}}(x,\bar{x}) = \langle \mathcal{O}_1(0) \mathcal{O}_2(x) \mathcal{O}_3(1) \mathcal{O}_4(\infty) \rangle. \tag{4.1}$$

Our main objective is to derive the operator product expansion by evaluating the small-$x$ expansion of $\mathcal{F}_{\mathrm{target}}$. If the amplitude $\mathcal{F}_{\mathrm{target}}(x,\bar{x})$ in the BCFT obeys the factorization condition, we should be able to expand it for $|x| < 1$ in powers of $x$ as

[18] As a trivial example, consider a harmonic oscillator and imagine the Hamiltonian acting on the state $|\psi\rangle = \Sigma(1/n)|n\rangle$.

[19] In an $n$-point function, we expect a divergence when $\Sigma j_i = k + n - 3$.

$$\mathcal{F}_{\rm target}(x,\bar{x})=\sum_{J,\bar{J}} x^{J-J_1-J_2}\bar{x}^{\bar{J}-\bar{J}_1-\bar{J}_2}\mathcal{C}_{\rm target}(J,\bar{J}), \quad (4.2)$$

where $(J,\bar{J})$ are the target space conformal weights and $\mathcal{C}_{\rm target}(J,\bar{J})$ is given in terms of three- and two-point functions as

$$\mathcal{C}_{\rm target}(J,\bar{J})=\langle \mathcal{O}_1(0)\mathcal{O}_2(1)\mathcal{O}_{J,\bar{J}}(\infty)\rangle \times\frac{1}{\langle \mathcal{O}_{J,\bar{J}}(\infty)\mathcal{O}_{J,\bar{J}}(0)\rangle} \times\langle \mathcal{O}_{J,\bar{J}}(0)\mathcal{O}_3(1)\mathcal{O}_4(\infty)\rangle \quad (4.3)$$

and $\{\mathcal{O}_{J,\bar{J}}\}$ is a complete set of operators in BCFT.

Before we start the detailed computation, let us summarize our result. We will focus on the case in which the operators $\mathcal{O}_1,\ldots,\mathcal{O}_4$ correspond to short strings with $w=0$, i.e., they correspond to states in discrete representations $\mathcal{D}_j^0 \otimes \mathcal{D}_j^0$ of the current algebra $SL(2,R)\times SL(2,R)$. We find that, if their conformal weights $j_1,\ldots,j_4$ obey the inequalities

$$j_1+j_2<\frac{k+1}{2}, \quad j_3+j_4<\frac{k+1}{2}, \quad (4.4)$$

the string amplitude (4.1) can indeed be expanded in powers of $x$ as Eq. (4.3), and the intermediate states $\mathcal{O}_{J,\bar{J}}$ are either short strings with $w=0$ and in the range (3.2), long strings with $w=1$, or two-particle states of short strings. All other physical states do not appear. In Sec. V, we will show that this is because the three-point functions in Eq. (4.3) vanish for the other cases. If (4.4) is not obeyed, then there are terms in the $x$ expansion that cannot be interpreted as coming from the exchange of physical states. We explain at the end of this section that this is due to the noncompactness of the target space of BCFT, and it is the physically correct behavior. For CFT's with compact target spaces, the operator product expansion (4.3) should always be valid. In our case, we expect it to hold only if (4.4) is obeyed. Now we proceed to explain these statements in more detail.

### A. The four-point function in the $SL(2,C)/SU(2)$ coset model

Each spacetime operator is associated to a worldsheet vertex operator $\mathcal{O}_i(x,\bar{x})\to\int d^2z\, \Phi_i(x,\bar{x};z,\bar{z})$. If we want to calculate the spacetime four-point function $\mathcal{F}_{\rm target}$, we should calculate the four-point function $\mathcal{F}_{\rm worldsheet}$ of the corresponding worldsheet vertex operators and integrate it over their positions. Using worldsheet conformal invariance, we can fix the worldsheet position of three of them, and the worldsheet correlator depends only on the cross ratio $z$. So we need to compute

$$\mathcal{F}_{\rm target}(x,\bar{x})=\int d^2z\, \mathcal{F}_{\rm worldsheet}(z,\bar{z};x,\bar{x}). \quad (4.5)$$

There are two factors that contribute to the worldsheet correlation function as

$$\mathcal{F}_{\rm worldsheet}(z,\bar{z};x,\bar{x})=\mathcal{F}_{SL(2)}(z,\bar{z};x,\bar{x})\mathcal{F}_{\rm internal}(z,\bar{z}), \quad (4.6)$$

where $\mathcal{F}_{SL(2)}$ is the correlation function of the $SL(2,C)/SU(2)$ coset model and $\mathcal{F}_{\rm internal}$ is that of the internal CFT.[20]

A closed-form expression of $\mathcal{F}_{SL(2)}$ is not known for generic values of $j_1,\ldots,j_4$ for the external states. We will use an expression for it given in [21], which involves an integral over a continuous family of solutions to the KZ equation (2.28). Let us review the derivation. The KZ equation (2.28) has an infinite number of solutions reflecting the fact that the Hilbert space of the $SL(2,C)/SU(2)$ model is decomposed into infinitely many representations of $SL(2,C)$. It turns out that there is a unique combination of these solutions that satisfies the factorization properties on the worldsheet, i.e., the $z$ expansion of the amplitude should be expressed as a sum over normalizable states when all four external operators, labeled by $j_1,\ldots,j_4$, are also normalizable (or close enough to normalizable). It was shown in [28] that the Hilbert space of the $SL(2,C)/SU(2)$ coset theory is a sum of the representations with $j=\frac{1}{2}+is$ ($s$: real, $>0$) with the conformal weight $\Delta(j)$. Therefore, it is reasonable to expect that the four-point function is a sum of products of the conformal block $\mathcal{F}_j(z,x)$ of the form

$$\mathcal{F}_j(z,x)=z^{\Delta(j)-\Delta(j_1)-\Delta(j_2)}x^{(j-j_1-j_2)}\sum_{n=0}^{\infty} f_n(x)z^n. \quad (4.7)$$

Substituting this into the KZ equation, one finds that $f_0(x)$ has to obey the hypergeometric equation in $x$ with two linearly independent solutions

$$F(j-j_1+j_2,j+j_3-j_4,2j;x), \qquad x^{1-2j}F(1-j-j_1+j_2,1-j+j_3-j_4 2-2j;x). \quad (4.8)$$

As we will discuss below, we need both solutions to construct a monodromy-invariant four-point function. Taking into account the factor $x^{j-j_1-j_2}$ in Eq. (4.7), one sees that the two solutions in (4.8) are related to each other by the reflection $j\to 1-j$, or $s\to -s$ if we write $j=\frac{1}{2}+is$. Therefore, instead of requiring $s>0$ and using both solutions, we can allow $s$ to be any real number and always pick the first solution in (4.8).

It was shown by Teschner that, for generic values of $j$, all other $f_n(x)$ $(n=1,2,\ldots)$ are determined iteratively by the KZ equation once we fix $f_0(x)$ as the initial condition at $z\to 0$. They take the form

$$f_n(x)=\sum_{m=-n}^{\infty} c_{nm}x^m. \quad (4.9)$$

Therefore, by demanding that $f_0(x)$ be given by the first solution in (4.8), we can uniquely determine $\mathcal{F}_j$ as a solution

[20] In general, $\mathcal{F}_{\rm worldsheet}$ could be a sum of such products.

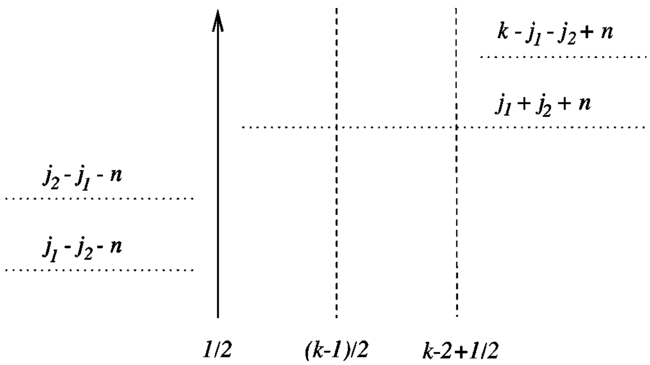


FIG. 4. The solid line indicates the integration contour for Eq. (4.10) in the $j$ complex plane. We highlighted the location of some poles in $C(j)$. Here all external $j_i$ are of the form $j_i=\frac{1}{2}+is_i$. There are similar poles with $j_1,j_2\to j_3,j_4$; there are also some other poles that will not be important for our purposes.

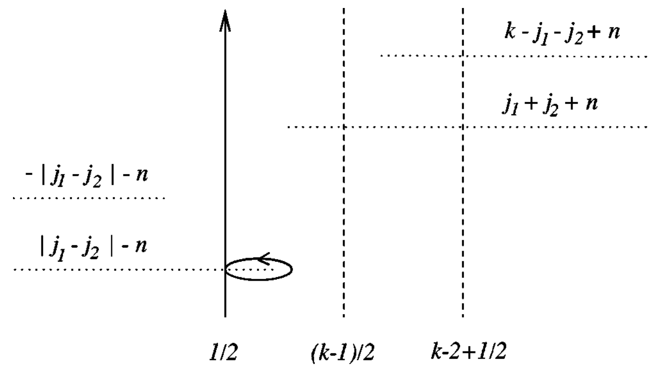


FIG. 5. The solid line indicates the integration contour after we analytically continue Eq. (4.10) in the external $j_i$. Some poles of the form $|j_1-j_2|-n$ have crossed the integration contour so we should include their residues. There are similar poles with $j_3,j_4$. We separated the poles along the imaginary direction for clarity, although they are all along the real axis when $j_i$ are real.

to the KZ equation. Note that, unlike $j_1,\ldots,j_4$, the parameter $j$ does not appear in the KZ equation (2.28), but it is used as a label of the solution of the KZ equation whose small $z$ behavior is as in Eq. (4.7).

The full four-point function $\mathcal{F}_{\mathrm{SL}(2)}(z,x)$ is then given by the worldsheet factorization ansatz [21] as

$$\mathcal{F}_{\mathrm{SL}(2)}(z,\overline{z};x,\overline{x})=\int_{(1/2)+iR}dj\,\mathcal{C}(j)|\mathcal{F}_j(z,\overline{z};x,\overline{x})|^2, \tag{4.10}$$

where the normalization factor $\mathcal{C}(j)$ is given by

$$\mathcal{C}(j)=C(j_1,j_2,j)\frac{1}{B(j)}C(j,j_3,j_4), \tag{4.11}$$

where $C(j_1,j_2,j_3)$ and $B(j)$ are defined in Eqs. (2.14) and (2.10). The integral is over $j=\frac{1}{2}+is$ with $s\in R$. As we mentioned, the $j$ integral covers both solutions (4.8) because of the reflection symmetry $j\to 1-j$ of the integration region. As shown in [21], including both solutions is necessary in order for the four-point function to be monodromy-invariant around $x=1$ and $\infty$. In Appendix B we argue that the integral over $j$ in Eq. (4.10) is convergent.

The expression (4.10) is valid if all external labels $j_1,\ldots,j_4$ are close to the line $j=\frac{1}{2}+is$. The expression for other values of $j_1,\ldots,j_4$ is defined by analytic continuation. When we do this, some poles in the integrand cross the integration contour. The four-point function is then Eq. (4.10) plus the contribution of all poles that have crossed the integration contour. We need to know the pole structure of $\mathcal{C}(j)$ and $\mathcal{F}_j(z,x)$. As we discussed in earlier sections, the three-point function $C(j_1,j_2,j)$ in Eq. (4.11) has poles at

$$j=1-j_1-j_2-j_p,\quad j_1+j_2+j_p,\quad \pm(j_1-j_2)-j_p \tag{4.12}$$

(see Fig. 4) where

$$j_p=n+m(k-2),\quad -(n+1)-(m+1)(k-2)$$

$$(n,m\geqslant 0).$$

To compute the correlation function of short strings with $w=0$, we need to analytically continue $j_1,\ldots,j_4$ from the line $j_i=\frac{1}{2}+is$ to the interval $\frac{1}{2}<j_i<(k-1)/2$ on the real axis. The poles that cross the contour of the $j$ integral in Eq. (4.10) are of the form

$$j=|j_1-j_2|-n,\quad n=0,1,2,\ldots, \tag{4.13}$$

with $j>\frac{1}{2}$. There are similar poles in $C(j,j_3,j_4)$ at

$$j=|j_3-j_4|-n,\quad n=0,1,2,\ldots. \tag{4.14}$$

There are no poles in $B(j)^{-1}$ and $\mathcal{F}_j$ that cross the contour when we do the analytic continuation. Therefore, after the analytic continuation in $j_1,\ldots,j_4$, the correlation function $\mathcal{F}_{\mathrm{SL}(2)}$ is defined by the integral (4.10) plus the contribution from the poles at Eqs. (4.13) and (4.14). Stated in another way, the contour of the $j$ integral is deformed from the line $j=\frac{1}{2}+is$ to avoid these poles. See Fig. 5.

This completes the specification of $\mathcal{F}_{\mathrm{SL}(2)}(z,x)$. The next task is to multiply the factor $\mathcal{F}_{\mathrm{internal}}(z,\overline{z})$ coming from the internal CFT and integrate the resulting expression over the $z$ plane as in Eq. (4.5). We will find it useful to deform the contour of the $j$ integral. We will deform the contour of the $j$ integration in Eq. (4.10) within the region

$$\frac{1}{2}\leqslant \mathrm{Re}\,j\leqslant\frac{k-1}{2}. \tag{4.15}$$

In this process, we will pick up poles in $\mathcal{C}(j)$ and $\mathcal{F}_j$, so it is useful to list them here. Among the poles (4.12) in $C(j,j_1,j_2)$, the relevant ones in the region (4.15) are of the form

$$\mathrm{Poles}_1:\quad j=j_1+j_2+n,$$

$$\mathrm{Poles}_2:\quad j=k-j_1-j_2+n, \tag{4.16}$$

$$n=0,1,2,\ldots.$$

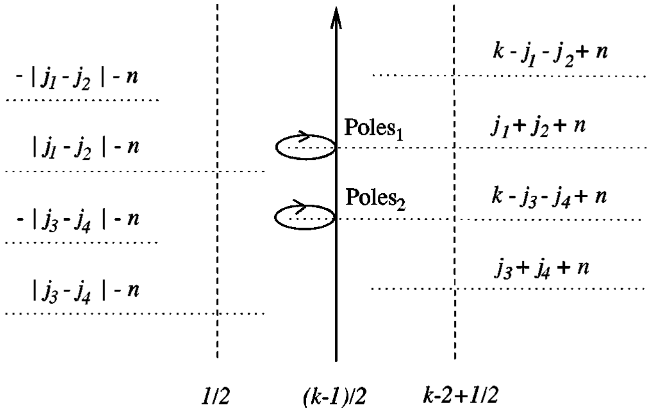


FIG. 6. We shifted the integration contour to $j=(k/2)-\frac{1}{2}+is$. We picked up contributions from Poles$_1$ and Poles$_2$. This figure represents the case in which $j_1+j_2<k/2$ and $j_3+j_4>k/2$.

Here we are assuming that $j_1,j_2$ are in the physical range $\frac{1}{2}<j_1,j_2<(k-1)/2$. Note that (4.15) imposes a constraint on allowed values of $n$ in Eqs. (4.16). The poles (4.13) are also in the region (4.15), but the contour of the $j$ integral is defined to avoid these poles, as we discussed in the previous paragraph; see Fig. 6. There are similar poles in $C(j,j_3,j_4)$ given by exchanging $j_1,j_2 \to j_3,j_4$. From Eq. (2.10), we can see that $1/B(j)$ has no poles in the region (4.15).

One may also ask if there is a pole coming from the conformal block $\mathcal{F}_j$. It turns out that there is no such pole in the region (4.15). This has been shown in [21] using properties of the Kac-Kazhdan determinant. To see this explicitly, it is useful to rearrange the expansion (4.7) as

$$\mathcal{F}_j(z,x)=x^{\Delta(j)-\Delta(j_1)-\Delta(j_2)+j-j_1-j_2} \times u^{\Delta(j)-\Delta(j_1)-\Delta(j_2)}\sum_{m=0}^{\infty} g_m(u)x^m, \quad (4.17)$$

where $u=z/x$. This expansion will also be used in the next subsection to evaluate the $z$ integral in the region where $|z|<1$. If we substitute this expansion in the KZ equation, we find that the first term $g_0(u)$ in the expansion should obey the hypergeometric equation in $u$. The solution which agrees with the initial condition (4.7) for small $z$ is

$$g_0(u)=F(j_1+j_2-j,j_3+j_4-j,k-2j;u). \quad (4.18)$$

By looking at the standard formula for the Taylor expansion of the hypergeometric function, one can check explicitly that $g_0(u)$ has no poles in the region (4.15). Given that $g_0(u)$ has no poles, we can prove inductively that the same is true for all $g_m(u)$, $m\geqslant 1$. The proof of this statement is given in Appendix B.

In the following subsections, we consider the case $|x|<1$ and expand the expression (4.10) in powers of $x$. We will then integrate it over $z$. We will not impose a restriction on $z$ since we must integrate over $z$ to obtain the physical string amplitude. We will divide the range of $z$ into two regions:

$$\text{region I:} \quad |z|<1,$$

$$\text{region II:} \quad |z|>1.$$

Since $\mathcal{F}_{SL(2)}(z,x)$ has the singularity (2.32) at $z=x$, one may consider dividing the region I further into two regions where $0<|z|<|x|$ and $|x|<|z|<1$, but it turns out to be unnecessary to do so, as we shall see below.

### B. Integral over the region I

To integrate the four-point function over the region I, it is useful to define the variable $u=z/x$ and use the expansion (4.17). We will mostly concentrate on the first term $g_0(u)$ of the expansion. As we mentioned, the KZ equation implies that the first term $g_0(u)$ obeys the hypergeometric equation whose solutions are $F_j(u)$ and $F_{k-1-j}(u)$, where $F_j$ is defined by

$$F_j(u)\equiv F(a,b,c;u), \quad a=j_1+j_2-j, \quad b=j_3+j_4-j, \quad c=k-2j. \quad (4.19)$$

At $u=1$, these solutions behave as $c_1+c_2(u-1)^{k-\Sigma j_i}$, where the coefficients $c_1,c_2$ are both nonzero for generic values of $j_1,\ldots,j_4$. It is therefore clear that the first solution (4.19) on its own is not monodromy-invariant at $u=1$. For a given $j$, there is a unique monodromy-invariant combination given by[21]

$$G_{j,0}(u,x)=|x^{\Delta(j)-\Delta(j_1)-\Delta(j_2)+j-j_1-j_2}u^{\Delta(j)-\Delta(j_1)-\Delta(j_2)}|^2 \times[|F_j(u)|^2+\lambda|u^{1-c}F_{k-1-j}(u)|^2], \quad (4.20)$$

where

$$\lambda=-\frac{\gamma(c)^2\gamma(a-c+1)\gamma(b-c+1)}{(1-c)^2\gamma(a)\gamma(b)}, \quad (4.21)$$

and $\gamma(x)$ is given in Eq. (2.11). The subindex 0 is there to remind us that we are examining the first term in the $x$ expansion in Eq. (4.17). It is useful to note that we can write it as

$$\mathcal{C}(j)G_{j,0}(u)=\mathcal{C}(j)|\mathcal{F}_{j,0}(u,x)|^2 +\mathcal{C}(k-1-j)|\mathcal{F}_{k-1-j,0}(u,x)|^2, \quad (4.22)$$

where

$$\mathcal{F}_{j,0}(u,x)\equiv x^{\Delta(j)-\Delta(j_1)-\Delta(j_2)+j-j_1-j_2} \times u^{\Delta(j)-\Delta(j_1)-\Delta(j_2)}F_j(u) \quad (4.23)$$

is the first term in the $x$ expansion of $\mathcal{F}_j$ in Eq. (4.17). We can show Eq. (4.22) by using the identities

$$C(k-1-j)=\lambda\mathcal{C}(j),$$

[21]Note that $j$ is not complex conjugated in this expression. In other words, $|a(j)x^{f(j)}|^2\equiv a^2(j)|x|^{2f(j)}$.

$$\Delta(k-1-j)+(k-1-j)=\Delta(j)+j, \tag{4.24}$$

$$\Delta(k-1-j)=\Delta(j)+1-c.$$

The problem with the monodromy-invariant combination (4.20) is that it does not satisfy the small-$z$ expansion condition (4.7) because of the factor $u^{1-c}$ in the second term in the parentheses. On the other hand, the solution (4.19) satisfies the expansion (4.7) but is not monodromy-invariant around $z=x$. This puzzle is resolved by performing the $j$ integral. We can show that, after the $j$ integral, the amplitude (4.10) is monodromy-invariant. To see this, we need to deform the contour from $j=\frac{1}{2}+is$ to $j=\frac{1}{2}+is+[(k-2)/2]$; see Fig. 6. The new contour is such that, if in includes the point $j$, it also includes the point $k-1-j$. Therefore, we write the integral of the solution (4.19) as $\frac{1}{2}$ of the integral of the monodromy-invariant combination $G_{j,0}(u,x)$. As we deform the contour, we pick up some residue contributions from the poles at Eq. (4.16). It turns out that each of those contributions is monodromy-invariant by itself. This can be seen by noting that, for the values of $j$ in Eq. (4.16), the coefficient $\lambda$ in Eq. (4.20) vanishes. More specifically, we find that the contributions from $\text{Poles}_1$ in Eq. (4.16) are nonsingular at $u=1$, while those of $\text{Poles}_2$ in Eq. (4.16) contain only the singular solution at $u=1$, and therefore both are monodromy-invariant by themselves. We can now express Eq. (4.10) in the manifestly monodromy-invariant form as

$$\int_{(1/2)+iR} dj\,\mathcal{C}(j)\mathcal{F}_j(z,x)=\int_{[(k-1)/2]+iR}\mathcal{C}(j)\mathcal{F}_j(z,x)+(\text{contribution from Poles}_1\text{ and Poles}_2)$$

$$=\frac{1}{2}\int_{[(k-1)/2]+iR} dj\,\mathcal{C}(j)[G_{j,0}(u,x)+\cdots]+(\text{contribution from Poles}_1\text{ and Poles}_2), \tag{4.25}$$

where the dots represent higher-order terms in the $x$ expansion. It is convenient to combine the integrand into the monodromy-invariant form $G_{j,0}(u,x)$ given by Eq. (4.20) because, in the following, we will perform the $z$ integral before the $j$ integral. (We will be careful about justifying the exchange of the $j$ integral and the $z$ integral by regularizing the $z$ integral.) In conclusion, we have shown that after integrating over $j$, Teschner's expression (4.10) for the four-point function is monodromy-invariant around $z=x$.

The contribution from $\text{Poles}_1$ is of the form $x^{j-j_1-j_2}f(z,\bar{z})$ with $j=j_1+j_2+n$. Since the integral of $f(z,\bar{z})$ times $\mathcal{F}_{\text{internal}}(z,\bar{z})$ is independent of $x$, we conclude that the conformal weight of the intermediate states is $J=j=j_1+j_2+n$. These conformal weights can be identified with the conformal weights of two-particle contributions. In other words, when we compute the spacetime operator product expansion, the intermediate operators could be two-particle operators. There can be other contributions with these quantum numbers in the intermediate channel which come from two disconnected sphere diagrams in string perturbation theory. The $z$ integral of this contribution contains divergences at small $z$. They are canceled by another contribution which will be discussed later.

If Eq. (4.4) is satisfied, Eq. (4.25) does not receive any contributions from $\text{Poles}_2$ in Eq. (4.16).

Before we perform the integral over the $z$ plane, we need to multiply $\mathcal{F}_{SL(2)}(z,x)$ by a four-point function $\mathcal{F}_{\text{internal}}(z,\bar{z})$ of the internal CFT. In region I, i.e., $|z|<1$, we can expand $\mathcal{F}_{\text{internal}}$ as

$$\mathcal{F}_{\text{internal}}(z,\bar{z})=\sum_{h,\bar{h}} z^{(h-h_1-h_2)}\bar{z}^{\bar{h}-h_1-h_2}\times\mathcal{C}_{\text{internal}}(h,\bar{h}), \tag{4.26}$$

where the coefficient is given by

$$\mathcal{C}_{\text{internal}}(h,\bar{h})=C_{\text{internal}}(h_1,h_2,h)\frac{1}{B_{\text{internal}}(h\bar{h})}\times C_{\text{internal}}(h,h_3,h_4), \tag{4.27}$$

and $B$ and $C$ are given by the two- and three-point functions of the internal CFT.

Now we are ready to integrate $\mathcal{F}_{\text{worldsheet}}=\mathcal{F}_{SL(2)}\times\mathcal{F}_{\text{internal}}$ over $z$ in region I, namely over the region $|u|\leqslant|x|^{-1}$. One problem is that this integral might diverge at $u=0$. This would not be a problem if we were actually integrating $\mathcal{F}_{\text{worldsheet}}$ since we can remove the divergence by analytic continuation, which is the standard procedure in string theory computation. The problem arises if we try to do the $z$ integral before the $j$ integral in Eq. (4.10) since these two integrals may not commute if there are divergences. In fact, it is necessary to keep track of these possible divergences and to be careful about the exchange of the $z$ and $j$ integrals in order to recover the correct pole structure. The two integrals commute if we regularize the $z$ integral by introducing a cutoff $\epsilon$ and integrate over $\epsilon\leqslant|u|\leqslant|x|^{-1}$. We will keep track of the $\epsilon$ dependence and send $\epsilon\to 0$ after we perform the $j$ integral. In practice, what we do is first integrate over the whole $u$ plane and define the integral by analytic continuation. We then subtract the contributions from $|u|<\epsilon$ and $|x|^{-1}<|u|$. If we use the same analytic continuation technique to evaluate the integrals over these three regions, the result after the subtraction of the two contributions gives the regularized integral over $\epsilon<|u|<|x|^{-1}$.

### *1. Integral over the whole u plane*

Let us start with the integral over the whole $u$ plane:

$$R_1 \equiv \int dz^2 \mathcal{F}_{\mathrm{SL}(2)} \mathcal{F}_{\mathrm{internal}}$$

$$= \sum_{h\bar{h}} \int dj\, \mathcal{C}(j) \mathcal{C}_{\mathrm{internal}}(h,\bar{h}) \times \sum_{m,\bar{m}=0}^{\infty} x^m \bar{x}^{\bar{m}} I^{h,\bar{h}}_{j;m,\bar{m}}(x). \tag{4.28}$$

The first term in the $x$ expansion is given by

$$I^{h,\bar{h}}_{j,0,0}(x) = x^{\Delta(j)+h-1+j-j_1-j_2} \bar{x}^{\Delta(j)+\bar{h}-1+j-j_1-j_2} \times \frac{1}{2}\int d^2u\, u^{d-1}\bar{u}^{\bar{d}-1}(|F_j|^2 + \lambda |u^{1-c} F_{k-1-j}|^2), \tag{4.29}$$

where

$$d = \Delta(j) + h - 1, \quad \bar{d} = \Delta(j) + \bar{h} - 1. \tag{4.30}$$

This integral can be done using the formula (C1) in Appendix C. We find

$$R_1 = \mathcal{C}_{\mathrm{internal}}(h,\bar{h}) \int_{[(k-1/2)]+iR} dj\, \mathcal{C}(j) x^{d+j-j_1-j_2} \bar{x}^{\bar{d}+j-j_1-j_2} \times \frac{\pi}{2} \frac{\Gamma(d)\Gamma(a-\bar{d})\Gamma(b-\bar{d})\Gamma(1-c+d)}{\Gamma(1-\bar{d})\Gamma(1-a+d)\Gamma(1-b+d)\Gamma(c-\bar{d})} \times \frac{\gamma(c)}{\gamma(a)\gamma(b)} + \cdots, \tag{4.31}$$

where the dots indicate terms with higher integer powers of $x,\bar{x}$. By looking at the powers of $x,\bar{x}$, we can read off the conformal weight of the intermediate states as

$$J = d + j = \Delta(j) + j + h - 1 = \frac{k}{4} + \frac{s^2 + \frac{1}{4}}{k-2} + h - 1, \tag{4.32}$$

where $j = [(k-1)/2] + is$ and a similar expression for $\bar{J}$ obtained by replacing $h \to \bar{h}$ in Eq. (4.32). We conclude that Eq. (4.31) represents the contribution of long strings with winding number $w=1$ in the intermediate channel. In Sec. V, we will show that the coefficient in Eq. (4.31) is precisely what we expect from Eqs. (4.2) and (4.3).

The subleading terms $I^{h,\bar{h}}_{j;m,\bar{m}}$ with $(m,\bar{m}) \neq (0,0)$ in the $x$ expansion (4.28), represented by the the dots in Eq. (4.31), are identified as coming from the global $\mathrm{SL}(2,R) \times \mathrm{SL}(2,R)$ descendents of the long strings considered above. Indeed their $J_0^3$ and $\bar{J}_0^3$ eigenvalues are

$$J_0^3 = J + m, \quad \bar{J}_0^3 = \bar{J} + \bar{m} \tag{4.33}$$

with $J$ as in Eq. (4.32). In principle, there could be new contributions from conformal primary fields with these quantum numbers, but they seem hard to disentangle from the descendent contributions.

### *2. Integral over* $|u|<\epsilon$

From the integral (4.28) that we just computed, we need to subtract contributions from $|u|<\epsilon$ and from $|x|^{-1}<|u|$. Here we will evaluate the integral over $|u|<\epsilon$. As in the case of $R_1$ [Eq. (4.28)], let us focus on the leading term in the $x$ expansion in Eq. (4.17). The integral we need to evaluate is

$$-\sum_{h\bar{h}} \int_{[(k-1)/2]+iR} dj\, \mathcal{C}(j) C_{\mathrm{int}}(h,\bar{h}) x^{d+j-j_1-j_2} \bar{x}^{\bar{d}+j-j_1-j_2} \times \int_{|u|<\epsilon} d^2u\, u^{d-1}\bar{u}^{\bar{d}-1} |F(a,b,c;u)|^2. \tag{4.34}$$

Here we used the reflection symmetry $j \to k-1-j$ of the contour at $[(k-1)/2]+is$ ($s$ real) to combine the two terms in Eq. (4.20) into one. We can carry out the $u$ integral by expanding $F(a,b,c;u)$ in powers of $u$,

$$\int_{|u|<\epsilon} u^{d-1}\bar{u}^{\bar{d}-1} |F(a,b,c;u)|^2 = \sum_{n,\bar{n}=0}^{\infty} \frac{\pi}{d+n} \epsilon^{2(d+n)} \delta_{n+h,\bar{n}+\bar{h}} \times \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(a+\bar{n})\Gamma(b+\bar{n})\Gamma(c)^2}{\Gamma(a)^2\Gamma(b)^2\Gamma(c+n)\Gamma(c+\bar{n})}. \tag{4.35}$$

Note that the condition $h+n = \bar{h}+\bar{n}$ is imposed by the angular integral over $u$. In order to take the limit $\epsilon \to 0$, we move the contour to $j = \frac{1}{2} + is$ with $s$ real. There the exponent $d+n$ of $\epsilon$ is positive (if we ignore the tachyon) since

$$d + n = \frac{s^2 + \frac{1}{4}}{k-2} + h + n - 1. \tag{4.36}$$

Thus the contribution from the contour integral along $j = \frac{1}{2} + is$ vanishes in the limit $\epsilon \to 0$. This does not mean that the original integral (4.34) vanishes in the limit $\epsilon \to 0$. As we are going see, the integral picks up pole residues as we move the contour from $j = [(k-1)/2] + is$ to $j = \frac{1}{2} + is$.

There are four types of poles that contribute when we deform the contour of the $j$ integral in Eq. (4.34) from $j$

$=[(k-1)/2]+is$ to $\frac{1}{2}+is$. The first type of poles comes from the zeros of $d+n$ in Eq. (4.35). At the pole, we have

$$d+n=-\frac{j(j-1)}{k-2}+h+n-1=0. \tag{4.37}$$

The $x$ dependence of the pole contribution is $x^{j-n-j_1-j_2}$ so that the spacetime conformal weight of the corresponding operator is $J=j-n$. We can identify this state as coming from a particular current algebra descendent of a $w=0$ short string representation of the form

$$(J^-_{-1})^n(\overline{J}^-_{-1})^{\overline{n}}|j,j\rangle \tag{4.38}$$

in the SL(2,$R$) WZW model times an operator of dimension $h,\overline{h}$ in the the internal CFT. In fact Eq. (4.37) is the $L_0=1$ condition for such an intermediate state. The $L_0=\overline{L}_0$ condition follows from the condition $h+n=\overline{h}+\overline{n}$ in Eq. (4.35). In Sec. V, we will check that the coefficient in Eqs. (4.34) and (4.35) evaluated at the pole (4.37) exactly agrees with what we expect from the operator product expansion (4.2) and (4.3). The states in (4.38) are global SL(2,$R$)×SL(2,$R$) primaries, although those with $n\geqslant 1$ are descendents of the current algebra. Higher-order terms in the $x$ expansion (4.28) produce terms which have the quantum number of descendents of the states in (4.38) under the global SL(2,$R$)×SL(2,$R$). Note that due to the fact that we only shifted the contour within the range (4.15), the values of $j$ of these discrete state contributions to the OPE are naturally bounded by (4.15). This reproduces the constraint on the spectrum of the short string found in [1,2].

The second type of poles is at $j=j_1+j_2+n$ ($n=0,1,2,\ldots$). These cancel the $\epsilon$ dependence of the contribution from $\text{Poles}_1$ that emerged when we originally moved the contour from $j=\frac{1}{2}+is$ to $[(k-1)/2]+is$. Thus the net result is that we can compute the $z$ integral for the contribution from $\text{Poles}_1$ by the standard analytic continuation method. The resulting contribution can be interpreted as a contribution to the OPE from two-particle operators.

Similarly, the third type of poles is at $j=k-j_1-j_2+n$ ($n=0,1,2,\ldots$). These cancel the $\epsilon$ dependence of the contributions from $\text{Poles}_2$. These poles do not appear if (4.4) is obeyed.

Finally, the fourth type of poles is at $j=|j_1-j_2|-n$. In the original contour of Eq. (4.10), we avoided these poles since they crossed the contour when we performed the analytic continuation in $j_1,\ldots,j_4$. We now pick up contributions from these poles since we have to move the contour all the way to the line at $j=\frac{1}{2}+is$. The contributions from these poles have explicit $\epsilon$ dependence. We believe that these should be explicitly subtracted.

All that we said regarding $j_1$, $j_2$ should be repeated for $j_3$, $j_4$.

To summarize, the integral over $|u|<\epsilon$ reproduces the exchange of short string states with $w=0$ and mixing with two-particle states. These are the only contributions to the integral as long as (4.4) is satisfied.

#### 3. Integral over $|u|>|x|^{-1}$

Finally, let us evaluate the integral over $|u|>|x|^{-1}$ and subtract it from $R_1$. It is convenient to use the expansion of Eq. (4.20) for large $u$. It is given by

$$G_{j,0}=|x^{\Delta(j)-\Delta(j_1)-\Delta(j_2)+j-j_1-j_2}u^{\Delta(j)-\Delta(j_1)-\Delta(j_2)}|^2\Bigg\{\Bigg[\mathcal{C}(j)\frac{\Gamma(j_1+j_2-j_3-j_4)^2\Gamma(k-2j)^2}{\Gamma(j_3+j_4-j)^2\Gamma(k-j-j_1-j_2)^2}+(j\to k-1-j)\Bigg]$$
$$\times\left|\left(\frac{z}{x}\right)^{j-j_1-j_2}F\left(j_1+j_2-j,j_1+j_2-k+j+1,j_1+j_2-j_3-j_4+1;\frac{x}{z}\right)\right|^2+[(j_1,j_2)\leftrightarrow(j_3,j_4)]\Bigg\}. \tag{4.39}$$

Note that this is the large-$u$ expansion of the leading term (4.20) in the $x$ expansion in region I (4.17). The large-$u$ expansion of the full KZ solution is different and will be discussed later when we study the integral in the region II. In Eq. (4.28), we integrated this leading term over the whole plane. Thus we need to subtract the integral over $|u|>|x|^{-1}$ using the same integrand to obtain an approximate expression for the integral of the full solution of the KZ equation over $|u|<|x|^{-1}$. Using Eq. (4.39), we find that the integral gives terms of the form

$$x^{\Delta(j)+h-1+j-j_1-j_2}\overline{x}^{\Delta(j)+\overline{h}-1+j-j_1-j_2}$$
$$\times\int_{|u|>|x|^{-y}}du^2u^{d-1}\overline{u}^{\overline{d}-1}(a_{n\overline{n}}u^n\overline{u}^{\overline{n}}|u|^{2(j-j_1-j_2)}$$
$$+b_{n\overline{n}}u^n\overline{u}^{\overline{n}}|u|^{2(j-j_3-j_4)})$$
$$\sim\tilde{a}_{n\overline{n}}x^n\overline{x}^{\overline{n}}+\tilde{b}_{n\overline{n}}x^{j_3+j_4-j_1-j_2+n}\overline{x}^{j_3+j_4-j_1-j_2+\overline{n}}, \tag{4.40}$$

for some $a_{n,\bar{n}}$, $b_{n,\bar{n}}$, $\tilde{a}_{n,\bar{n}}$, and $\tilde{b}_{n,\bar{n}}$. From the exponents of $x$, we see that these terms all have the form of two-particle contributions. It seems possible that we could shift the contour of integration in $j$ to a region where it becomes convergent. This shift might produce extra contributions, but they all have these powers of $x$ and therefore will be of the form of two-particle exchanges.

This completes the evaluation of the $z$ integral in region I.

### C. Integral over region II

It remains now to do the integral over the region II. In this region, we can expand any solution of the KZ equation as

$$F(z,x) \sim x^{\alpha} \sum_{m=0}^{\infty} \tilde{g}_m(z) x^m. \tag{4.41}$$

Substituting this into the KZ equation, we find that $\alpha=0$ or $\alpha=j_3+j_4-j_1-j_2$. This means that the full contribution from this region is interpreted as two-particle contributions. In this region, we also have to expand the internal part in a different way. But in any case, the $x$ dependence is just that of the two-particle contributions.

Thus we have completed the computation of the integral over the $z$ plane with the results summarized at the beginning of the section. The intermediate states in the small-$x$ expansion are identified and are found to be consistent with the operator product expansion in BCFT interpreted in the standard way as in Eq. (4.2), provided (4.4) is satisfied. Note that as long as (4.4) is satisfied, the three-point functions that appear in the factorization on intermediate discrete states automatically obey the constraint $\Sigma j_i < k$. This is consistent with our previous statement that only those three-point functions make sense in the theory.

### D. When the OPE does not factorize

Let us now discuss what happens when (4.4) is not satisfied. In this case, besides the terms we discussed above, we get contributions from the residues of $\text{Poles}_2$ in Eq. (4.16). If we were to read off naively the dimension $J$ of an intermediate operator from the power of $x$ appearing in these contributions, we would find $J=k-j_1-j_2+n$ (or a similar expression with $j_3, j_4$). For generic values of $k$, $j_1, \ldots, j_4$, there is no physical operator with this value of $J$. Therefore, these contributions do not have an interpretation as exchange of intermediate physical states as in Eq. (4.2). Their presence signals a breakdown in the operator product expansion.

One may naively interpret this as saying that we need to include more physical states in the theory. We claim this is not the correct interpretation. Instead we propose that, in this case, the operator product expansion is not well defined in the target space theory. This is due to the noncompactness of the target space of BCFT. To clarify this issue, it is useful to go back to the simple quantum mechanics example we gave in Sec. III B, i.e., that of a quantum particle moving in a one-dimensional space with coordinate $x$ under a potential that is zero for $|x| \gg 1$ such that the wave function of the ground state decays as $\langle x|0\rangle = \psi(x) \sim e^{-(\kappa/2)x}$ for large $x$. In these circumstances, we consider the operators $\mathcal{O}_i(t) = e^{\lambda_i x(t)}$ and try to evaluate their correlator $\langle 0|\mathcal{O}_4(t_4)\mathcal{O}_3(t_3)\mathcal{O}_2(t_2)\mathcal{O}_1(t_1)|0\rangle$. This correlation function is well defined if $\Sigma \lambda_i < \kappa$. Now we can try to perform the OPE when $t_1 \to t_2$ and $t_3 \to t_4$. Naively one may expect to find normalizable (and also continuum-normalizable) states running in the intermediate channel. It is easy to see that this will be the case only if $\lambda_1+\lambda_2<\kappa/2$ and $\lambda_3+\lambda_4<\kappa/2$. These conditions are analogous to (4.4). If these conditions are not obeyed, the intermediate state is not in the Hilbert space of the theory. In other words, the product $\mathcal{O}_1\mathcal{O}_2$ maps the state $|0\rangle$ outside the Hilbert space. This is effect is not a UV divergence; rather it is an IR divergence in the target space of the quantum-mechanical system.

These contributions from $\text{Poles}_2$ that we are discussing are important for reproducing the general properties of the amplitude that we explained in Sec. III. The four-point function should have a pole at $\Sigma j_i - k = 1$. This pole is absent from all the terms in the amplitude that can be written as Eq. (4.3). But it is present in the term coming from $\text{Poles}_2$, as can be checked explicitly by performing the integral over $z$ for the $\text{Poles}_2$ contribution. Note that (4.4) cannot be obeyed if we are at the pole at $\Sigma j_i - k = 1$, so we definitely have $\text{Poles}_2$ contributions in this region.

Note that we have assumed that all the $j$'s involved in the computation of the OPE are generic enough so that there are no coincident poles. Coincident poles can produce terms involving $\log x$. These were studied in [37,38], and they have the same interpretation here as they had in their case.

## V. TWO- AND THREE-POINT FUNCTIONS WITH SPECTRAL FLOWED STATES

In the preceding section, we showed that the four-point function of short strings with $w=0$ is factorized into a sum of products of three-point functions. We found that the intermediate states are long strings with $w=1$, short strings with $w=0$, and two-particle states. These intermediate states are identified by evaluating the $x$ expansion of the amplitude and by comparing exponents of $x$ with the spectrum of physical states of the short and long strings. One of the purposes of this and the next sections is to prove that the coefficients in the $x$ expansion are what we expect from the factorization of BCFT. To this end, we need to compute two- and three-point functions involving spectral flowed states. We will also explain the origin of the constraint on the winding number violation. In Appendix D, we will use the representation theory of the $SL(2,R)$ current algebra to show that two short strings with $w=0$ can only be mixed with short strings with $w=0,1$ or long strings with $w=1$. This almost accounts for the winding number violation rule we saw in the factorization of the four-point function, but leaves out the question of why short strings with $w=1$ do not appear in the intermediate channel. In this section, we will show that, if we normalize the vertex operators so that their target space two-point functions are finite, the three-point function of two short strings with $w=0$ and one short string with $w=1$ vanishes identically, thereby explaining the additional constraint on the winding number violation. We will also discuss other aspects of these correlation functions.

In [1], it is shown how to construct vertex operators for the spectral flowed representations. This can be done most easily in the $m$ basis, where the generators $(J_0^3,\bar{J}_0^3)$ of the global SL(2,$C$) isometry are diagonalized. On the other hand, in Eqs. (2.9) and (2.13), we used the $x$ basis to express the two- and three-point functions. Therefore, to compute correlation functions involving spectral flowed states, we first have to convert Eqs. (2.9) and (2.13) into the $m$ basis, perform the spectral flow operation as described in [1], and then transform the result back in the $x$ basis.

One thing we need to be careful about in this procedure is that the spectral flow changes the way the global SL(2,$C$) isometry acts on states since the currents are transformed as

$$J_0^{\pm}=\tilde{J}^{\pm}_{\mp w},\quad J_0^3=\tilde{J}_0^3+\frac{k}{2}w. \tag{5.1}$$

For example, consider a representation of the current algebra whose worldsheet energy $\tilde{L}_0$ is bounded from below. [$\mathcal{D}_j^{w=0}$ and $\mathcal{C}_{j,\alpha}^{w=0}$ are an example of such representations, but here we do not assume that the lowest-energy states of the representation make a unitary representation of the global SL(2,$R$).] We then pick one of the lowest-energy ($\tilde{L}_0$) states $|\psi\rangle$, satisfying[22]

$$\tilde{J}_n^{\pm,3}|\psi\rangle=0,\quad n=1,2,3,\dots,$$

$$\tilde{J}_0^3|\psi\rangle=m|\psi\rangle[-\tilde{J}_0^3\tilde{J}_0^3+\tfrac{1}{2}(\tilde{J}_0^+\tilde{J}_0^-+\tilde{J}_0^-\tilde{J}_0^+)]|\psi\rangle$$
$$=-j(j-1)|\psi\rangle. \tag{5.2}$$

If $m=\pm(j+n)$ for a nonzero integer $n$, the state $|\psi\rangle$ belongs to the discrete representation $d_j^{\pm}$ with respect to the SL(2,$R$) algebra generated by $\tilde{J}_0^a$. Otherwise it is in the continuous representation $c_{j,\alpha}$, where $m=\alpha$ (mod integer).[23] If $w$ is positive, the same state $|\psi\rangle$ is seen in the spectral flowed frame (5.1) as obeying

$$J_0^-|\psi\rangle=0,\quad J_0^3|\psi\rangle=\left(m+\frac{k}{2}w\right)|\psi\rangle. \tag{5.3}$$

With respect to the global SL(2,$R$) algebra generated by $J_0^a$, the state $|\psi\rangle$ is the lowest weight state of a discrete representation $d_J^+$ with $J=m+(k/2)w$, independently of whether $|\psi\rangle$ was in $d_j^{\pm}$ or $c_{j,\alpha}$ of the SL(2,$R$) algebra generated by $\tilde{J}_0^a$. Similarly, spectral flow with $w<0$ turns $|\psi\rangle$ into the highest weight state of $d_J^-$ with $J=-m+(k/2)|w|$. In our physical application, we identify the SL(2,$R$) algebra generated by $J_0^a$ with the spacetime isometries of the background and the global SL(2) symmetries of the BCFT. In what follows, we will indicate by $J$ and $M$ the global SL(2) spin and $J_0^3$ eigenvalue, respectively (see Fig. 7).

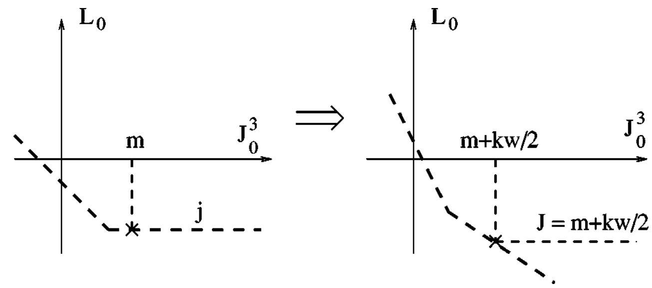


FIG. 7. Under the spectral flow, a global SL(2,$C$) descendant $|\psi\rangle$ of spin $J_0^3=m$ among the lowest energy states in $\mathcal{D}_j^{w=0}$ or $\mathcal{C}_{j,\alpha}^{w=0}$ turns into the lowest weight state of the discrete representation $d_J^+$ with $J=m+(k/2)w$. The figure shows the flow of $\mathcal{D}_j^{w=0}$. The resulting operator is denoted by $\Phi^{w,j}_{J,\bar{J}}(x,z)$.

The transformation between the $x$ basis and the $m$ basis is carried out as follows. Consider an operator $\Phi_{J,\bar{J}}(x,\bar{x})$ in the $x$ basis, with the spacetime conformal weights $(J,\bar{J})$. In general, the difference $(J-\bar{J})$ has to be an integer in order for their correlation functions to be single-valued in the $x$ space, and we will consider such cases only. The integral transform

$$\Phi_{J,M;\bar{J},\bar{M}}=\int\frac{d^2x}{|x|^2}x^{J-M}\bar{x}^{\bar{J}-\bar{M}}\Phi_{J,\bar{J}}(x,\bar{x}) \tag{5.4}$$

turns the operator into the $M$ basis where $M$ and $\bar{M}$ are eigenvalues of $J_0^3$ and $\bar{J}_0^3$, respectively.[24] Note that $(\bar{M},\bar{M})$ are not necessarily a complex conjugate of $(J,M)$. Since $(J-\bar{J})$ is an integer, the integral vanishes unless $(M-\bar{M})$ is also an integer and we will assume this in the following.

In practice, the $x$ integral in Eq. (5.4) is carried out after computing correlation functions and using analytic continuation in the variables, $J,M\dots$. When $J$ is real, we have to keep in mind that the $x$ integral gives poles at $M=J+n$ and $\bar{M}=\bar{J}+\bar{n}$, with non-negative integers $n,\bar{n}$. We will see this explicitly in the two- and three-point function computations in the following.[25] These are precisely the values at which the operator $\Phi_{J,M;\bar{J},\bar{M}}$ belongs to a discrete representation $d_J^+\otimes d_J^+$ of the global SL(2,$R$)×SL(2,$R$) symmetry. In such cases, we have to keep track of this additional divergent factor. There are also similar poles when $M=-J-n,\bar{M}=-\bar{J}-\bar{n}$ with non-negative integers $n,\bar{n}$ and they form $d_J^-\otimes d_J^-$. We will call the poles with positive $M$ as "incoming states" and the poles with negative $M$ as "outgoing states." In this way, we see that the single operator $\Phi_{J,\bar{J}}$ in the $x$ basis

[22] Here $j$ is what we called $\tilde{j}$ in [1].

[23] We are using the symbols $d_j^{\pm}$ and $c_{j,\alpha}$ to label representations of an SL(2,$R$) algebra, to distinguish them from the representations of the full current algebra.

[24] We reserve the small case letters $m,\bar{m}$ to denote eigenvalues of $J_0^3,\bar{J}_0^3$ in the $w=0$ sector, i.e., for states before we perform the spectral flow.

[25] Although there are two conditions on $M$ and $\bar{M}$, the pole is only in one variable of the form $(M+\bar{M}-J-\bar{J}-n-\bar{n})^{-1}$. The second condition is imposed by the angular integral in the $x$ space.

gives rise to both $d_J^+$ and $d_J^-$, depending on the value of $M$ we choose in evaluating the integral transform (5.4).

Correlation functions of spectral flowed operators are then evaluated as follows. We start with $n$-point correlation functions in the $w=0$ sector, which are known for $n=2$, 3, and 4. We perform the integral transform (5.4) to turn them into expressions in the $m$ basis. We then use the spectral flow operator to find expressions for $w\neq 0$ (as described in detail in the following subsections). Finally, we use Eq. (5.4) to transform the expressions back into the $x$ basis.

Alternatively, one can perform the spectral flow operation directly in the $x$ basis. In the case of $w=1$, the spectral flowed operator $\hat{\Phi}^{w=1,j}_{J,\bar{J}}(x,z)$ is constructed from $\Phi_j(x,z)$ in the $w=0$ sector as

$$\hat{\Phi}^{w=1,j}_{J,\bar{J}}(x,z)\equiv \lim_{\epsilon\to 0}\epsilon^{m}\bar{\epsilon}^{\bar{m}}\int d^2y\, y^{j-m-1}\bar{y}^{j-\bar{m}-1}\times\Phi_j(x+y,z+\epsilon)\Phi_{k/2}(x,z). \tag{5.5}$$

Here we put a caret on the spectral flowed operator since its normalization is different from the one naturally defined by going through the $m$ basis as described in the above paragraph. In Appendix E, we will prove that Eq. (5.5) in fact defines the spectral flowed operator by showing that it has the correct operator product expansions with the currents $J^{3,\pm}$. We will then use Eq. (5.5) to compute their two- and three-point functions.

In this section, we will use the spectral flowed operator defined through the $m$ basis. This approach has an advantage of being able to treat all values of $w$ simultaneously.

### A. Two-point functions

Let us start with a two-point function in $x$ space for generic values of $J$, $\bar{J}$. The two-point functions in the following typically take the form

$$\langle \Phi_{J,\bar{J}}(x_1)\Phi_{J,\bar{J}}(x_2)\rangle = \frac{D(J,\bar{J})}{x_{12}^{2J}\bar{x}_{12}^{2\bar{J}}}, \tag{5.6}$$

where we have suppressed a possible $z$ dependence. Performing the integral using the formula (C5) in Appendix C, we find

$$\langle \Phi_{J,M;\bar{J},\bar{M}}\Phi_{J,M;\bar{J},\bar{M}'}\rangle = \delta^2(M+M')\frac{\pi\Gamma(1-2\bar{J})\Gamma(J+M)\Gamma(\bar{J}-\bar{M})}{\Gamma(2J)\Gamma(1-J+M)\Gamma(1-\bar{J}-\bar{M})}D(J,\bar{J}). \tag{5.7}$$

The delta function $\delta^2(M)$ is the standard delta function for the sum $(M+\bar{M})$ and the Kronecker delta for the difference $(M-\bar{M})$, which is an integer. Using the formula $\Gamma(x)\Gamma(1-x)=\pi/\sin \pi x$ and the fact that $(J-\bar{J})$ and $(M-\bar{M})$ are both integers, one can check that the expression (5.7) is symmetric under exchange of $(J,M)$ and $(\bar{J},\bar{M})$.

Conversely, if we are given the expression (5.7), we can turn it back into the form (5.6) in the $x$ basis. To do this, it is not necessary to know the expression for all possible values of $M$, $\bar{M}$. For example, the expression (5.7) has a pole at $M=J$ and $\bar{M}=\bar{J}$, and the residue is equal to $D(J,\bar{J})$ times a simple factor. Thus it is sufficient to know the pole residue there in order to recover the $x$ space expression (5.6). Similarly, we can reconstruct Eq. (5.6) from the residue of the pole at $M=-J$ and $\bar{M}=-\bar{J}$. In the following, we will encounter such situations.

We now consider the two-point function of $w=0$ states given by Eq. (2.9) and convert it into a two-point function with $w\neq 0$ states. As we mentioned, we first turn the expression into the $m$ basis, perform the spectral flow, and then transform this back into the $x$ basis. In transforming the second term (2.9) into the $m$ basis, we can use Eq. (5.7) with $D(j,j)=\delta(j-j')B(j)$; the $x$ integral of the first term is easy to do directly. In the $m$ basis, it is straightforward to apply the spectral flow. As explained in [1], the only change in the two-point function is that the power of $z$ is modified in an $m$-dependent fashion reflecting the change in the worldsheet conformal weight,

$$\Delta(j)\to\Delta(j)-wm-\frac{k}{4}w^2, \tag{5.8}$$

without any modification to the coefficient. We should also remember that the assignment of the global SL(2,$C$) charges is changed according to the discussion after Eq. (5.1). To perform the spectral flow explicitly, we bosonize the $J^3$ current as $J^3=i\sqrt{(k/2)}\partial\varphi$ and write an operator with $J^3$ charge $m$ as

$$\Phi_{j,m}\sim e^{im\sqrt{2/k}\varphi}\psi_{j,m}. \tag{5.9}$$

The operator $\psi_{j,m}$ carries no $J^3$ charge and is analogous to the parafermion field in the SU(2) WZW model. We then make the replacement

$$e^{im\sqrt{(2/k)}\varphi}\to e^{i[m+w(k/2)]\sqrt{(2/k)}\varphi}, \tag{5.10}$$

and similarly for $\bar{m}$. As explained in [1], the operator we find in this way has $J=M=m+(k/2)w$, $\bar{J}=\bar{M}=\bar{m}+(k/2)w$, namely, it is the lowest weight state in the representation $d_J^+\otimes d_{\bar{J}}^+$ of the global SL(2,$C$) isometry. Including the modified $z$ dependence that comes from applying the spectral flow operator, we obtain the two-point function [1]

$$\langle \Phi^{w,j}_{J,M;\bar{J},\bar{M}}(z_1)\Phi^{-w,j'}_{J,M';\bar{J},\bar{M}'}(z_2)\rangle = \frac{1}{z_{12}^{2[\Delta(j)-wM+(k/4)w^2]}\bar{z}_{12}^{2[\Delta(j)-w\bar{M}+(k/4)w^2]}}\delta^2(M+M')\left[\delta(j+j'-1)+\delta(j-j')\right.$$
$$\left.\times\frac{\pi B(j)}{\gamma(2j)}\frac{\Gamma(j+m)}{\Gamma(1-j+m)}\frac{\Gamma(j-\bar{m})}{\Gamma(1-j-\bar{m})}\right], \quad (5.11)$$

where $J=M=m+(k/2)w$ and $\bar{J}=\bar{M}=\bar{m}+(k/2)w$. Note that $j$ is the spacetime conformal weight of the original $w=0$ operator and it should be distinguished from $J,\bar{J}$ for the operator we get after spectral flow. The amount of spectral flow of the second operator is $-w$; this is necessary in order to preserve the total $J^3_0$ charge. If $w,m>0$, we can interpret the first operator as an incoming state and the second as an outgoing state.

We would like to convert Eq. (5.11) back to the $x$ basis. According to our previous discussion, this can be done by evaluating the pole residue at $J=M$ and $\bar{J}=\bar{M}$. Unlike a generic two-point function such as Eqs. (5.6) and (5.7), the expression (5.11) is finite at this location.[26] The pole that we are missing here comes from the divergent integral of the form $\int d^2z/|z|^2$. We recognize that it has the same form as the volume $V_{\rm conf}$ of the conformal group of $S^2$ with the two-point fixed function,

$$\int \frac{d^2z}{|z|^2} = \frac{\int \frac{d^2z\, d^2w\, d^2u}{|z-w|^2|w-u|^2|u-x|^2}}{\int \frac{d^2z\, d^2w}{|z-w|^4}} = V_{\rm conf}. \quad (5.12)$$

Since evaluating the pole residue of Eq. (5.11) at $J=M$, $\bar{J}=\bar{M}$ is the same as evaluating it at the pole and dividing it by $V_{\rm conf}$ (with an appropriate regularization of the $z$ integral), we can interpret Eq. (5.11) as resulting from a two-point function in the $x$ basis of the form

$$\langle \Phi^{w,j}_{J,\bar{J}}(x_1,z_1)\Phi^{w,j'}_{J\bar{J}}(x_2,z_2)\rangle = \frac{1}{V_{\rm conf}}\left[\delta(j+j'-1)+\delta(j-j')\frac{\pi B(j)}{\gamma(2j)}\frac{\Gamma(j+m)}{\Gamma(1-j+m)}\frac{\Gamma(j-\bar{m})}{\Gamma(1-j-\bar{m})}\right]$$
$$\times\frac{1}{x_{12}^{2J}\bar{x}_{12}^{2\bar{J}}z_{12}^{2[\Delta(j)-wM+(k/4)w^2]}\bar{z}_{12}^{2[\Delta(j)-w\bar{M}+(k/4)w^2]}}. \quad (5.13)$$

The factor $V^{-1}_{\rm conf}$ will eventually be canceled in the string theory computation that follows. In going from Eq. (5.11) to Eq. (5.13), we have switched the sign of $w$ in the second operator. This is due to the fact that an outgoing state with negative $w$ is the same as an incoming state with positive $w$. In other words, in the $x$ basis we can label the operators with $w\geq 0$.

Some readers may be disturbed by the appearance of the infinite factor $V_{\rm conf}$ in our computation. We can avoid the use of $V_{\rm conf}$ altogether if we work directly in the $x$ basis using Eq. (5.5). This will be explained in Appendix E. For $w=1$, both approaches give the same result. For $w>1$, computations in the $x$ basis become cumbersome. For this reason, we will continue to work in the $m$ basis in this section so that we can find expressions for all $w$ at once.

So far, we have taken $j$ to be arbitrary. Let us now set $j=\frac{1}{2}+is$, so that we have a continuous representation at $w=0$. In this case, the spectral flowed expression (5.13) gives the two-point function of the vertex operator for the long string with $w=1$. In order to compute the spacetime two-point function, we need to take into account the contribution from the internal CFT. We choose the internal conformal weight $(h,\bar{h})$ such that the long string obeys the physical state condition

$$\Delta(j)-wM+\frac{k}{4}w^2+h=1,$$
$$\Delta(j)-w\bar{M}+\frac{k}{4}w^2+\bar{h}=1. \quad (5.14)$$

Assuming that the operator in the internal CFT is unit normalized, its effect is to multiply the factor $z^{-2h}\bar{z}^{-2\bar{h}}$ to Eq. (5.13). We then need to integrate over $z$ and divide it by the volume of the conformal group on the sphere. This produces another factor of $V^{-1}_{\rm conf}$. By changing the normalization of the operator as $\hat{\Phi}=V_{\rm conf}\,\Phi$, the two-point function in the target space is given by

[26]There is an important exception when the $w=0$ operator is in a discrete representation, in which case $\bar{m}=j+\bar{n}$ and there is a pole. We will come back to this point later.

$$\langle \hat{\Phi}^{w,j}_{J,\bar{J}}(x_1)\hat{\Phi}^{w,j'}_{J,\tilde{J}}(x_2)\rangle_{\text{target}} = \frac{1}{V_{\text{conf}}} V^2_{\text{conf}} \langle \Phi^{w,j}_{J,\bar{J}}(x_1,z_1=0)\Phi^{w,j}_{J,\bar{J}}(x_2,z_2=1)\rangle_{\text{worldsheet}}$$
$$\sim \left[ \delta(s+s') + \delta(s-s') \frac{\pi B(j)}{\gamma(2j)} \frac{\Gamma\left(j-\frac{k}{2}w+J\right)}{\Gamma\left(1-j-\frac{k}{2}w+J\right)} \frac{\Gamma\left(j+\frac{k}{2}w-\bar{J}\right)}{\Gamma\left(1-j+\frac{k}{2}w-\bar{J}\right)} \right] \frac{1}{x_{12}^{2J}\bar{x}_{12}^{2\bar{J}}}, \quad (5.15)$$

where

$$j=\tfrac{1}{2}+is,$$
$$J=\frac{k}{4}w+\frac{1}{w}\left(\frac{\frac{1}{4}+s^2}{k-2}+h-1\right), \quad (5.16)$$

and a similar expression for $\bar{J}$ in terms of $\bar{h}$. As far as the two-point function is concerned, we can of course normalize the operator $\Phi$ as we like. All we are saying here is that this normalization removes the divergent factor $V_{\text{cont}}$ and keeps the target space two-point function finite. In the next subsection, we will see that the rescaling $\hat{\Phi}=V_{\text{conf}}\Phi$ also gives finite results for the three-point functions that appear in the factorization of the four-point function.

We would like to make a couple of comments about the two-point function of the long strings (5.15). Unlike the case of the short string, the on-shell condition does not require $j=j'$. However, the two-point function has the delta functions $\delta(s+s')$ and $\delta(s-s')$, giving constrains on the labels $s,s'$. For the operator before the spectral flow, the term proportional to $\delta(s+s')$ in Eq. (2.9) is multiplied by $\sim\delta^2(x_{12})$, i.e., it is a contact term in BCFT. After the spectral flow (5.15), the corresponding term contributes to the long-range correlation of the two operators $\sim x_{12}^{-2J}\bar{x}_{12}^{-2\bar{J}}$ in the same way as the term proportional to $\delta(s-s')$. Thus, when we discuss the factorization of the four-point function, we need to take into account both the first and the second terms in Eq. (5.15). Another remark we would like to make is that the factor multiplying the $\delta(s-s')$ in the second term in Eq. (5.15) is a pure phase $e^{i\delta(s)}$, see Eq. (3.6). We can interpret it as the phase shift for a scattering experiment where we let a long string come from infinity of $AdS_3$, shrink to the origin, and go back to infinity again [1]. In fact, the operators labeled by $s$ and $-s$ are not independent, and they are related by the reflection coefficient $\Phi^{(1/2)+is}\sim e^{i\delta(s)}\Phi^{(1/2)-is}$ as shown in [19].

Now let us turn to discrete representations. We start with a global SL$(2,C)$ descendent with $m=j+q$ and $\bar{m}=j+\bar{q}$, where $q,\bar{q}$ are non-negative integers. After the flow, we obtain a state with $J=M=j+q+(k/2)w$, $\bar{J}=\bar{M}=j+\bar{q}+(k/2)w$. In this case, we get a pole from one of the $\Gamma$ functions in Eq. (5.13), and it cancels the factor $V_{\text{conf}}^{-1}$. Thus the expression in the $x$ space is finite. As in the case of a long string, turning this into a string theory two-point function generates an additional factor of $V_{\text{conf}}^{-1}$, but this is also canceled by $\delta(j-j')$ in Eq. (5.13) evaluated at $j=j'$.[27]

With all the factors $V_{\text{conf}}$ canceled out, we have a finite correlation function in the target space. There is one subtlety here since there is a possibility that a $j$-dependent factor appears when we cancel $\delta(j-j')$ at $j=j'$ with the volume of the conformal group $V_{\text{conf}}$. We claim that, in fact, a finite factor of the form $|2j-1+(k-2)w|$ remains after the cancellation. One heuristic way to see this is the following. (A more rigorous derivation of this factor in the case of $w=0$ is given in Appendix A.) If we regularize the computation by taking $j$ to be slightly away from on-shell $L_0(j)-1=0$ and introduce a cutoff $\epsilon$ in the $z$ integral, the volume $V_{\text{conf}}$ of the conformal group would be $\delta_\epsilon(L_0(j)-1)$, where $\delta_\epsilon$ is a Gaussian with a short tail which becomes the delta function in the limit $\epsilon\to 0$. This is the factor that cancels the $\delta(j-j')$ term in the worldsheet two-point function. Thus we expect that the cancellation of the two divergences leaves the finite factor given by

$$\left|\frac{\partial L_0(j)}{\partial j}\right| \sim |2j-1+(k-2)w|, \quad (5.17)$$

up to a $k$-dependent coefficient. Taking this into account, the two-point function of the short string with winding number $w\neq 0$ is of the form

$$\langle \Phi^{w,j}_{JJ}(x_1)\Phi^{w,j}_{JJ}(x_2)\rangle_{\text{target}} = \frac{1}{V_{\text{conf}}}\langle \Phi^{w,j}_{J,\bar{J}}(x_1,z_1=0) \times\Phi^{w,j}_{J,\bar{J}}(x_2,z_2=0)\rangle_{\text{worldsheet}}$$
$$\sim |2j-1+(k-2)w| \times\frac{\Gamma(2j+q)\Gamma(2j+\bar{q})}{\Gamma(2j)^2 q!\bar{q}!}\frac{B(j)}{x_{12}^{2J}\bar{x}_{12}^{2\bar{J}}}, \quad (5.18)$$

where $q=J-j-(k/2)w$, $\bar{q}=\bar{J}-j-(k/2)$. Unlike the case of the long string, we do not have to rescale the operator $\Phi_{J,\bar{J}}$.

[27] For a short string, the physical spectrum of $j$ is discrete and we need to evaluate the $\delta$ function right at $j=j'$ rather than leaving the delta functions $\delta(s+s')$ and $\delta(s-s')$ as in the case of long strings in Eq. (5.15).

We note that the coefficient in Eq. (5.18) is positive as long as $j$ is in the physical range $\frac{1}{2}<j<(k-1)/2$. This of course is consistent with the positivity of the physical Hilbert space of the string in $AdS_3$. When $w=0$, the two-point function is given by

$$\langle \Phi_j^{w=0}(x_1)\Phi_j^{w=0}(x_2)\rangle_{\rm target}\sim(2j-1)\frac{B(j)}{|x_{12}|^{4j}}. \tag{5.19}$$

Later in this section, we will show that this additional factor of $(2j-1)$ is precisely what one needs in order to reproduce the factorization of the four-point function onto the short string with $w=0$. In general, we have to be careful about a possible $j$-dependent factor that could appear when we go from the worldsheet expression to the target space expression, and Eq. (5.19) is an example of this.

For a short string, another useful computation one can do is to evaluate the two-point functions of operators $\hat{\Phi}_{J,\bar{J}}^{j;q,\bar{q}}$ corresponding to the state of the form

$$(J_{-1}^-)^p(\bar{J}_{-1}^-)^{\bar{p}}|j;m=\bar{m}=j\rangle, \tag{5.20}$$

where $J=j-p$ and $\bar{J}=j-\bar{p}$ are the spacetime conformal weights under global $SL(2,C)$. Although they are descendants of the current algebra, they are the lowest weight states of the global $SL(2,C)$. These states appear in the intermediate channel of the factorization of the four-point amplitude discussed later in this section, so it is useful to compute their two-point functions here. They are computed in the following way. Let us view these states as given by performing one unit of spectral flow on the lowest energy states as in $\mathcal{D}_{(k/2)-j}^{-0}\rightarrow\mathcal{D}_{(k/2)-j}^{-w=1}=\mathcal{D}_j^{+0}$. We start with the state $|j';m=-j'-p,\bar{m}=-j'-\bar{p}\rangle$ with $j'=(k/2)-j$. Under one unit of spectral flow, this state is mapped into a state of the form (5.20). So we first compute the correlation function of the state labeled by $j'$ in the $m$ basis, perform spectral flow using the formulas (5.11), and finally we go to the $x$ basis as in Eq. (5.18). We find

$$\langle \Phi_{J\bar{J}}^{jp\bar{p}}(x_1)\Phi_{J\bar{J}}^{jp\bar{p}}(x_2)\rangle \sim(2j-1)\frac{\Gamma(k-2j+p)\Gamma(k-2j+\bar{p})}{\Gamma(k-2j)^2p!\bar{p}!}\frac{B\left(\frac{k}{2}-j\right)}{x_{12}^{2(j-p)}\bar{x}_{12}^{2(j-\bar{p})}}, \tag{5.21}$$

where again we have assumed that the amplitude is multiplied by a unit-normalized primary field in the internal CFT operator so that the total worldsheet conformal weight of the vertex operator is 1, and we integrated the resulting two-point function over the worldsheet. We have taken into account the factor $(2j-1)$ discussed at Eq. (5.19). Notice that, up to a $k$-dependent factor, $B[(k/2)-j]$ is equal to $B(j)^{-1}$ as one can see from Eq. (2.10). If we set $p=\bar{p}=0$ in Eq. (5.21), we recover the original result (5.19) but with a different normalization; instead of $B(j)$, we have $B^{-1}(j)$. What this shows is that the natural normalization of the operator in $\mathcal{D}_j^{w=0}$ and that of the operator in $\mathcal{D}_{(k/2)-j}^-$ are different. It is therefore more convenient to define the operator corresponding to the state (5.20) as

$$\hat{\Phi}_{J\bar{J}}^{jp\bar{p}}(x)\sim B(j)\Phi_{J\bar{J}}^{jp\bar{p}}(x). \tag{5.22}$$

In this way, for $p=\bar{p}=0$, we recover the $w=0$ SL(2) current algebra primaries with the standard normalization (2.10). Their two-point function is then given by

$$\langle \hat{\Phi}_{J\bar{J}}^{jp\bar{p}}(x_1)\hat{\Phi}_{J\bar{J}}^{jp\bar{p}}(x_2)\rangle \sim(2j-1)\frac{\Gamma(k-2j+p)\Gamma(k-2j+\bar{p})}{\Gamma(k-2j)^2p!\bar{p}!}\frac{B(j)}{x_{12}^{2(j-p)}\bar{x}_{12}^{2(j-\bar{p})}}. \tag{5.23}$$

We will use this formula in Sec. V E, where we examine effects of intermediate short strings with $w=0$ in the four-point function.

### B. Three-point functions in *m* basis

Let us now turn to three-point functions. In the case of the two-point functions, the winding number $w$ is preserved in the $m$ basis (5.11). This simply reflects the fact that the worldsheet Hamiltonian $L_0+\bar{L}_0$ can be diagonalized by states carrying fixed amounts of $w$. However, the winding number can be violated by string interactions. In this subsection, we will compute the four-point function with three vertex operators and one spectral flow operator. This computation has been done in [20], and we reproduce it here. In [26], this was done using the free field theory approach. In the next subsection, we will use this result to derive the three-point functions with winding number violations.

The spectral flow operator changes the winding number of another operator by one unit. According to [1], we can view it as the lowest weight state in $d_j^+\otimes d_j^+$ with $j=k/2$. This operator is outside the allowed range (3.2) for physical operators in the target space theory. We will not use this operator by itself for an operator in the target space theory, but it is used in an intermediate step to construct physical operators with nonzero winding numbers. A very important property of the spectral flow operator is that it has a null descendant of the form

$$J_{-1}^-|j=k/2;m=k/2\rangle=0. \tag{5.24}$$

We can then compute a four-point function where one of the operators is $|j=k/2,m=k/2\rangle$ since it obeys the differential equation which follows from the existence of the null state (5.24). The equation turns out to have a unique solution up to an overall normalization, and we can use it to derive a three-point function with winding number violation. This computation also serves as a simple example where we can find an explicit expression for $\mathcal{F}_{SL(2)}$ in Eq. (4.6) (though in the nongeneric case) and it gives us some intuition about how four-point functions look in general. In particular, we will find that the solution indeed has a singularity at $z=x$ with the exponent for $(z-x)$ expected from the general argument given in Sec. II D.

We want to compute the four-point function

$$\begin{aligned}&\langle\Phi_{j_1}(x_1,z_1)\Phi_{k/2}(x_2,z_2)\Phi_{j_3}(x_3,z_3)\Phi_{j_4}(x_4,z_4)\rangle\\&=|z_{43}|^{2(\Delta_2+\Delta_1-\Delta_4-\Delta_3)}|z_{42}|^{-4\Delta_2}|z_{41}|^{2(\Delta_3+\Delta_2-\Delta_4-\Delta_1)}|z_{31}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}\\&\quad\times|x_{43}|^{2(j_2+j_1-j_4-j_3)}|x_{42}|^{-4j_2}|x_{41}|^{2(j_3+j_2-j_4-j_1)}|x_{31}|^{2(j_4-j_1-j_2-j_3)}\times\tilde{C}(j_1,j_3,j_4)|\mathcal{F}(z,x)|^2,\end{aligned}\tag{5.25}$$

where the coefficient $\tilde{C}(j_1,j_3,j_4)$ will be determined later. We have written the dependence on the cross ratios $z=z_{21}z_{43}/z_{31}z_{42}$ and $x=x_{21}x_{43}/x_{31}x_{42}$ of the worldsheets and the target space coordinates in the form of a square of some homomorphic function $\mathcal{F}$ in Eq. (5.25), anticipating that there is only one state in the intermediate channel. This fact will be derived by explicitly solving the differential equation below. The null state condition (5.24) for the operator at $z_2$ implies the equation

$$\left\{\left[\frac{x}{z}-\frac{x-1}{z-1}\right]x(x-1)\partial_x-\kappa\left[\frac{x^2}{z}-\frac{(x-1)^2}{z-1}\right]-\frac{2j_1x}{z}-\frac{2j_3(x-1)}{z-1}\right\}\mathcal{F}(z,x)=0.\tag{5.26}$$

Here $j_2=k/2$ and $\kappa=j_4-j_1-j_2-j_3$. On the other hand, since

$$L_{-1}\left|j=\frac{k}{2}\right\rangle=-J^3_{-1}\left|j=\frac{k}{2}\right\rangle,\tag{5.27}$$

the KZ equation takes the form

$$\partial_z\mathcal{F}=-\left\{\frac{x(x-1)}{z(z-1)}\partial_x+\kappa\left[\frac{x}{z}-\frac{x-1}{z-1}\right]+\frac{j_1}{z}+\frac{j_3}{z-1}\right\}\mathcal{F}.\tag{5.28}$$

Using Eq. (5.26), we can eliminate $\partial_x$ from Eq. (5.28), and we obtain

$$\partial_z\mathcal{F}=\left\{\frac{j_1}{z}+\frac{j_3}{z-1}-\frac{(j_1+j_3+j_4-k/2)}{z-x}\right\}\mathcal{F}.\tag{5.29}$$

This equation can be easily integrated and we can insert the resulting general solution in Eq. (5.26) to determine the $x$ and $z$ dependence of $\mathcal{F}$ completely. We find

$$\mathcal{F}=z^{j_1}(z-1)^{j_3}(z-x)^{-j_1-j_3-j_4+(k/2)}\times x^{2j_3+\kappa}(x-1)^{2j_1+\kappa}.\tag{5.30}$$

The solution is unique up to an overall normalization, and the four-point function is indeed given by the absolute value squared of this function as anticipated in Eq. (5.25). Note also that there is a singularity at $z=x$ with precisely the expected form.

We also need to determine the coefficient $\tilde{C}(j_1,j_3,j_4)$ in Eq. (5.25). We use the same method as in [20,19]. The standard operator product expansion formula gives $C(j_1,k/2,j)B(j)^{-1}C(j,j_3,j_4)$, where $j$ is for the intermediate state. As we mentioned earlier in Eq. (2.21), the factor $C(j_1,k/2,j)$ is equal to the delta function $\delta(j_1+j-(k/2))$ modulo a $k$-dependent ($j_1$-independent) coefficient. This is consistent with the fact that, in Eq. (5.30), only the state with $j=(k/2)-j_1$ is propagating in the intermediate channel for $z\to0$. Thus the coefficient $\tilde{C}$ is determined as

$$\begin{aligned}\tilde{C}(j_1,j_3,j_4)&\sim B\left(\frac{k}{2}-j_1\right)^{-1}C\left(\frac{k}{2}-j_1,j_3,j_4\right)\\&\sim B(j_1)C\left(\frac{k}{2}-j_1,j_3,j_4\right)\end{aligned}\tag{5.31}$$

modulo a $k$-dependent factor. Here we used Eq. (2.10).

As shown in [1] and reviewed in the preceding subsection, the spectral flow operator is given by the operator $\Phi_{k/2}$ in the $m$ basis. Thus we need to perform the integral transform (5.4) on Eq. (5.25) and set $m_2=-k/2$. As in the case of the two-point function, setting this value of $m_2$ generates a pole in the amplitude so the spectral flow operator is defined by

$$e^{-i\sqrt{(k/2)}\varphi}\sim\frac{1}{V_{\rm conf}}\Phi_{k/2,-k/2;k/2,-k/2},\tag{5.32}$$

where the operator $\Phi_{k/2,-k/2;k/2,-k/2}$ is normalized as in Eq. (5.4). The factor $1/V_{\rm conf}$ is there to remind us that we have to extract a pole residue at $m=-k/2$. This residue can be evaluated by taking the limit $x_2\to\infty$ of $|x_2|^{2k}$ times Eq. (5.25). After performing the $x_i$ integrals, we find [20]

$$\int \prod_{i=1,3,4} d^2x_i x_i^{j_i-m_i-1}\bar{x}_i^{j_i-\bar{m}_i-1}\left\{\lim_{x_2\to\infty}|x_2|^{2k}(5.25)\right\}$$
$$=\tilde{C}(j_1,j_3,j_4)\,\delta^2\left(-\frac{k}{2}+m_1+m_3+m_4,-\frac{k}{2}+\bar{m}_1+\bar{m}_3+\bar{m}_4\right)$$
$$\times\prod_{i=1,3,4}(z_2-z_i)^{m_i}z_{13}^{\Delta_4-\Delta_1-\Delta_3+(k/4)+m_4}z_{34}^{\Delta_1-\Delta_3-\Delta_4+(k/4)+m_1}z_{41}^{\Delta_3-\Delta_4-\Delta_1+(k/4)+m_3}$$
$$\times\prod_{i=1,3,4}(\bar{z}_2-\bar{z}_i)^{\bar{m}_i}\bar{z}_{13}^{\Delta_4-\Delta_1-\Delta_3+(k/4)+\bar{m}_4}\bar{z}_{34}^{\Delta_1-\Delta_3-\Delta_4+(k/4)+\bar{m}_1}$$
$$\times\bar{z}_{41}^{\Delta_3-\Delta_4-\Delta_1+(k/4)+\bar{m}_3}\frac{1}{\gamma\left(j_1+j_3+j_4-\frac{k}{2}\right)}\prod_{i=1,3,4}\frac{\Gamma(j_i-m_i)}{\Gamma(1-j_i+\bar{m}_i)}, \tag{5.33}$$

where ''cyclic'' means a cyclic permutation of the labels 134. The $z_2$ dependence is what we expect for the operator (5.32). We can now extract the action of the spectral flow operator on $\Phi_{j_1}$. This is done by taking the limit of $z_2\to z_1$ and extracting the coefficient of the term which goes like $z_{12}^{m_1}\bar{z}_{12}^{\bar{m}_1}$. This performs spectral flow on the operator inserted at $z_1$ by $-1$ unit.[28] According to the rules (5.1) of the spectral flow, the new spacetime quantum numbers of the operator at $z_1$ are $M=m_1-(k/2)$ and $\bar{M}=\bar{m}_1-(k/2)$, and its global $SL(2,C)$ left and right conformal weights are $J=|M|$ and $\bar{J}=|\bar{M}|$. Finally, we find

$$\langle\Phi^{w=-1,j_1}_{J,M,\bar{J},\bar{M}}(z_1)\Phi_{j_3,m_3,\bar{m}_3}(z_3)\Phi_{j_4,m_4\bar{m}_4}(z_4)\rangle$$
$$=\tilde{C}(j_1,j_3,j_4)\,\delta^2\left(-\frac{k}{2}+m_1+m_3+m_4,-\frac{k}{2}+\bar{m}_1+\bar{m}_3+\bar{m}_4\right)z_{13}^{\Delta_4-\hat{\Delta}_1-\Delta_3}z_{34}^{\hat{\Delta}_1-\Delta_3-\Delta_4}z_{41}^{\Delta_3-\Delta_4-\hat{\Delta}_1}$$
$$\times\bar{z}_{13}^{\Delta_4-\bar{\hat{\Delta}}_1-\Delta_3}\bar{z}_{34}^{\bar{\hat{\Delta}}_1-\Delta_3-\Delta_4}\bar{z}_{41}^{\Delta_3-\Delta_4-\bar{\hat{\Delta}}_1}\frac{1}{\gamma\left(j_1+j_3+j_4-\frac{k}{2}\right)}\frac{\Gamma(j_1-m_1)}{\Gamma(1-j_1+\bar{m}_1)}\frac{\Gamma(j_3-m_3)}{\Gamma(1-j_3+\bar{m}_3)}\frac{\Gamma(j_4-\bar{m}_4)}{\Gamma(1-j_4+m_4)}, \tag{5.34}$$

with

$$\hat{\Delta}_1=\Delta(j_1)+m_1-\frac{k}{4},\quad \bar{\hat{\Delta}}_1=\Delta(j_1)+\bar{m}_1-\frac{k}{4},$$
$$J=-M=-m_1+\frac{k}{2},\quad \bar{J}=-\bar{M}=-\bar{m}_1+\frac{k}{2}, \tag{5.35}$$

where we used the $\delta$ function in $m_i$ to go from Eq. (5.33) to Eq. (5.34).[29] This indeed has the expected $z$ dependence for the correlation function of one spectral flowed operator with two unflowed operators.

### C. Three-point functions in the $x$ basis

In this subsection, we will discuss how to go from the $m$ basis to the $x$ basis for three-point functions. We want to rewrite Eq. (5.34) in the $x$ basis. This is similar to what we did for the two-point functions.

We start with a general three-point function in the $x$ basis,

$$\langle\Phi_{J_1,\bar{J}_1}(x_1)\Phi_{J_2\bar{J}_2}(x_2)\Phi_{J_3\bar{J}_3}(x_3)\rangle=\frac{D(J_1,J_2,J_3)}{x_{12}^{J_1+J_2-J_3}x_{13}^{J_1+J_3-J_2}x_{23}^{J_2+J_3-J_1}\bar{x}_{12}^{\bar{J}_1+\bar{J}_2-\bar{J}_3}\bar{x}_{13}^{\bar{J}_1+\bar{J}_3-\bar{J}_2}\bar{x}_{23}^{\bar{J}_2+\bar{J}_3-\bar{J}_1}}, \tag{5.36}$$

[28]We have $-1$ unit of spectral flow because we extracted the $m_2=-k/2$ component of the spectral flow operator in Eq. (5.33). The resulting operator represents an outgoing state carrying away one unit of winding number.

[29]We also used properties of the $\Gamma$ function to absorb the sign $(-1)^{m_4-\bar{m}_4}$ that came from the powers of $z_{14}$ in going from Eq. (5.33) to Eq. (5.34).

where the $J_i,\bar{J}_i$ label the conformal weight under global SL$(2,C)$. We can compute the integral transform of this expression to go to the $m$ basis. The integral can be written using the Barnes hypergeometric function [27]. For our purposes, we do not need to compute the most general expression since the three-point function (5.34) is really the residue of the pole at $J_1=-M_1,\bar{J}_1=-\bar{M}_1$ in the $x$ integral of Eq. (5.36). This pole comes from the region where $x_1$ is very large. We are interested in the coefficient of this pole. This is obtained by taking the $x_1\to\infty$ limit of $x_1^{2J_1}\bar{x}_1^{2\bar{J}_1}$ times Eq. (5.36) and then performing the integral transform with respect to $x_2$ and $x_3$. We obtain

$$\langle \Phi_{J_1=-M_1,\bar{J}_1=-\bar{M}_1}\Phi_{J_2\bar{J}_2,M_2,\bar{M}_2}\Phi_{J_3\bar{J}_3,M_3,\bar{M}_3}\rangle \sim V_{\rm conf}\delta^2(M_1+M_2+M_3)D(J_1,J_2,J_3)\frac{\Gamma(\bar{J}_3-\bar{M}_3)}{\Gamma(1-J_3+M_3)}\frac{\Gamma(J_2-M_2)}{\Gamma(1-\bar{J}_2+\bar{M}_2)}\frac{\Gamma(1+\bar{J}_1-\bar{J}_2-\bar{J}_3)}{\Gamma(J_2+J_3-J_1)}, \tag{5.37}$$

where the $V_{\rm conf}$ is there to remind us that the rest is the residue of a pole. Notice that only properties under global SL$(2,C)$ have been used to derive this formula.

By comparing Eq. (5.34) to Eq. (5.37) [and changing the labels $(2,3)\to(3,4)$ in Eq. (5.37) in the obvious way], we find that the three-point function in $x$ space is given by

$$\langle \Phi^{w=1,j_1}_{J,\bar{J}}(x_1)\Phi_{j_3}(x_3)\Phi_{j_4}(x_4)\rangle \sim \frac{1}{V_{\rm conf}}B(j_1)C\left(\frac{k}{2}-j_1,j_3,j_4\right)\frac{\Gamma(j_3+j_4-J)}{\Gamma(1+\bar{J}-j_3-j_4)}\frac{\Gamma\left(j_1+J-\frac{k}{2}\right)}{\Gamma\left(1-j_1-\bar{J}+\frac{k}{2}\right)}\frac{1}{\gamma\left(j_3+j_4+j_1-\frac{k}{2}\right)}$$
$$\times\frac{1}{x_{13}^{J+j_3-j_4}x_{14}^{J+j_4-j_3}x_{34}^{j_3+j_4-J}\bar{x}_{13}^{\bar{J}+j_3-j_4}\bar{x}_{14}^{\bar{J}+j_4-j_3}\bar{x}_{34}^{j_3+j_4-\bar{J}}}, \tag{5.38}$$

where

$$J_1=-m_1+\frac{k}{2},\quad \bar{J}_1=-\bar{m}_1+\frac{k}{2}, \tag{5.39}$$
$$J_{3,4}=\bar{J}_{3,4}=j_{3,4}.$$

In the case of $j=\frac{1}{2}+is$, when the first operator corresponds to a long string, this factor of $1/V_{\rm conf}$ is canceled since the long string operator $\hat{\Phi}$ comes with the extra factor of $V_{\rm conf}$ as in Eq. (5.15). Thus we conclude that the three-point function of two short strings with $w=0$ and one long string with $w=1$ is nonzero. In the following subsection, we compare the expression (5.38) with the factorization of the four-point function.

In Appendix D, we will show, using the representation theory of the SL$(2,R)$ current algebra, that two short strings with $w=0$ can only be mixed with short strings with $w=0,1$ or long strings with $w=1$. One may ask why we did not see short strings with $w=1$ in the factorization of the four-point function. In fact, there is an additional reason for the vanishing of the three-point amplitude with two short strings with $w=0$ and one short string with $w=1$. If $j_1$ is real and $m_1,\bar{m}_1<0$, the operator $\Phi^{w=1,j_1}_{J,\bar{J}}$ in Eq. (5.38) corresponds to a short string with $w=1$. For this operator, the two-point function is finite as we saw in Eq. (5.18), and we do not have to rescale the operator as we did for the long string. Thus we interpret the factor of $1/V_{\rm conf}$ in Eq. (5.38) as saying that the three-point function vanishes. This gives the additional constraint on the winding number violation stating that two short strings with $w=0$ cannot produce a short string with $w=1$.

As a check that we are interpreting this factor of $1/V_{\rm conf}$ correctly and as a further application of Eq. (5.38), let us consider the case in which $j_1$ is real and $m_1,\bar{m}_1>0$, $m_1=j_1+p$, $\bar{m}_1=j_1+\bar{p}$. This can be interpreted as doing the spectral flow of a discrete representation by $-1$ unit, thus producing the operator described at Eq. (5.20) with $j=(k/2)-j_1$. This state is just a descendant in a discrete representation with $w=0$. Thus, in this case, we do not expect the three-point function to vanish. Indeed we find that, as we set $m_1=j_1+p$, one of the $\Gamma$ functions in Eq. (5.38) develops a pole, thereby canceling the factor $1/V_{\rm conf}$ in Eq. (5.38). Finally, we obtain

$$\langle \hat{\Phi}^{jp\bar{p}}_{J\bar{J}}(x_1)\Phi_{j_3}(x_3)\Phi_{j_4}(x_4)\rangle \sim (-1)^{p+\bar{p}}C(j,j_3,j_4)\frac{\Gamma(j_3+j_4-j+p)}{p!\Gamma(j_3+j_4-j)}\frac{\Gamma(j_3+j_4-j+\bar{p})}{\bar{p}!\Gamma(j_3+j_4-j)}$$
$$\times\frac{1}{x_{13}^{J+j_3-j_4}x_{14}^{J+j_4-j_3}x_{34}^{j_3+j_4-J}\bar{x}_{13}^{\bar{J}+j_3-j_4}\bar{x}_{14}^{\bar{J}+j_4-j_3}\bar{x}_{34}^{j_3+j_4-\bar{J}}}. \tag{5.40}$$

Note that $j=(k/2)-j_1$, where $j_1$ is the label appearing in Eq. (5.38) and $J=j-p,\bar{J}=j-\bar{p}$. We have also normalized the operator as in Eq. (5.22). If we set $p=\bar{p}=0$, we indeed find that this is the same as the correlation function of three $w=0$ discrete states. This is an interesting consistency check of what we are doing. Moreover, we will see that the expression (5.40) exactly matches with the factorization of the four-point function in the target spacetime.

## VI. FACTORIZATION OF FOUR-POINT FUNCTIONS

In the preceding section, we computed the two- and three-point functions including spectral flowed operators. In this section, we will use these results to show that the coefficients of the powers of $x$ appearing in the spacetime operator product expansion computed in Sec. IV are precisely what are expected, i.e., each of them is a product of two three-point functions involving the intermediate state divided by the two-point function of that intermediate state.

### A. Factorization on long strings

Let us first examine the coefficient for the continuous representations appearing in Eq. (4.31). In the expression (4.31), the integration contour runs along $j_c=k/2-\frac{1}{2}+is=k/2-j$, where $j=\frac{1}{2}-is$. We are denoting the SL(2,$C$) spin along the contour by $j_c$, and $j$ is introduced for convenience. Then we define

$$J=j_c+d(j_c)=\frac{s^2+\frac{1}{4}}{k-2}+h-1,$$
$$\bar{J}=j_c+\bar{d}(j_c)=\frac{s^2+\frac{1}{4}}{k-2}+\bar{h}-1. \tag{6.1}$$

From the power of $x$ in Eq. (4.31), we conclude that $J$ is the spacetime conformal weight of the intermediate state. The coefficient of this power of $x$ is Eq. (4.31),

$$\frac{\Gamma\left(J-\frac{k}{2}+j\right)}{\Gamma\left(1-\bar{J}+\frac{k}{2}-j\right)}\frac{\Gamma(j_1+j_2-\bar{J})}{\Gamma(1-j_1-j_2+J)}\frac{\Gamma(j_3+j_4-\bar{J})}{\Gamma(1-j_3-j_4+J)}\frac{\Gamma\left(1+J-j-\frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}+j-\bar{J}\right)}$$
$$\times\frac{\gamma(2j)}{\gamma\left(j_1+j_2+j-\frac{k}{2}\right)\gamma\left(j_3+j_4+j-\frac{k}{2}\right)}\frac{C\left(j_1,j_2,\frac{k}{2}-j\right)C\left(\frac{k}{2}-j,j_3,j_4\right)}{B\left(\frac{k}{2}-j\right)}. \tag{6.2}$$

It can be shown that this coefficient is given by the product of two of the three-point functions divided by the two-point function. This can be seen explicitly by writing Eq. (6.2) as

$$B(j)C\left(\frac{k}{2}-j,j_1,j_2\right)\frac{\Gamma(j_1+j_2-J)}{\Gamma(1-j_1-j_2+\bar{J})}\frac{\Gamma\left(j+J-\frac{k}{2}\right)}{\Gamma\left(1-j-\bar{J}+\frac{k}{2}\right)}\frac{1}{\gamma\left(j_1+j_2+j-\frac{k}{2}\right)}\frac{\gamma(2j)}{B(j)}\frac{\Gamma\left(1-j-\frac{k}{2}+J\right)}{\Gamma\left(j+\frac{k}{2}-\bar{J}\right)}\frac{\Gamma\left(1-j+\frac{k}{2}-\bar{J}\right)}{\Gamma\left(j-\frac{k}{2}+J\right)}$$
$$\times B(j)C\left(\frac{k}{2}-j,j_3,j_4\right)\frac{\Gamma(j_3+j_4-J)}{\Gamma(1-j_3-j_4+\bar{J})}\frac{\Gamma\left(j+J-\frac{k}{2}\right)}{\Gamma\left(1-j-\bar{J}+\frac{k}{2}\right)}\frac{1}{\gamma\left(j_3+j_4+j-\frac{k}{2}\right)}. \tag{6.3}$$

Note that we used that $B[(k/2)-j]\sim B(j)^{-1}$. We see that this has the form of the product of two three-point functions (5.38) divided by the two-point function (5.15) with $w=1$. Note that, in Eq. (5.15), there are two terms, one proportional to $\delta(s+s')$ and the second proportional to $\delta(s-s')$. Physically, $s$ and $-s$ describe the same operator. So when we consider the inverse of the two-point function, it is convenient to restrict the integral over $s$ so that $s\geqslant 0$. This is possible in Eq. (4.31) since the expression is symmetric under $s\to -s$. In this prescription, only the term proportional to $\delta(s-s')$ in the two-point function needs to be inverted. We have checked that, if we took the other term in the two-point function, the one proportional to $\delta(s+s')$, we will find the same result provided we switch the sign of $s$ in one of the three-point functions in Eq. (6.3), precisely as required.

### B. Factorization on short strings

We now consider Eq. (4.34) where, as we explained before, we should shift the $j$ contour of integration. This picks up some poles explicitly displayed in Eq. (4.35). These poles are at $d=-n,\bar{d}=-\bar{n}$. From their $x$ dependence, we conclude that the spacetime conformal weight of the intermediate operator is $J=j-n,\bar{J}=j-\bar{n}$. The residue of the pole is

$$\frac{1}{\partial d/\partial j}\frac{C(j_1,j_2,j)C(j,j_3,j_4)}{B(j)}\frac{\Gamma(j_1+j_2-j+n)}{n!\Gamma(j_1+j_2-j)}\frac{\Gamma(j_3+j_4-j+n)}{n!\Gamma(j_3+j_4-j)}\frac{n!\Gamma(k-2j)}{\Gamma(k-2j+n)}$$
$$\times\frac{\Gamma(j_1+j_2-j+\bar{n})}{\bar{n}!\Gamma(j_1+j_2-j)}\frac{\Gamma(j_3+j_4-j+\bar{n})}{\bar{n}!\Gamma(j_3+j_4-j)}\frac{\bar{n}!\Gamma(k-2j)}{\Gamma(k-2j+\bar{n})}. \tag{6.4}$$

The factor $[\partial d(j)/\partial j]^{-1}$ appears here since the pole we picked up in Eq. (4.35) is of the form $\sim[d(j)+n]^{-1}$ and we are evaluating residues in the $j$ integral in Eq. (4.34). We see that this has precisely the expected form for a state like Eq. (5.20) propagating in the intermediate channel. Indeed we can write Eq. (6.4) as the product of two three-point functions (5.40) divided by the coefficient of the two-point function (5.23), including the factor involving $\partial d/\partial j\sim(2j-1)$, which we discussed at Eq. (5.19) as

$$(-1)^{n+\bar{n}}C(j,j_1,j_2)\frac{\Gamma(j_1+j_2-j+n)}{n!\Gamma(j_1+j_2-j)}\frac{\Gamma(j_1+j_2-j+\bar{n})}{\bar{n}!\Gamma(j_1+j_2-j)}\frac{1}{(2j-1)B(j)}\frac{n!\Gamma(k-2j)}{\Gamma(k-2j+n)}\frac{\bar{n}!\Gamma(k-2j)}{\Gamma(k-2k+\bar{n})}$$
$$\times(-1)^{n+\bar{n}}C(j,j_3,j_4)\frac{\Gamma(j_3+j_4-j+n)}{n!\Gamma(j_3+j_4-j)}\frac{\Gamma(j_3+j_4-j+\bar{n})}{\bar{n}!\Gamma(j_3+j_4-j)}. \tag{6.5}$$

In other words, we need to correct the two-point function by the factor $(2j-1)$ as in Eq. (5.19) in order to get the right spacetime factorization properties. This completes the test of the factorization of the four-point function.

## VII. FINAL REMARKS

Most of what we said in this paper refered to the Euclidean theory, both on the worldsheet and on target space. These computations can also be interpreted as describing string theory on a Lorentzian target space. Note that string theory in Lorentzian $AdS_3$ can be thought of in terms of the usual $S$-matrix formulation, where the asymptotic states are the long strings. Short strings appear as poles in the long string amplitudes. We did not compute this precisely but this is the expected picture. It would be interesting to expand the four-point function for two long strings with $w=1$ and two with $w=-1$, and see that indeed we produce only long and short strings in accordance with the winding violation rule described in the Appendix D. In this way, the theory on the Lorentzian $AdS_3$ can be interpreted either in terms of an $S$ matrix or in terms of a BCFT, albeit one with a noncompact target space. The $S$-matrix computation of long strings is in fact describing scatterings in the Lorentzian BCFT. It is amusing to note that this singular BCFT is reproducing some features which seem characteristic to strings in flat space, such as having an $S$-matrix description. This may give us a hint as to how to construct a holographic description of flat space physics.

This BCFT is rather peculiar due to the noncompactness of its target space. All the computations we have been defining were for the case in which the BCFT is on $S^2$. These computations are well defined when properly interpreted, as we discussed in this paper. The only peculiarity is that we cannot insert too many discrete state operators, but this should not be surprising since we also saw simple quantum-mechanical models where that is true. If we put the BCFT on a torus, we will find divergences in one-loop computations as we have shown explicitly in [2]. In [2], these divergences were regulated by adding a volume cutoff near the boundary, but strictly speaking the one-loop free energy is infinite. We would find a similar result in the quantum-mechanical example we discussed in Sec. III B. This BCFT would not be well defined on a higher genus Riemann surface.

The SL(2,$R$) WZW model has an interesting algebraic structure which should be explored further.

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## APPENDIX A: TARGET SPACE TWO-POINT FUNCTION OF SHORT STRING WITH $w=0$

In Sec. V, we computed the target space two-point function starting with the worldsheet two-point function and dividing it by the volume of worldsheet conformal symmetry which fixes the two points. This process involved some subtlety since we have to cancel two divergent factors, leaving the finite coefficient $|2j-1+(k-2)w|$ for a short string.[30] In this appendix, we present an alternative derivation of the target space two-point function in the case of $w=0$.

The idea is to use the target space Ward identity. We assume that there is some current algebra symmetry in the BCFT and use it to relate the three-point function including the conserved current to the two-point function that we want to compute.[31] One may view this as the string theory version of the computation in [37] where a similar factor for the two-point function was derived using the Ward identity in the supergravity approximation.

The global symmetry of the BCFT comes from a current algebra symmetry in the internal CFT on the worldsheet. According to [9], the vertex operator for the target space current is given by $J(x,z)\overline{L(\bar{z})}\Phi_{j=1}(x,z)$, where

$$J(x,z)=-J^-(z)+2xJ^3(z)-x^2J^+(z), \tag{A1}$$

and $\overline{L}(\bar{z})$ is the current algebra generator in the internal CFT. Thus, to compute the Ward identity for the target space two-point function, we need to evaluate a three-point function $\langle\Phi_{j_1}\Phi_{j_2}\Phi_{j=1}\rangle$. Due to the fact that the two-point function of the internal CFT is nonzero only between operators with the same conformal dimensions, the on-shell condition requires $j_1=j_2$ and we can focus our attention to this case. We then find that the $AdS_3$ part of the correlation function is of the form

$$\begin{aligned}\langle\Phi_j(x_1,z_1)\Phi_j(x_2,z_2)\Phi_1(x_3,z_3)\rangle &= \frac{G(-2j)}{2\pi^2\nu^{2j}\gamma\left(\frac{k-1}{k-2}\right)G(1-2j)}\frac{1}{|z_{12}|^{2\Delta}|x_{12}|^{2(2j-1)}|x_{23}|^2|x_{31}|^2}\\ &= \frac{1}{2\pi^2\nu^{2j}\gamma\left(\frac{k-1}{k-2}\right)\gamma\left(\frac{2j-1}{k-2}\right)}\frac{1}{|z_{12}|^{4\Delta}|x_{12}|^{2(2j-1)}|x_{23}|^2|x_{31}|^2}\\ &= \frac{1}{\nu\cdot 2\pi(k-2)\gamma\left(\frac{k-1}{k-2}\right)}\frac{B(j)}{|z_{12}|^{4\Delta}|x_{12}|^{2(2j-1)}|x_{23}|^2|x_{31}|^2},\end{aligned} \tag{A2}$$

where $\Delta=-j(j-1)/(k-2)$. We then multiply the current generator $J(x_3,z_3)$ on $\Phi_1(x_3,z_3)$. Using the operator product expansion

$$J(x,z)\Phi_j(y,w)\sim\frac{1}{z-w}\left((x-y)^2\frac{\partial}{\partial y}-2j(x-y)\right)\Phi_j(y,w), \tag{A3}$$

we find

$$\langle\Phi_j(x_1,z_1)\Phi_j(x_2,z_2)[J(x_3,z_3)\Phi_1(x_3,z_3)]\rangle\sim\frac{(2j-1)B(j)}{|z_{12}|^{4\Delta}|x_{12}|^{4j}}\left(\frac{1}{\bar{x}_3-\bar{x}_1}-\frac{1}{\bar{x}_3-\bar{x}_2}\right)\left(\frac{1}{z_3-z_1}-\frac{1}{z_3-z_2}\right), \tag{A4}$$

where we ignored a constant independent of $j$. To obtain the spacetime three-point function, we choose an operator $\phi_h$ of

[30] There is no such factor for a long string.

[31] If there is no current algebra symmetry, one can use the energy-momentum tensor, which exists in any CFT. It is straightforward to generalize the following computation with the energy-momentum tensor, and one obtains the same normalization for the target space two-point function.

dimension $h$ in the internal CFT so that $\Delta+h=1$ and multiply it to $\Phi_j$. Similarly, we multiply the current generator $\bar{L}$ of the internal CFT to $\Phi_1$. We find

$$\langle[\Phi_j(x_1,z_1)\phi_h(z_1)][\Phi_j(x_2,z_2)\phi'_h(z_2)][J(x_3,z_3)\bar{L}^a(\bar{z}_3)\Phi_1(x_3,z_3)]\rangle \sim \frac{1}{|z_{12}|^2|z_{23}|^2|z_{31}|^2}\left(\frac{q_1}{\bar{x}_3-\bar{x}_1}+\frac{q_2}{\bar{x}_3-\bar{x}_2}\right)\frac{(2j-1)B(j)}{|x_{12}|^{4j}}, \tag{A5}$$

where $q_1$ and $q_2$ are the $R$ charges $\phi_h(z_1)$ and $\phi'_h(z_2)$, respectively, and we used the charge conservation, $q_1+q_2=0$. Comparing this with what we expect for the target space Ward identity, we find that the spacetime two-point function is given by

$$\langle\Phi_j(x_1)\Phi(x_2)\rangle\sim\frac{(2j-1)B(j)}{|x_{12}|^{4j}}, \tag{A6}$$

reproducing the result we have obtained using the heuristic argument in Sec. V A.

It is easy to see that if we insert in Eq. (A6) the operator $J\bar{J}\Phi_1(x_3)$, we obtain Eq. (A6) times an extra factor of $(2j-1)$ in agreement with the arguments in [24].

## APPENDIX B: SOME PROPERTIES OF THE CONFORMAL BLOCKS

In this appendix, we will prove that the conformal block $\mathcal{F}_j(z,x)$ of the four-point function has no poles in $j$ when $\frac{1}{2}\leqslant \mathrm{Re}\, j \leqslant (k-1)/2$. We also argue that the integral over $j$ in Eq. (4.10) is convergent.

### 1. Proof of no poles in $\mathcal{F}_j$ in $\frac{1}{2}\leqslant\mathrm{Re}\, j\leqslant(k-1)/2$

This has been shown in [21] using properties of the Kac-Kazhdan determinant. Here we present a direct proof of the absence of poles.

We use the expansion (4.17) as

$$\mathcal{F}_j(z,x)=x^{\Delta(j)-\Delta(j_1)-\Delta(j_2)+j-j_1-j_2} \times u^{\Delta(j)-\Delta(j_1)-\Delta(j_2)}\sum_{n=0}^{\infty} g_n(u)x^n, \tag{B1}$$

where $u=z/x$. As we discussed in Sec. IV, the KZ equation and the boundary condition for small $z$ determine that $g_0(u)$ is given by the hypergeometric function

$$g_0(x)=F(j_1+j_2-j,j_3+j_4-j,k-2j;u). \tag{B2}$$

The standard Taylor expansion of the hypergeometric function shows that the $u$ expansion of $g_0(u)$ has no poles in the region (4.15).

Let us write

$$r\equiv\frac{k-1}{2}-j \tag{B3}$$

and

$$a(r)\equiv-\frac{r^2}{k-2}, \tag{B4}$$

which is defined so that

$$\Delta(j)+j=a(r)+\frac{k}{4}+\frac{1}{4(k-2)}. \tag{B5}$$

We then look for a solution to the KZ equation in the power series expansion of the form

$$F(x,u)=x^{a(r)+(k/4)+[1/4(k-2)]-\Delta(j_1)-\Delta(j_2)-j_1-j_2} \times u^{b(r)-[(k-2)/4]+[1/4(k-2)]-\Delta(j_1)-\Delta(j_2)} \times\sum_{m,n=0}^{\infty}\tilde{c}_{m,n}u^m x^n, \tag{B6}$$

where $b$ is some constant which will be determined below. We have chosen $m,n=0,1,\ldots$ so that the expansion is consistent with Eqs. (4.7), (4.9), and (B1). The fact that $g_0(u)$ has no poles means that $\tilde{c}_{m,n=0}$ has no poles. We will then show inductively that this is also the case for all $\tilde{c}_{m,n}$ with $n\geqslant1$.

Substituting this into the KZ equation, we find the recursive equation for the coefficient $c_{n,m}$ of the form

$$P[a(r)+n,b+m]\tilde{c}_{m,n} =(\text{linear combination of } \tilde{c}_{m',n'},\ m'<m,\ n'\leqslant n) \tag{B7}$$

for some function $P(a,b)$, which is quadratic in $b$. The right-hand side contains no poles in $j$. For $n=0,m=0$, this gives the condition $P(a(r),b)=0$. This is merely the characteristic equation for the hypergeometric equation on $g_0(u)$, and we know that this determines $b$ to be $b=b_{\pm}(r)=a(r)\pm r$. In Eq. (B2), we have chosen the + root in order to fit it with the boundary condition (4.7). With this choice of $b$, we want to show that $P[a(r)+n,b_+(r)+m]$ is nonzero for any $n\geqslant1$ and $m\geqslant0$ in the region (4.15), or equivalently $0\leqslant\mathrm{Re}\, r\leqslant(k-2)/2$. If this is true, by recursive application of Eq. (B7), we can show that $\tilde{c}_{m,n}$ has no pole.

Our strategy is to look for a solution to $P[a(r)+n,b']=0$ for $n\geqslant1$ and show that it can never be of the form $b'=b_+(r)+m$ for any $m\geqslant0$. Let us write $a(r)+n=a(r')$ for some $r'$. We know that the zeros of $P(a(r'),b')=0$ are given by $b'=b_{\pm}(r')$. Let us first consider the solution of $b'=b_+(r')$. Since

$$b_+(r')=a(r')+r'=a(r)+r'+n, \tag{B8}$$

then $b'$ could be equal to $b_+(r)+m=a(r)+r+m$ if and only if $r'=r+m-n$. On the other hand, $r'$ was defined by $a(r)+n=a(r')$. Eliminating $r'$, we find the condition

$$-2mr-(n-m)^2=n(k-2-2r). \tag{B9}$$

This cannot be satisfied by $r$ in the range $0\leqslant \mathrm{Re}\, r \leqslant (k-2)/2$ for $(n,m)\neq(0,0)$ since the real part of the left-hand side is negative while it is positive on the right-hand side. For the other solution $b'=b_-(r')$, we also find the same equation (B9). Thus we have shown that $P[a(r)+n,b_+(r)+m]$ never vanishes for $n\geqslant 1, m\geqslant 0$ if $r$ is in this range. This proves that $\mathcal{F}_j(z,x)$ has no pole in the region of our interest.

### 2. Convergence of the expression for the four-point function

To see that the $j$ integral (4.10) is indeed convergent, we note that, as a function of $j=\frac{1}{2}+is$, the coefficient $|\mathcal{C}(j)|$ behaves as $\sim e^{\alpha s}$ for large $s$ where $\alpha$ is some constant. This can be deduced from the expression for the two- and three-point functions (2.10), (2.14), and (2.15), using the asymptotic formulas of the $\Gamma$ function and the Barnes $\Gamma$ function. Due to the factor $u^{a(r)}$ in Eq. (B6), each term in the $u$ expansion of $\mathcal{C}(j)|\mathcal{F}_j|^2$ decays as $e^{-\beta s^2}$ for $s\to\pm\infty$ as long as $|u|<1$. We can see, using Eq. (B7), that the coefficients in the $u$ and $x$ expansion do not grow more than polynomiallyin $n$, $m$, so that these sums will converge if $|u|<1,|x|<1$. For other values of $u,x$, Eq. (4.10) is defined by analytic continuation.

## APPENDIX C: A USEFUL FORMULA

In this appendix, we derive the formula

$$\begin{aligned} I(a,b,c,d,\bar{d}) &= \int d^2u\, u^{d-1}\bar{u}^{\bar{d}-1}\left(|F(a,b,c;u)|^2+\lambda|u^{1-c}F(1+b-c,1+a-c,2-c;u)|^2\right)\\ &= \pi\frac{\Gamma(d)\Gamma(a-\bar{d})\Gamma(b-\bar{d})\Gamma(1-c+d)}{\Gamma(1-\bar{d})\Gamma(1-a+d)\Gamma(1-b+d)\Gamma(c-\bar{d})}\frac{\gamma(c)}{\gamma(a)\gamma(b)}, \end{aligned} \tag{C1}$$

where

$$\lambda=-\frac{\gamma(c)^2\gamma(a-c+1)\gamma(b-c+1)}{(1-c)^2\gamma(a)\gamma(b)}, \tag{C2}$$

and $\gamma(x)$ is defined in Eq. (2.11). The formula (C1) is obtained as follows. Let us first prove the following identity:

$$|F(a,b,c;u)|^2+\lambda|u^{1-c}F(1+b-c,1+a-c,2-c;u)|^2=\frac{\gamma(c)}{\pi\gamma(b)\gamma(c-b)}|u^{1-c}|^2\int d^2t|t^{b-1}(u-t)^{c-b-1}(1-t)^{-a}|^2. \tag{C3}$$

This is based on the following formula:

$$\begin{aligned} \int d^2t|t^a(u-t)^c(1-t)^b|^2 &= \frac{\sin(\pi a)\sin(\pi c)}{\sin(\pi(a+c))}\left|\int_0^u dt\, t^a(u-t)^c(1-t)^b\right|^2\\ &\quad+\frac{\sin(\pi b)\sin(\pi(a+b+c))}{\sin(\pi(a+c))}\left|\int_1^\infty dt\, t^a(u-t)^c(1-t)^b\right|^2. \end{aligned} \tag{C4}$$

A derivation of this formula can be found, for example, in [41], where it appears in the context of the free boson realization of the $c<1$ conformal field theory. There the variable $t$ corresponds to the location of the screening operator. Using the fact that the $t$ integrals on the right-hand side of Eq. (C4) can be expressed in terms of the hypergeometric function, we obtain Eq. (C3). The integral $I(a,b,c,d,\bar{d})$ of the hypergeometric functions can then be expressed as the following double integral:

$$I(a,b,c;d,\bar{d})=\frac{\gamma(c)}{\pi\gamma(b)\gamma(c-b)}\int d^2u\, d^2t\, u^{d-1}\bar{u}^{\bar{d}-1}|t^{b-1}(u-t)^{c-b-1}(1-t)^{-a}|^2. \tag{C5}$$

It turns out that both $u$ and $t$ integrals can be carried out using the formula

$$\int d^x|x|^{2a}|1-x|^{2b}x^n(1-x)^m=\pi\frac{\Gamma(a+n+1)\Gamma(b+m+1)\Gamma(-a-b-1)}{\Gamma(-a)\Gamma(-b)\Gamma(a+b+m+n+2)}. \tag{C6}$$

A derivation of this formula can be found, for example, in Sec. 7.2 of [42]. Thus we have proven the formula (C1).

## APPENDIX D: CONSTRAINTS ON WINDING NUMBER VIOLATION

We have seen in [1] that representations of the $SL(2,R)$ current algebra are parameterized in terms of an integer $w$. For long strings, this integer could be interpreted as the winding number of the long string. For short strings, it is just a parameter of states with no obvious semiclassical interpretation.

Let us clarify the meaning of $w$ for the short string. The short string wave function, when expanded at large $\rho$, has components on all winding numbers. An explicit discussion of this in an expansion around $\rho=0$ can be found in [1]. By an abuse of notation, we will still call $w$ the winding number of short strings, but it should be kept in mind that it is *not* the winding number in the semiclassical sense. It is not even the winding number of the largest component of the wave function at infinity. For example, when $k$ is large, the wave function for a $w=0$ state can be expanded at large $\rho$ as

$$\Psi=e^{-2j\rho}\psi_0+e^{-2(k/2-j)\rho}\psi_1+\cdots, \tag{D1}$$

where we separated the radial dependence, and the indices on $\psi_0,\psi_1,\ldots$ indicate the actual winding numbers at $\rho=\infty$. As $j\to k/2$, we see that the second term with winding number 1 becomes more dominant even though we are still studying the wave function with $w=0$. This second component of the wave function is responsible for giving the divergences in the two- and three-point functions, which we discussed in Sec. II. The winding number has a semiclassical meaning at infinity. However, since the circle is contractible, we do not expect that it should be conserved. In fact, it is not. We find, however, that there is an interesting pattern in winding number violations. It essentially says that the possible amounts of winding violation are restricted by the number of operators in a way that we will make precise below. This was first observed in [20]. Below, we will make a precise statement, and we will prove it using the properties of the representations of the $SL(2,R)$ current algebra.

Let us work in the $m$ basis. The states are labeled by $|d,\tilde{j},w\rangle$ and $|c,\tilde{j},w\rangle$, as well as some $m$ that we do not indicate since it will not be important in what follows. Here the letters *d,c* indicate discrete or continuous representations. We will think of $d$ as $d^+$ and we construct $d^-$ by considering $d^+$ with $w<0$. The winding number $w$ can have any sign. The sign of $w$ distinguishes an incoming states and an outgoing state in the Lorentzian picture. The sign of $m$ is correlated with the sign of $w$.[32] These representation are such that there is a ''lowest weight'' state that obeys the conditions

$$J^+_{w+n}|d,\tilde{j},w\rangle=J^-_{-w+n-1}|d,\tilde{j},w\rangle=0,$$
$$J^+_{w+n}|c,\tilde{j},w\rangle=J^-_{-w+n}|c,\tilde{j},w\rangle=0, \tag{D2}$$
$$n\geqslant 1.$$

[32]This is true in our case, but it might not be true in some quotients of $AdS_3$ [43].

All states in the representation can be generated by acting with the generators that do not annihilate the states. Furthermore, for operators with $j$ in the physical ranges for continuous and discrete representations, there are no null states in the representation.

Now we will consider the following state:

$$\prod_{i=1}^{n_d}\Phi^d_{w_i}(z_i)\prod_{j=1}^{n_c}\Phi^c_{w_j}(z_j)|0\rangle, \tag{D3}$$

where $n_d,n_c$ is the number of continuous and discrete representations. The state $|0\rangle$ does not quite make sense, but after we act with any of the operators, we get a state that does make sense. Now we want to consider the state (D3) and decompose it into representations of $SL(2,R)$. For this we pick a circle $|z|=A$ sufficiently large so that all the points where the operators are inserted are left inside the circle. We consider $SL(2,R)$ generators that are defined by integrating the $SL(2,R)$ currents on this contour times appropriate powers of $z$. In other words, $J^{\pm}_n\sim\oint dz\,J^{\pm}(z)z^n$. Now let us show that some combination of the form $J^P=J^+_a+c_1J^+_{a-1}+\cdots$ annihilates the state (D3). The precise combination is

$$J^P=\oint dz\prod_{i=1}^{n_t}(z-z_i)^{w_i+1}J^+(z), \tag{D4}$$

where $n_t=n_c+n_d$. We see that $a=\Sigma w_i+n_t$. We see that this combination annihilates the state (D3) after using Eq. (D2). We can now decompose Eq. (D3) into $SL(2,R)$ representations with definite $w$. This implies that Eq. (D4) will annihilate each of the states with definite winding number independently. Now we will show that this implies that the state will carry a winding number less than or equal to $a-1=\Sigma w_i+n_t-1$. Suppose that there was a state with winding number $a$. Then Eq. (D4) would annihilate it. But, on the other hand, we know that all operators in Eq. (D4) act as creation operators on the Fock space due to Eq. (D2). Since there are no null states in the representation, we conclude that this cannot happen. To be more precise, let us expand the hypothetical state with winding number $w\geqslant a$ in such a way that we fix $J^3_0$ and we look at the state with fixed $J^3_0$ with a minimum value of $L_0$ (though $L_0$ is not bounded below, it is bounded below if we consider fixed $J^3_0$). Let us denote this state by $|h\rangle$. It is clear that $J^+_a|h\rangle=0$ since there is no other state with which it could mix. This is inconsistent with the idea that there are no null vectors. Therefore, the state must have a winding number less than or equal to $a-1$.

Now we can similarly form the combination $J^N=J^-_b+c_1J^-_{b-1}+\cdots$, which annihilates the state. The precise combination is

$$J^N=\oint dz\prod_{i=1}^{n_d}(z-z_i)^{-w_i}\prod_{j=1}^{n_c}(z-z_j)^{-w_j+1}J^-(z) \tag{D5}$$

so that $b=-\Sigma w_i+n_c$. We see using Eq. (D2) that Eq. (D5) annihilates Eq. (D3). Now we show that the total winding number of the state should be bigger than $-b$. Suppose, to the contrary, that the winding number of the state is smaller than or equal to $-b$. Then Eq. (D2) implies as above that $J_b^-$ will annihilate at least one state. Actually, the precise statement will depend on whether the state we consider is discrete or continuous. If the state is discrete, then the statement is a bit weaker, so $w$ should be bigger than $-b-1$.

If we expand Eq. (D3) in irreducible representations of the SL(2,$R$) current algebra, it becomes a sum of discrete and continuous states whose winding numbers are restricted as

$$\begin{aligned} -n_c+1 &\leqslant w-\sum w_i \leqslant n_t-1, \quad \text{continuous,} \\ -n_c &\leqslant w-\sum w_i \leqslant n_t-1, \quad \text{discrete.} \end{aligned} \tag{D6}$$

In terms of correlation functions of operators, we need to take the inner product of Eq. (D3) with $\langle 0|\Phi(z)$, where $\Phi$ could be a discrete or continuous representation. Notice that, in our conventions, when we take the adjoint of a discrete representation we take $w \to -1-w$ while for a continuous representation we take $w \to -w$. We conclude that correlators will obey the winding number violation rule

$$\begin{aligned} -N_t+2 &\leqslant \sum w_i \leqslant N_c-2, \quad \text{at least one continuous,} \\ -N_d+1 &\leqslant \sum w_i \leqslant -1, \quad \text{all discrete,} \end{aligned} \tag{D7}$$

where now $N_t = N_c + N_d$, and $N_c, N_d$ is the total number of operators in the continuous and the discrete representations appearing in the correlation function. Note that throughout this discussion, we were thinking of the correlators in the $m$ basis, and the discrete states were taken with $\tilde{m} = \pm \tilde{j}$.

Now let us consider the operators in the $x$ basis. The labels $w_i$ of all operators can be taken to be non-negative. In that case, it is easy to show that in an $N$-point function the winding numbers should obey

$$w_i - \sum_{j\neq i} w_j \leqslant N-2. \tag{D8}$$

Note that an operator $\mathcal{O}^w(x,z)$ obeys simple OPE expansion rules for the currents $J^a(x,z) = e^{xJ_0^+} J^a(z) e^{-xJ_0^+}$ (see [29]). Since $J^+(x,z) = J^+(z)$, the analysis done with the operator (D4) goes through as before and leads to (D8) if we put the $i$th operator at $z = x = \infty$. This shows that for a three-point function, the winding violation is only by one unit, so that the correlation function of two discrete $w=0$ states with any state with $w>1$ vanishes in $x$ space.

## APPENDIX E: ANOTHER DEFINITION OF THE SPECTRAL FLOWED OPERATORS

In Sec. V, we defined the operator corresponding to the spectral flowed representation by (i) starting with the operator $\Phi_{j,\bar{j}}(x,\bar{x})$ in the regular representation in the $x$ basis, (ii) going to the $m$ basis by the integral transform (5.4), (iii) multiplying $e^{w(k/2)\sqrt{(2/k)}\varphi}$ with $J^3 = i\sqrt{(k/2)}\partial\varphi$ as in Eq. (5.10), and (iv) going back to the $x$ basis to obtain expressions such as in Eqs. (5.18) and (5.40).

Here we will describe a way to define the spectral flowed operator $\Phi_{J,\bar{J}}^{w,j}(x)$ without going through the $m$ basis. This approach has an advantage that we do not have to deal with the infinite factor $V_{\rm conf}$ as we did in Sec. V. We will compute the two- and three-point functions including $\Phi_{J,\bar{J}}^{w,j}(x)$, and show that they agree with the results in Sec. V when $w=1$.

### 1. Definition in the $x$ basis

The definition, in the case of $w=1$, is given by the fusion of $\Phi_j$ with the spectral flow operator $\Phi_{k/2}$ as

$$\begin{aligned} \Phi_{J,\bar{J}}^{w=1,j}(x,z) \equiv \lim_{\epsilon\to 0} \epsilon^m \bar{\epsilon}^{\bar{m}} \int d^2y\, y^{j-m-1} \bar{y}^{j-\bar{m}-1} \\ \times \Phi_j(x+y, z+\epsilon) \Phi_{k/2}(x,z), \end{aligned} \tag{E1}$$

where $J = m + (k/2)$ and $\bar{J} = \bar{m} + (k/2)$. This equality is understood to hold inside of any correlation functions.

First we need to show that the limit $\epsilon \to 0$ in Eq. (E1) exists, i.e., the result of the $y$ integral scales as $\epsilon^{-m}\bar{\epsilon}^{-\bar{m}}$ for small $\epsilon$. We will prove this for a correlation function where there are at least two more operators besides $\Phi_{J,\bar{J}}^{w,j}(x)$. There is a subtlety with the argument when there is only one additional operator in the correlation function, i.e., when we consider a two-point function including $\Phi_{J,\bar{J}}^{w=1,j}$. This does not cause a problem since $\Phi_{J,\bar{J}}^{w=1,j}$ has a nonzero two-point function only with another operator with $w=1$, which actually is a composite of two operators as in Eq. (E1). In fact, we will be able to compute the two-point function using Eq. (E1).

For simplicity, we set $x=0$ and $z=0$ and consider a correlation function

$$\mathcal{F} = \langle \Phi_{j_1}(x_1,z_1)\cdots\Phi_{j_N}(x_N,z_N)\Phi_j(y,\epsilon)\Phi_{k/2}(0,0)\rangle. \tag{E2}$$

For $|y| \ll |x_i|$ and $|\epsilon| \ll |z_i|$ $(i=1,\ldots,N)$, with finite $\epsilon/y$, we can show that this behaves as

$$\mathcal{F} \sim \epsilon^j (\epsilon - y)^{-2j-\mathcal{D}} y^{\mathcal{D}} f(x_3,\ldots,x_N; z_3,\ldots,z_N), \tag{E3}$$

where $\mathcal{D}$ is a differential operator acting on $x_3,\ldots,x_N$. We have set $(x_1,z_1)=(1,1)$ and $(x_2,z_2)=(\infty,\infty)$ by using the SL(2,$C$) symmetries on both worldsheet and the target space. For $N=2$, we can check this explicitly by using the formula (5.30) for the four-point function with the spectral flow operator. (In this case, $\mathcal{D}$ is a number depending on $j, j_1, j_2$.) This can be generalized for any correlation function with $N \geqslant 2$ as follows. The spectral flow operator $\Phi_{k/2}$ obeys the null state condition

$$J^-(z)\Phi_{k/2}(x,z) = 0. \tag{E4}$$

Using this, the KZ equation is simplified as

$$\frac{\partial}{\partial z}\Phi_{k/2}(x,z) = -J^3(z)\Phi_{k/2}(x,z). \tag{E5}$$

Let us evaluate the KZ equation in the correlation function (E2). When $|\epsilon| \ll |z_1|,\ldots,|z_N|$, we can ignore the operator product singularities of $J^3(z)$ at $z=0$ with the operators at

$z_1,\ldots,z_N$, and we only have to consider the contribution from $\Phi_j(y,\epsilon)$. We then find that the KZ equation (E5) leads to

$$\frac{\partial}{\partial\epsilon}\mathcal{F}=-\frac{1}{\epsilon}\left(y\frac{\partial}{\partial y}+j\right)\mathcal{F}. \tag{E6}$$

To evaluate the null state condition (E4) in the limit of our interest, we need to use the global SL$(2,C)$ invariance of $\mathcal{F}$ to turn derivatives with respect to $x_i$, for example $\partial_{x_1}$ and $\partial_{x_2}$, into a derivative with respect to $y$. This is where we need to assume that there are at least two more operators in the correlation function. Setting $(x_1,z_1)=(1,1)$ and $(x_2,z_2)=(\infty,\infty)$ after this procedure, and taking the limit $\epsilon,y\to 0$ keeping $\epsilon/y$ finite, we find that the null state condition (E4) leads to the equation

$$\left\{\frac{y}{\epsilon}\left(-(\epsilon-y)\frac{\partial}{\partial y}+2j\right)+\mathcal{D}\right\}\mathcal{F}=0, \tag{E7}$$

with some differential operator $\mathcal{D}$ acting on $x_3,\ldots,x_n$. Here $\epsilon^{-1}y^2\partial_y$ acting on $\mathcal{F}$ comes from the operator product expansion of $J^-(0,0)$ with $\Phi_j(y,\epsilon)$, and the other terms are obtained from $J^-(0,0)$ with $\Phi_{j_1}(x_1,z_1)\cdots\Phi_{j_N}(x_N,z_N)$ and by converting $\partial_{x_i}$'s into $\partial_y$ by using the SL$(2,C)$ invariance in the target space. We can then show that a general solution to Eqs. (E6) and (E7) is given by Eq. (E3) (besides the contact term solution discussed in the footnote later).

Now we can estimate the $y$ integral in Eq. (E1). From the discussion in the above paragraph, we see that the product of the operators $\Phi_j(y,\epsilon)\Phi_{k/2}(0,0)$ can be expanded, in the leading order in $\epsilon\to 0$, as

$$\langle\Phi_j(y,\epsilon)\Phi_{k/2}(0,0)\cdots\rangle\sim|\epsilon^j(\epsilon-y)^{-2j-\mathcal{D}}y^{\mathcal{D}}|^2\sum_{n,\bar{n}=0}^{\infty}f_{n,\bar{n}}(x_3,\ldots,x_N)y^n\bar{y}^{\bar{n}} \tag{E8}$$

for some operators $\mathcal{O}_{n,\bar{n}}$. The $y$ integral for each term in the expansion (E8) can then be estimated as

$$|\epsilon|^{2j}\int d^2y\, y^{j-m-1+\mathcal{D}+n}\bar{y}^{j-\bar{m}-1+\mathcal{D}+\bar{n}}|\epsilon-y|^{-2(2j+\mathcal{D})}\sim\epsilon^{-m+n}\bar{\epsilon}^{-\bar{m}+\bar{n}}, \tag{E9}$$

where we assumed $m-\bar{m}\in\mathbf{Z}$. Thus the limit $\epsilon\to 0$ in Eq. (E1) is well defined. Only the $n=\bar{n}=0$ survives in the limit. Note that, although the differential operator $\mathcal{D}$ has dropped out from the exponent of $\epsilon$, there is a product of $\Gamma$ functions whose arguments include $\mathcal{D}$. When this operator acts on the finite term left over, it modifies its $z_i$ and $x_i$ dependence for $i=3,\ldots,N$, but does give rise to additional $\epsilon$ dependence.

Next we need to show that the operator defined by Eq. (E1) is indeed in the flowed representation. We do this by checking that it has the correct OPE with the SL$(2,R)$ currents. To show this, we start with the standard operator product expansion for operators with $w=0$,

$$J(x',z')\Phi_j(x+y,z+\epsilon)\Phi_{k/2}(x,z)=\left\{\frac{1}{z'-z-\epsilon}\left[(x+y-x')^2\frac{\partial}{\partial y}+2j(x+y-x')\right]+\frac{1}{z'-z}\left[(x-x')^2\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)+k(x-x')\right]\right\}\Phi_j(x+y,z+\epsilon)\Phi_{k/2}(x,z). \tag{E10}$$

Applying this to Eq. (E1) and performing the integration by parts in $y$, we obtain

$$\begin{aligned}J(x',z')\Phi^{w=1,j}_{J,\bar{J}}(x,z)&=\lim_{\epsilon,\bar{\epsilon}\to 0}\epsilon^m\bar{\epsilon}^{\bar{m}}\int d^2y\, y^{j-m-1}\bar{y}^{j-\bar{m}-1}\left\{\frac{1}{z'-z-\epsilon}[-(j-m-1)y^{-1}(x+y-x')^2+(2j-2)(x+y-x')]\right.\\&\quad\left.+\frac{1}{z'-z}\left[(x-x')^2\frac{\partial}{\partial x}+(j-m-1)y^{-1}(x-x')^2+k(x-x')\right]\right\}\Phi_j(x+y,z+\epsilon)\Phi_{k/2}(x,z)\\&=\lim_{\epsilon,\bar{\epsilon}\to 0}\epsilon^m\bar{\epsilon}^{\bar{m}}\int d^2y\, y^{j-m-1}\bar{y}^{j-\bar{m}-1}\left\{-\frac{j-m-1}{(z'-z)^2}\frac{\epsilon}{y}(x-x')^2\right.\\&\quad\left.+\frac{1}{z'-z}\left[(x-x')^2\frac{\partial}{\partial x}+2\left(m+\frac{k}{2}\right)(x-x')\right]\right\}\Phi_j(x+y,z+\epsilon)\Phi_{k/2}(x,z)\\&=-(j-m-1)\frac{(x-x')^2}{(z'-z)^2}\Phi^{w=1,j}_{J+1,\bar{J}}(x,z)+\frac{1}{z'-z}\left[(x-x')^2\frac{\partial}{\partial x}+2\left(m+\frac{k}{2}\right)(x-x')\right]\Phi^{w=1,j}_{J,\bar{J}}(x,z).\end{aligned} \tag{E11}$$

This means that the corresponding state

$$|w=1,j;J,\overline{J}\rangle=\Phi^{w=1,j}_{J,\overline{J}}(x=0,z=0)|0\rangle \tag{E12}$$

obeys

$$J^3_0|w=1,j;J,\overline{J}\rangle=\left(m+\frac{k}{2}\right)|w=1,j;J,\overline{J}\rangle,$$
$$J^3_n|w=1,j;J,\overline{J}\rangle=0,J^{\pm}_{n\pm 1}|w=1,j;J,\overline{J}\rangle=0 \quad (n=1,2,\dots). \tag{E13}$$

This is the correct highest weight condition for a state with $w=1$.

### 2. Three-point function

Now we can use the definition (E1) to compute correlation functions with spectral flowed states. First let us study the three-point function. We start with the four-point function with a spectral flow operator,

$$\langle\Phi_{j_1}(x_1)\Phi_{k/2}(x_2)\Phi_{j_3}(x_3)\Phi_{j_4}(x_4)\rangle=|z_{43}|^{2(\Delta_2+\Delta_1-\Delta_4-\Delta_3)}|z_{42}|^{-4\Delta_2}|z_{41}|^{2(\Delta+\Delta_2-\Delta_4-\Delta_1)}|z_{31}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}|z|^{2j_1}|1-z|^{2j_3}$$
$$\times|x_{43}|^{2(k/2+j_1-j_4-j_3)}|x_{42}|^{-2k}|x_{41}|^{2(j_3+k/2-j_4-j_1)}|x_{31}|^{2(j_4-j_1-k/2-j_3)}\times B(j_1)C(k/2$$
$$-j_1,j_3,j_4)|z-x|^{2(-j_1-j_3-j_4+k/2)}|x|^{2(-j_1+j_3+j_4-k/2)}|x-1|^{2(j_1-j_3+j_4-k/2)}. \tag{E14}$$

Setting $x_1=x_2+w$,

$$x=\frac{x_{21}x_{43}}{x_{31}x_{42}}=\frac{wx_{43}}{(w-x_{32})x_{42}},$$
$$1-x=\frac{(w-x_{42})x_{32}}{(w-x_{32})x_{42}}, \tag{E15}$$
$$z-x=\frac{(zx_{42}-x_{43})w-zx_{32}x_{42}}{(w-x_{32})x_{42}}.$$

Substituting this into Eq. (E14), we find

$$\langle\Phi_{j_1}(x_1)\Phi_{k/2}(x_2)\Phi_{j_3}(x_3)\Phi_{j_4}(x_4)\rangle=|z_{43}|^{2(\Delta_2+\Delta_1-\Delta_4-\Delta_3)}|z_{42}|^{-4\Delta_2}|z_{41}|^{2(\Delta_3+\Delta_2-\Delta_4-\Delta_1)}|z_{31}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}$$
$$\times|z|^{2j_1}|1-z|^{2j_3}B(j_1)C(k/2-j_1,j_3,j_4)|x_{42}|^{2(j_1+j_3-j_4-k/2)}|x_{32}|^{2(j_1-j_3+j_4-k/2)}$$
$$\times|w|^{2(-j_1+j_3+j_4-k/2)}|(zx_{42}-x_{43})w-zx_{32}x_{42}|^{-j_1-j_3-j_4+k/2}. \tag{E16}$$

We then multiply the factor $|w|^{2(j_1-m_1-1)}$ and integrate over $w$. We find

$$\int dw^2|w|^{2(j_1-m_1-1)}\langle\Phi_{j_1}(x_1)\Phi_{k/2}(x_2)\Phi_{j_3}(x_3)\Phi_{j_4}(x_4)\rangle$$
$$=(\text{standard powers of } z_i)B(j_1)C(k/2-j_1,j_3,j_4)|x_{42}|^{2(j_1+j_3-j_4-k/2)}|x_{32}|^{2(j_1-j_3+j_4-k/2)}$$
$$\times\int d^2w|w|^{2(j_3+j_4-m_1-k/2-1)}|(zx_{42}-x_{43})w-x_{32}x_{42}|^{-j_1-j_3-j_4+k/2}$$
$$=|z_{43}|^{2(\Delta_2+\Delta_1-\Delta_4-\Delta_3)}|z_{42}|^{-4\Delta_2}|z_{41}|^{2(\Delta_3+\Delta_2-\Delta_4-\Delta_1)}|z_{31}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}|1-z|^{2j_3}|z|^{-2m_1}B(j_1)C(k/2-j_1,j_3,j_4)$$
$$\times|x_{42}|^{2(j_3-j_4-m_1-k/2)}|x_{32}|^{2(-j_3+j_4-m_1-k/2)}|zx_{42}-x_{43}|^{2(-j_3-j_4+m_1+k/2)}$$
$$\times\pi\frac{\Gamma(j_3+j_4-m_1-k/2)\Gamma(-j_1-j_3-j_4+k/2+1)\Gamma(j_1+m_1)}{\Gamma(1-j_3-j_3+m_1+k/2)\Gamma(j_1+j_3+j_4-k/2)\Gamma(1-j_1-m_1)}. \tag{E17}$$

Now we multiply by $|z_{21}|^{2m_1}$ and send $z_{21}\to 0$. We find

$$\lim_{z_{21}\to 0}|z_{21}|^{2m_1}\int dw^2|w|^{2(j_1-m_1-1)}\langle\Phi_{j_1}(x_1)\Phi_{k/2}(x_2)\Phi_{j_3}(x_3)\Phi_{j_4}(x_4)\rangle$$
$$=B(j_1)C(k/2-j_1,j_3,j_4)\pi\frac{\Gamma(j_3+j_4-J)}{\Gamma(1-j_3-j_4+J)}\frac{\Gamma(j_1+J-k/2)}{\Gamma(1-j_1-J+k/2)}\frac{1}{\gamma(j_3+j_4+j_1-k/2)}$$
$$\times|x_{42}|^{2(j_3-j_4-J)}|x_{32}|^{2(j_4-j_3-J)}|x_{43}|^{2(J-j_3-j_4)}|z_{43}^{\hat\Delta_1-\Delta_3-\Delta_4}z_{42}^{\Delta_3-\hat\Delta_1-\Delta_4}z_{32}^{\Delta_4-\hat\Delta_1-\Delta_3}|^2, \tag{E18}$$

where

$$\hat\Delta_1=\Delta(j_1)-m_1-\frac{k}{4},\quad J=m_1+\frac{k}{2}. \tag{E19}$$

Due to the limit in Eq. (E18), we can neglect higher powers of $z$ appearing at various places.

The result (E18) is in agreement with Eq. (5.38), which we computed by going through the $m$ basis.[33] We should point out that the factor $1/V_{\rm conf}$ in Eq. (5.38) is absent in Eq. (E18). Thus the definition (E1) includes the rescaling $\Phi\to\hat\Phi=V_{\rm conf}\Phi$ that we performed for the long string.

### 3. Worldsheet two-point function

To compute the two-point function with spectral flowed operators, we start with the four-point function with two insertions of spectral flow operators, say $j_2=j_4=k/2$. The KZ equation and the null state conditions imply that $j_1=j_3$. To see this, we notice that the four-point function should be symmetric under $2\leftrightarrow 4$, leaving 1 and 3 unchanged. This changes $z\to 1-z$, $x\to 1-x$. Taking into account also the prefactors, we find

$$\frac{4\mathrm{pt}(1,2,3,4)}{4\mathrm{pt}(1,4,3,2)}=\left|\left(\frac{z}{1-z}\right)^{\Delta_1-\Delta_3+j_1-j_3}\left(\frac{1-x}{x}\right)^{j_1-j_3}\right|^2. \tag{E20}$$

Demanding that this is 1, we find $j_1=j_3$.[34]

[33] Note that $m$ in Eq. (E19) is $-m$ in Eq. (5.39).

[34] A solution with $j_1=1-j_3$ appears to come from a contact term for the four-point function. In fact, the function $z^{-j_1}\delta^2(x-z)$ is a solution to Eqs. (E6) and (E7), with $j_3=1-j_1$ and $\mathcal{D}=1-2j_1$, which is the value that appears in the four-point function equation when $j_2=j_4=k/2$. [Note that $\delta^2(x-z)$ is not a standard contact term, for $x=z$ is not a coincidence limit of two operators. If we use the relation between the four-point function $\mathcal{F}_{\mathrm{SL}(2)}(z,x)$ in the $\mathrm{SL}(2,C)/\mathrm{SU}(2)$ coset model and a five-point function in the Liouville model, recently pointed out in [33], one can interpret $\delta^2(x-z)$ as a contact term coming from the coincidence limit involving the extra operator one inserts in the Liouville model. It would be interesting to find a direct interpretation of such a contact term in the $\mathrm{SL}(2,C)/\mathrm{SU}(2)$ model.] Inserting $z^{-j_1}\delta^2(x-z)$ into the $x$ integral we describe below and doing the same change of variables, we see that we recover the term proportional to $\delta(j_1+j_2-1)$ in the two-point function.

Now let us apply Eq. (E1) to extract the two-point function of the spectral flowed state. As we explained in the above, we expect $j_1=j_3$ from the null vector equations. In fact, the factor $C(k/2-j_1,j_3,j_4)$ in Eq. (E14) with $j_4=k/2$ vanishes for $j_1\neq j_3$ and is infinite at $j_1=j_3$. We can regularize the infinity by slightly modifying the spectral flow operator as $k/2\to k/2+i\epsilon$. Indeed, in the limit $\epsilon\to 0$, we recover the $\delta$ function enforcing $j_1=j_3$,

$$C\left(\frac{k}{2}-j_1,j_3,\frac{k}{2}\right)$$
$$=(k\text{-dependent coefficient})\,\delta(j_1-j_3). \tag{E21}$$

Thus the four-point function in this case reduces to

$$\langle\Phi_j(x_1,z_1)\Phi_{k/2}(x_2,z_2)\Phi_j(x_3,z_3)\Phi_{k/2}(x_4,z_4)\rangle$$
$$=|z_{42}^{k/2}z_{31}^{-2\Delta}z^j(1-z)^jx_{42}^{-k}x_{31}^{-2j}(z-x)^{-2j}|^2$$
$$\times B(j)\,\delta(j-j'), \tag{E22}$$

where we ignored a $k$-dependent overall coefficient. Now we set $x_1=w_1+x_2$, $x_3=w_3+x_4$, multiply by $|w_1|^{2(j-m_1-1)}|w_3|^{2(j-m_3-1)}$, and integrate over $w_1$ and $w_3$. It is convenient to introduce new variables $u_1$ and $u_3$ defined by $w_i=x_{42}u_i$, $i=1,3$. The integrated correlation function becomes

$$|z|^{2j}\int d^2u_1d^2u_2$$
$$\times|u_1^{j-m_1-1}u_2^{j-m_3-1}[u_1u_3-z(u_{31}+1)]^{-2j}|^2, \tag{E23}$$

where we set $z=0$ in the term with $(1-z)$ since we are going to be interested in the small-$z$ behavior of Eq. (E22). Here we omitted the standard factors of $x_{42}$ and $z_{42}$. It is convenient to change the integration variables as $u_1=\sqrt{s}y$, $u_3=\sqrt{s}y^{-1}$. After rescaling $s=zt$, we find that (E23) goes as

$$|z^{-1/2(m_1+m_3)}|^2\int d^2y\,d^2s|y^{m_3-m_1-1}|^2$$
$$\times|s^{j-(1/2)(m_1+m_3)-1}[s+\sqrt{sz}(y-y^{-1})-1]^{-2j}|^2. \tag{E24}$$

Since we are interested in the leading term in the $z$ expansion, we set $z=0$ in the last factor. The integral over $y$ then gives $\delta^2(m_1-m_3)$, and the integral over $s$ gives a combination of $\Gamma$ functions,

$$2\pi\frac{\Gamma(j-\bar{m}_1)\Gamma(j+m_1)}{\gamma(2j)\Gamma(1-j+\bar{m}_1)\Gamma(1-j-m_1)}. \tag{E25}$$

Combining this with the factor $B(j)\delta(j_1-j_3)$ in Eq. (E22), we have reproduced the expression for the two-point function in Eq. (5.13).

Finally, let us note that, instead of the definition (E1), we could also define the spectral flowed operator via

$$\Phi^{w=1,j}_{J,\bar{J}}(x,z)\equiv\lim_{y\to 0} y^{j-m}\bar{y}^{j-\bar{m}}\int d^2\epsilon\,\epsilon^{m-1}\bar{\epsilon}^{\bar{m}-1}\times\Phi_j(x+y,z+\epsilon)\Phi_{k/2}(x,z), \tag{E26}$$

for $J=m+(k/2)$. Instead of integrating over $y$, here we are taking an integral over $\epsilon$. In this definition of the flowed operator, the expression is manifestly local in $x$. On the other hand, the definition (E1) is manifestly local in $z$. In order to show that the two definitions are equivalent, we note that the relevant part of the correlation function behaves as Eq. (E9). Then, with the previous definition in Eq. (E1), we find that the spectral flowed correlator goes as

$$\int d^2w|w^{j-m-1+\mathcal{D}}(1-w)^{-2j-\mathcal{D}}|^2 \tag{E27}$$

after we rescale $w\to zw$ and taking the $z\to 0$ limit. Similarly from Eq. (E26), we obtain

$$\int d^2t|t^{j+m-1}(1-t)^{-2j-\mathcal{D}}|^2 \tag{E28}$$

after rescaling $z=xt$ and and taking the $x\to 0$ limit. We see that after the change of variables $t=1/w$, the two integrals become the same. This shows that the two definitions (E1) and (E26) give the same results in general.

#### 4. Target space two-point function

Let us turn to the target space two-point function for the state with $w=1$. We apply the method used in Appendix A for $w=0$ and use the Ward identity to determine the normalization of the two-point function. We start with the following identity for the three-point function:

$$\begin{aligned}
&\langle\Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\Phi^{w=1,j_2}_{J_2,\bar{J}_2}(x_2,z_2)J(x_3,z_3)\Phi_1(x_3,z_3)\rangle\\
&\quad=-(j_1-m_1-1)\frac{(x_3-x_1)^2}{(z_3-z_1)^2}\langle\Phi^{w=1,j_1}_{J_1+1,\bar{J}_1}(x_1,z_1)\Phi^{w=1,j_2}_{J_2,\bar{J}_2}(x_2,z_2)\Phi_1(x_3,z_3)\rangle-(j_2-m_2-1)\\
&\qquad\times\frac{(x_3-x_2)^2}{(z_3-z_2)^2}\langle\Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\Phi^{w=1,j_2}_{J_2+1,\bar{J}_2}(x_2,z_2)\Phi_1(x_3,z_3)\rangle+\left\{\frac{1}{z_3-z_1}\left[(x_1-x_3)^2\frac{\partial}{\partial x_1}+2\left(m_1+\frac{k}{2}\right)\right]+\frac{1}{z_3-z_2}\right.\\
&\qquad\left.\times\left[(x_2-x_3)^2\frac{\partial}{\partial x_2}+2\left(m_2+\frac{k}{2}\right)\right]\right\}\langle\Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\Phi^{w=1,j_2}_{J_2,\bar{J}_2}(x_2,z_2)\Phi_1(x_3,z_3)\rangle.
\end{aligned} \tag{E29}$$

We then have to compute the three-point functions on the right-hand side of this equation. We start with the expression for the five-point function with two spectral flow operators, obtained in [20],

$$\begin{aligned}
&\langle\Phi_{k/2}(x_1,z_1)\Phi_{k/2}(x_2,z_2)\Phi_{j_1}(y_1,\zeta_1)\Phi_{j_2}(y_2,\zeta_2)\Phi_1(y_3,\zeta_3)\rangle\\
&\quad=B(j_1)B(j_2)C\left(\frac{k}{2}-j_1,\frac{k}{2}-j_2,1\right)|(x_1-x_2)^{j_1+j_2+1-k}\mu_1^{j_1-j_2-1}\mu_2^{j_2-j_1-1}\mu_3^{1-j_1-j_2}|^2,
\end{aligned} \tag{E30}$$

where

$$\mu_i=\frac{(x_1-y_{i+1})(x_2-y_{i+2})}{(z_1-\zeta_{i+1})(z_2-\zeta_{i+2})}-\frac{(x_1-y_{i+2})(x_2-y_{i+1})}{(z_1-\zeta_{i+2})(z_2-\zeta_{i+1})}. \tag{E31}$$

We have neglected the $z$- and $\zeta$-dependent factors. When $j_1=j_2$, the factor $B^2C$ in Eq. (E30) is equal to $B(j_1)$ up to a $k$-dependent factor as

$$C\left(\frac{k}{2}-j,\frac{k}{2}-j,1\right)=\frac{G(2j-k)}{2\pi^2\nu^{k-2j}\gamma\left(\frac{k-1}{k-2}\right)G(1+2j-k)}=\left(\frac{k-2}{2\pi}\right)^2\nu^{2-k}\frac{1}{B(j)}. \tag{E32}$$

We apply Eq. (E1) to Eq. (E30) and integrate over $\zeta_1$ and $\zeta_2$. It is convenient to set $x_1=z_1=0$, $x_2=z_2=1$, $y_3=\zeta_3=\infty$, $y_1=u\zeta_1$, $y_2-1=v(\zeta_2-1)$, and take $\zeta_1$, $1-\zeta_2$ to be small. In this limit we find

$$\mu_1=1-\nu, \quad \mu_2=1-u, \quad \mu_3=uv-1. \tag{E33}$$

The integral we need to evaluate, in order to compute $\langle \Phi^{w=1,j_1}_{J_1,\bar{J}_1}\Phi^{w=1,j_2}_{J_2,\bar{J}_2}\Phi_1\rangle$ is then

$$\int du^2dv^2u^{j_1-m_1-1}\bar{u}^{j_1-\bar{m}_1-1}v^{j_2-m_2-1}\bar{v}^{j_2-\bar{m}_2-1}|(u-1)^{j_1-j_2-1}(v-1)^{j_2-j_1-1}(uv-1)^{1-j_1-j_2}|^2. \tag{E34}$$

Let us consider the case of the long string. We then have $j_a=\frac{1}{2}+is_2$, and the integral gives a $\delta$-function singularity at $s_1=s_2$ coming from the region of the integral of $u\sim 1$ or $v\sim 1$. The term proportional to the $\delta$ function can be evaluated as

$$\begin{aligned}\delta(j_1-j_2)&\left(\int du^2u^{j_1-m_1-1}\bar{u}^{j_1-\bar{m}_1-1}|(u-1)^{-2j_1}|^2+(m_1,\bar{m}_1\to m_2,\bar{m}_2)\right)\\ &=\delta(j_1-j_2)\frac{\pi}{\gamma(2j_1)}\left(\frac{\Gamma(j_1-m_1)\Gamma(j_1+\bar{m}_1)}{\Gamma(1-j_1-m_1)\Gamma(1-j_1+\bar{m}_1)}+(m_1,\bar{m}_1\to m_2,\bar{m}_2)\right).\end{aligned} \tag{E35}$$

Since

$$J=\frac{k}{4}+\frac{\frac{1}{4}+s^2}{k-2}+h-1, \tag{E36}$$

the delta function $\delta(s_1-s_2)$ together with the condition $h_1=h_2$ in the internal CFT implies $J_1=J_2$ and therefore $m_1=m_2$. The correlator multiplying the double pole term in Eq. (E29) then gives

$$\begin{aligned}&\int du^2dv^2u^{j_1-m-2}\bar{u}^{j-\bar{m}-1}v^{j_2-m-1}\bar{v}^{j_2-\bar{m}-1}|(u-1)^{j_1-j_2-1}(v-1)^{j_2-j_1-1}(uv-1)^{1-j_1-j_2}|^2\\ &\sim\delta(j_1-j_2)\frac{1}{\gamma(j)}\left(\frac{-j_1-m}{j_1-m-1}+1\right)\frac{\Gamma(j_1-m)\Gamma(j_1+\bar{m})}{\Gamma(1-j_1-m)\Gamma(1-j_1+\bar{m})}\\ &\sim-\delta(j_1-j_2)\frac{1}{\gamma(j_1)}\frac{2m+1}{j_1+m-1}\frac{\Gamma(j_1-m)\Gamma(j_1+\bar{m})}{\Gamma(1-j_1-m)\Gamma(1-j_1+\bar{m})}.\end{aligned} \tag{E37}$$

On the other hand, the correlator multiplying the single pole term in Eq. (E29) gives

$$\delta(j_1-j_2)\frac{2m_1+k}{\gamma(j_1)}\frac{\Gamma(j_1-m_1)\Gamma(j_1+\bar{m}_1)}{\Gamma(1-j_1-m_1)\Gamma(1-j_1+\bar{m}_1)}. \tag{E38}$$

We combine them with Eq. (E32) and multiply by the correlation function $\langle \phi_h(z_1)\phi'_h(z_2)\overline{L}(\overline{z_3})\rangle$ in the internal CFT, as we did in Appendix A, to compute the on-shell three-point function involving the target space $R$ current. We find

$$\begin{aligned}&\langle[\Phi^{w=1,j_1}_{J,\bar{J}}(x_1,z_1)\phi_h(z_1)][\Phi^{w=1,j_2}_{J,\bar{J}}(x_2,z_2)\phi_h(z_2)][J(x_3,z_3)\overline{L}(\overline{z_3})\Phi_1(x_3,z_3)]\rangle\\ &\sim\frac{1}{|z_{12}|^2|z_{23}|^2|z_{31}|^2}\left(\frac{q_1}{\overline{x_3}-\overline{x_1}}+\frac{q_2}{\overline{x_3}-\overline{x_2}}\right)\delta(j_1-j_2)\frac{\Gamma(j_1-m)\Gamma(j_1+\bar{m})}{\gamma(j_1)\Gamma(1-j_1-m)\Gamma(1-j_1-\bar{m})}\frac{B(j_1)}{|x_{12}|^{4,J}},\end{aligned} \tag{E39}$$

where $q_1$ and $q_2$ are the $R$ charges of the two operators. From this, we find that the spacetime two-point function of $\Phi^{w=1,j}_{J,\bar{J}}\phi_h$ is

$$\frac{\Gamma(j_1-m_1)\Gamma(j_1+\bar{m}_1)}{\gamma(j_1)\Gamma(1-j_1-m_1)\Gamma(1-j_1-\bar{m}_1)}B(j_1).$$

We do not have the extra factor of $(2j-1)$ for the long string.

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