Energy of Infinitely Long, Cylindrically Symmetric Systems in General Relativity*

KIP S. THORNE†
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received 30 September 1964; revised manuscript received 7 December 1964)

A definition of energy is proposed for systems invariant under rotations about, and translations along, a symmetry axis. This energy (which is called "cylindrical energy" or "C energy") takes the form of a covariant vector P^i, which obeys the conservation law $P^i{}_{;i}=0$. C energy is localizable and locally measurable: The component of P^i along the world line of an observer is the C-energy density he measures. Near the symmetry axis of a static system, where strong gravitational fields are absent, C-energy density reduces to proper mass density $T_0{}^0$. C energy is propagated by Einstein-Rosen gravitational waves and by cylindrical electromagnetic waves. *In vacuo* and in the presence of electromagnetic fields the C energy on a space-like hypersurface is minimized when the system is static; and the difference between the "potential" part and the "kinetic" part is a Lagrangian for the Einstein-Maxwell field equations. C energy can be a powerful tool in the analysis of finite as well as infinite cylindrically symmetric systems. Here it is used to elucidate the nature of Einstein-Rosen gravitational radiation, and to suggest and support the conjecture of flux resistance to gravitational collapse: In any configuration of electromagnetic fields collapsing toward a singularity, each electric and magnetic-field line is either entirely ejected from the collapsing region or entirely swallowed by it as collapse proceeds; there can be no flux threading a collapsed region.

I. INTRODUCTION

NEARLY fifty years ago Einstein formulated his general theory of relativity. Since that time one of the greatest unsolved puzzles of the theory has been the nature of energy within the framework of relativity. As early as 1918, Einstein[1] gave a successful definition of the total energy of a body residing in an asymptotically flat space-time. Since then, there has been a great proliferation of alternative definitions,[2] most of which yield the same result as Einstein's, provided space-time becomes asymptotically flat sufficiently rapidly at large distances. However, during these first fifty years of relativity theory, nobody has successfully defined the total energy of a closed universe or of any other system around which space-time is *not* asymptotically flat; nor has anybody formulated an adequate definition of localized energy density within the framework of general relativity. Tentative definitions of these concepts have been given by Bergmann, Møller, and Komar,[2] but these have all been found wanting in some respect. It is often suggested (but nobody has demonstrated) that these concepts should not be well defined in relativity theory.

The purpose of this paper is to propose and justify a definition of localized energy density and total energy for a particular class of systems around which space-time is not asymptotically flat. These are systems with "whole-cylinder symmetry," i.e., systems invariant under rotation about and translation along a symmetry axis and under reflection in any plane containing the symmetry axis or perpendicular to it. We shall call our energy-like quantity "cylindrical energy," or *C energy*, to distinguish it from the multitude of energy-like quantities which have been discussed in the past.

C energy is defined in terms of a *contravariant C-energy flux vector* P^i, which satisfies the conservation law $P^i{}_{;i}=0$. The projection of P^i on the world line of a given observer is the C-energy density which he would measure in a local Lorentz reference frame. Consequently, we interpret the projection of P^i on the normal to a space-like hypersurface as the C-energy density in that hypersurface; and we interpret its projection on the normal to a time-like hypersurface as the C-energy flux (C energy flowing per second, per square centimeter) across that hypersurface.

It may be surprising that the C-energy flux tensor P^i is of first rank rather than of second rank and symmetric, like T^{ij}, the material stress-energy tensor. This is the price one must pay to have C energy simultaneously (1) localizable (defined in terms of a unique tensor), and (2) obey an integral conservation law,[3]

$$\int_{\text{(closed surface)}} P^i dS_i = 0 .$$

In the case of asymptotically flat space-time, Einstein[1] gave up localizability of energy in order to express it in terms of a symmetric entity of second rank (a "pseudotensor") which obeys an integral conservation law.

The justification for nominating C energy to the office of "the energy of whole-cylinder-symmetric systems" is that in almost every way thus far tested, C energy conforms to our prior experience of how energy should behave: It is covariant; it obeys an integral conservation law; it is localizable and locally measurable; near the symmetry axis of a static system, where strong gravitational fields are absent, it reduces to

* This work was supported in part by the U. S. Air Force Office of Scientific Research and by the National Science Foundation.
† Danforth Foundation Fellow 1963–64, National Science Foundation Predoctoral Fellow 1964–65.
[1] A. Einstein, Berlin Ber. 448 (1918).
[2] For a concise review and bibliography see C. W. Misner, Phys. Rev. **130**, 1590 (1963).
[3] For a discussion of integral conservation laws, see, e.g., J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Company, Amsterdam, 1960), Chap. VI.

proper mass; it is propagated by Einstein-Rosen gravitional waves and by cylindrical electromagnetic waves; *in vacuo* and in the presence of electromagnetic fields the C energy on a space-like hypersurface is absolutely minimized when the system is static, and the difference between the "potential" and the "kinetic" parts of the C energy is a Lagrangian for the combined Einstein-Maxwell field equations. However, there remains at least one more very important test which C energy must pass in order to earn the title of energy: Consider a toroidal body with major radius much larger than minor radius (thin ring). Near the ring space-time has whole-cylinder symmetry, and C energy is well defined. For C energy to deserve the title "energy" the total C energy of such a torus should be equal to or simply related to its Schwarzschild mass (the mass measured by examining the Keplerian motion of particles in orbits about the torus sufficiently distant that the field has spherical symmetry). The relationship between C energy and Schwarzschild mass is presently being investigated; but the analysis is difficult to carry out and is not yet complete, except in the Newtonian approximation. In that case, the integrated C energy and the Schwarzschild mass are equal, as is shown in Sec. III-C of this paper.

Regardless of whether or not C energy deserves the name "energy," its conservation law and absolute minimum properties make it a very useful tool in analyzing the dynamics of whole-cylinder-symmetric systems. For instance, in Sec. IV-C we use it to give a clear, simple description of the nature of Einstein-Rosen gravitational waves; in Sec. VI-B we use it to demonstrate the resistance of magnetic-field lines to cylindrical gravitational collapse; and in a separate paper[4] it is used to discuss the dynamics of Melvin's magnetic universe[5] when arbitrarily large radial perturbations are introduced.

In the remainder of this paper a detailed account of the definition and properties of C energy is presented.

In Sec. II we introduce a special set of coordinates which exploits the whole-cylinder symmetry of the system. We use the field equations in this coordinate system to motivate the selection of a certain scalar field, E, as a potential function for the C-energy flux vector. We then give an invariant geometric interpretation to E and express it in terms of the Killing vectors for whole-cylinder symmetry. From the scalar field E and the Killing vectors we construct the C-energy flux vector P^i; and in terms of P^i we give expressions for the local C-energy density and C-energy flux which would be measured by a given observer. Finally, we describe experiments which an observer could perform to measure the C-energy density in his neighborhood.

In Sec. III it is shown that the total C energy per unit length inside a solid cylinder of small mass is the same as its proper mass[6] per unit length. The mass of such a cylinder is also calculated according to Levi-Civita's prescription and found not always to agree with its proper mass. A particular physically acceptable cylinder is displayed, for which (C *energy*) = (*proper mass*) $= -(\frac{1}{2})$ $\times$(*Levi-Civita's mass*). Section III concludes with a proof of the equality of the C energy and the Schwarzschild mass of a static torus with small mass per unit length.

In Sec. IV the C-energy properties of a pure gravitational field are discussed. We show that the gravitational field around a cylinder contains—out to infinite distance—an infinite C energy per unit length, and that the gravitational C energy contained in any shell outside the cylinder is an absolute minimum (subject to the constraint of partially fixed metric on the boundaries) if and only if the metric in that shell is static. Then we show that C energy is propagated by Einstein-Rosen gravitational waves and that a gravitational wave pulse emitted by a cylinder of small mass per unit length carries a quantity of C energy equal to the decrease in the proper mass of the cylinder when the pulse is emitted. Finally, we generalize C energy to the case of cylindrical gravitational waves with 2 polarization states and show that they, too, carry a positive C energy.

Section V contains a discussion of the C energy of cylindrical electromagnetic universes. Here we divide the C energy into a potential part plus a kinetic part and show that their difference provides a variational basis for deriving the combined Einstein-Maxwell field equations, and that the unique static universe found by Melvin[5] gives an absolute minimum of the C energy contained inside any cylinder (subject to the constraints of fixed total magnetic flux inside the cylinder and partially fixed metric on its surface).

Section VI contains a discussion of the usefulness of C energy in the analysis of the dynamics of both locally and globally whole-cylinder-symmetric systems. In particular, C energy is used to suggest and support the conjecture of flux resistance to gravitational collapse.

II. DEFINITION OF *C* ENERGY

A. Coordinate Systems which Exploit Whole-Cylinder Symmetry

1. The Standard Coordinate System

Throughout this paper we consider only gravitating systems which exhibit what Melvin[7] calls "whole-cylinder symmetry": systems invariant under rotation about and translation along a symmetry axis, and under reflection in any plane containing the symmetry axis or perpendicular to it. If φ is an azimuthal angle (of period 2π) about this axis, and z is a coordinate measured along it, then the general line element exhibiting whole-

[4] K. S. Thorne, Phys. Rev. (to be published).

[5] M. A. Melvin, Phys. Letters 8, 65 (1964).

[6] By proper mass we mean $\int T_0{}^0({}^{(3)}g)^{1/2}d^3x$.

[7] M. A. Melvin, Phys. Rev. (to be published).

cylinder symmetry takes the form

$$ds^2=g_{00}d\tilde{t}^2+2g_{01}d\tilde{t}d\tilde{r}+g_{11}d\tilde{r}^2+g_{22}dz^2+g_{33}d\varphi^2\,, \quad (1)$$

where the metric coefficients depend only on $\tilde{t}$ and $\tilde{r}$. By an appropriate coordinate transformation on $\tilde{t}$ and $\tilde{r}$, this metric can always be transformed to

$$ds^2=e^{2(\gamma^*-\psi^*)}(dt^{*2}-dr^{*2})-e^{2\psi^*}dz^2-\alpha^2e^{-2\psi^*}d\varphi^2\,. \quad (2)$$

We shall call this the standard coordinate system or line element. It is defined up to transformations of the form

$$\begin{aligned}&z^*\to\pm z^*/\zeta+z_o;\quad (t^*,r^*)\to(\hat{t},\hat{r});\quad \varphi\to\pm\varphi+\varphi_o\,,\\ &\psi^*\to\psi^*+\ln\zeta;\quad \gamma^*\to\gamma^*-\tfrac{1}{2}\ln[\pm(\partial\hat{r}/\partial r^*)^2\\ &\qquad\qquad\mp(\partial\hat{r}/\partial t^*)^2]+\ln\zeta;\quad \alpha\to\zeta\cdot\alpha\,, \quad (3)\end{aligned}$$

where $\hat{t}$, $\hat{r}$ are any two functions of t^*, r^* satisfying

$$\begin{pmatrix}\partial\hat{r}/\partial t^*\\ \partial\hat{r}/\partial r^*\end{pmatrix}=\pm\begin{pmatrix}\partial\hat{t}/\partial r^*\\ \partial\hat{t}/\partial t^*\end{pmatrix},$$

and ζ, φ_o, z_o are arbitrary constants. The Einstein field equations for the standard line element are[7]

$$\begin{aligned}&\ddot{\alpha}-\alpha''=8\pi G(\mathfrak{T}_{0^*}{}^{0^*}+\mathfrak{T}_{r^*}{}^{r^*})\,,\\ &\ddot{\psi}^*+(\dot{\alpha}/\alpha)\dot{\psi}^*-\psi^{*\prime\prime}-(\alpha'/\alpha)\psi^{*\prime}\\ &\qquad=(4\pi G/\alpha)(\mathfrak{T}_{0^*}{}^{0^*}+\mathfrak{T}_{r^*}{}^{r^*}+\mathfrak{T}_\varphi{}^\varphi-\mathfrak{T}_z{}^z)\,,\\ &\gamma^{*\prime}=(\alpha'^2-\dot{\alpha}^2)^{-1}\{8\pi G(\alpha'\mathfrak{T}_{0^*}{}^{0^*}+\dot{\alpha}\mathfrak{T}_{0^*}{}^{r^*})\\ &\qquad+\alpha\alpha'(\dot{\psi}^{*2}+\psi^{*\prime 2})-2\alpha\dot{\alpha}\dot{\psi}^*\psi^{*\prime}+\alpha'\alpha''-\dot{\alpha}\dot{\alpha}'\}\,, \quad (4)\\ &\dot{\gamma}^*=(\alpha'^2-\dot{\alpha}^2)^{-1}\{8\pi G(-\dot{\alpha}\mathfrak{T}_{0^*}{}^{0^*}-\alpha'\mathfrak{T}_{0^*}{}^{r^*})\\ &\qquad-\alpha\dot{\alpha}(\dot{\psi}^{*2}+\psi^{*\prime 2})+2\alpha\alpha'\dot{\psi}^*\psi^{*\prime}-\dot{\alpha}\alpha''+\alpha'\dot{\alpha}'\}\,,\\ &\ddot{\gamma}^*-\gamma^{*\prime\prime}=(\psi^{*\prime 2}-\dot{\psi}^{*2})+(8\pi G/\alpha)\mathfrak{T}_\varphi{}^\varphi\,.\end{aligned}$$

Here $\dot{U}\equiv\partial U/\partial t^*$ and $U'\equiv\partial U/\partial r^*$; $\mathfrak{T}_i{}^j\equiv\sqrt{(-g)}T_i{}^j$; $T_i{}^j$ is the stress-energy tensor; and we set $c=1$.

If there is any coordinate system in which the metric of a whole-cylinder-symmetric system is static and invariant under time reversal, then there is a standard line element (2) which is static.[8]

2. *The Hyperbolic Canonical Coordinate System*

Suppose that, in addition to having whole-cylinder symmetry, the stress-energy tensor satisfies the condition of "pressure-energy equality"[9]

$$T_{0^*}{}^{0^*}+T_{r^*}{}^{r^*}=0 \quad (5)$$

in some standard coordinate system (a condition satisfied by the vacuum and by the electromagnetic systems of Sec. V). Then the field equation (4a) permits a transformation of the form (3) to a new standard coordinate system in which α is the radial variable.[10] The resultant metric is

$$ds^2=e^{2(\gamma-\psi)}(dt^2-dr^2)-e^{2\psi}dz^2-r^2e^{-2\psi}d\varphi^2\,. \quad (6)$$

This will be called the *hyperbolic canonical line element.*[11] It is uniquely defined up to the change of scale

$$\begin{aligned}&r\to\zeta r\,,\quad t\to\pm\zeta t+t_0\,,\quad z\to\pm z/\zeta+z_0\,,\\ &\varphi\to\pm\varphi+\varphi_0\,,\quad \psi\to\psi+\ln\zeta\,,\quad \gamma\to\gamma\,.\end{aligned} \quad (7)$$

In the hyperbolic canonical coordinate system the field equations (4) reduce to

$$\ddot{\psi}-(1/r)(r\psi')'=4\pi G(\mathfrak{T}_\varphi{}^\varphi-\mathfrak{T}_z{}^z)/r\,, \quad (8a)$$

$$\gamma'=r(\dot{\psi}^2+\psi'^2)+8\pi G\mathfrak{T}_0{}^0\,, \quad (8b)$$

$$\dot{\gamma}=2r\dot{\psi}\psi'+8\pi G\mathfrak{T}_r{}^0\,, \quad (8c)$$

$$\ddot{\gamma}-\gamma''=(\psi'^2-\dot{\psi}^2)+8\pi G\mathfrak{T}_\varphi{}^\varphi/r\,. \quad (8d)$$

If there are any coordinates in which the metric of a whole-cylinder-symmetric system satisfying condition (5) is static and invariant under time-reversal, then the canonical line element (6) is itself static.[8]

B. The Definition of C Energy

1. *Potential Function for the C-Energy Flux Vector*

We are now ready to define the C energy of systems with whole-cylinder symmetry. As an immediate motivation for the definition (the ultimate motivation consists of the useful energy-like properties demonstrated in later sections) consider the Einstein field equations (8) for the canonical coordinate system. Melvin[5,7] has emphasized that ψ here plays a role analogous to that of the Newtonian gravitational po-

[8] This is seen as follows: Let t be a time coordinate with respect to which the system is static. We can always use z and φ as two of our space coordinates and thereby obtain the line element $ds^2=g_{00}d\tilde{t}^2+g_{11}d\tilde{r}^2+g_{22}dz^2+g_{33}d\varphi^2$, where g_{ij} depend on $\tilde{r}$ only. Replacing $\tilde{t}$ and $\tilde{r}$ by t^* and r^* where $t^*=\tilde{t}$, $\sqrt{g_{00}}dr^*=\sqrt{(-g_{11})}d\tilde{r}$, transforms this into the *static standard form* (2). If $T_0{}^{*0^*}+T_r{}^{*r^*}=0$, the field equations (4) then give $\alpha=br^*+c$. The transformation $r=\alpha=br^*+c$, $t=bt^*$, $\gamma=\gamma^*-\ln b$ then puts the *static* line element in the *hyperbolic canonical form* (6).

[9] This condition is invariant under the transformations (3) among the standard coordinate systems.

[10] In the original coordinate system it is possible to have $\alpha'^2=\dot{\alpha}^2$ along a null surface of the form $r^*\pm t^*=$constant. The Jacobian, $\alpha'^2-\dot{\alpha}^2$, of the transformation between the old (r^*,t^*) and the new (r,t) coordinate systems vanishes on such a surface; so one of the coordinate systems must exhibit a coordinate singularity there. If the singularity is in the new (hyperbolic canonical) system, then r and t may reverse their roles as space-like and time-like coordinates as one crosses the singular surface [$e^{2(\gamma-\psi)}$ may change sign in Eq. (6)]. However, smoothness of the geometry demands that at the axis of symmetry r be space-like and t time-like. Throughout this paper we shall use hyperbolic canonical coordinates only when no coordinate singularities of this form arise. We shall reserve for later communications the discussion of Einstein-Rosen waves and electromagnetic systems for which the line element (6) has singular null surfaces. [See, e.g., K. S. Thorne, Ph.D. thesis, Princeton University, 1965 (unpublished); also, the discussion of Bertotti's magnetic universe and of thin-ring toruses constructed from it, in the chapter by K. S. Thorne, in the *Proceedings of the Second Texas Symposium on Relativistic Astrophysics* (The University of Chicago Press, Chicago, 1965)].

[11] Under the transformation $t\to iz$ and $z\to it$, the hyperbolic canonical line element becomes Weyl's canonical line element, which is used extensively in studying static axially-symmetric systems with $T_r{}^r+T_z{}^z=0$. [H. Weyl, Ann. Physik **54**, 117 (1918); H. Weyl and R. Bach, Math. Z. **13**, 134 (1922); T. Levi-Civita, Rend. Acc. Lincei **28**, 3 and 101 (1919).]

tential; it satisfies the inhomogeneous wave equation (8a) with a sum of components of the material stress-energy tensor as source. Once ψ is calculated, the solution for γ is reduced to quadrature [Eqs. (8b), (8c)]. Note that γ' is positive definite and is the sum of two terms: one—$r(\dot\psi^2+\psi'^2)$—is, aside from the factor r, the energy density usually associated with a scalar field; the other—$8\pi G\mathfrak{T}_0{}^0=8\pi Gre^{2(\gamma-\psi)}\cdot T_0{}^0$—is, aside from a multiplicative function of the metric, the material energy density in the canonical coordinate system. Similarly, $\dot\gamma$ is the sum of a term resembling the energy flux usually associated with a scalar field, and a term proportional to the energy flux $T_r{}^0$ of the material medium. This suggests that in the canonical coordinate system we take $P^0\sim\gamma'$ for the energy density (time component of C-energy flux vector) and $P^r\sim\dot\gamma$ for the energy flux (radial component of C-energy flux vector).

It might be objected that according to Newtonian gravitational theory, energy density for a whole-cylinder-symmetric system should look like $T_0{}^0-(\psi'^2+\dot\psi^2)$, not like $\gamma'\sim T_0{}^0+(\psi'^2+\dot\psi^2)$. However, the former choice of sign would lead to a negative C energy for Einstein-Rosen gravitational radiation ($T_0{}^0=0$), and it would destroy many of the useful, energy-like properties which C-energy possesses when the positive sign is chosen.

If $\gamma(r,t)$ is the key to C energy in the canonical coordinate system, what is the key in the more general standard coordinates? It should be a function of (r^*,t^*) which (1) is constructed from the metric, (2) reduces to γ for the special case of canonical coordinates, and (3) is invariant under the transformation (3) from one standard coordinate system to another. These three conditions uniquely determine the analog of γ for the standard coordinate system. It is

$$4GE(r^*,t^*)\equiv[\gamma^*-\ln(\alpha'^2-\dot\alpha^2)^{1/2}]. \tag{9}$$

The quantity $E(r^*,t^*)$ acts as a potential function from which to calculate the C-energy flux vector. For this reason, it is desirable to display the geometrical significance of E.

Let $\xi_{(z)}{}^i$ and $\xi_{(\varphi)}{}^j$ be the Killing vectors associated with the invariant translations and rotations of our whole-cylinder-symmetric system. Normalize $\xi_{(z)}{}^i$ by requiring

$$|\xi_{(z)}|\equiv[\xi_{(z)}{}^i\xi_{(z)i}]^{1/2}=1 \tag{10}$$

on the symmetry axis, at a moment of time when there is no gravitational radiation there.[12] Normalize $\xi_{(\varphi)}{}^i$ by requiring

$$|\xi_{(\varphi)}|\equiv[\xi_{(\varphi)}{}^i\xi_{(\varphi)i}]^{1/2}=(\text{proper circumference about axis of symmetry}). \tag{11}$$

Now, consider all invariant motions of the system of the form

$$x^i\to x^i+\lambda\xi_{(z)}{}^i+\eta\xi_{(\varphi)}{}^i,\quad 0\leqslant\lambda\leqslant\lambda_o,\quad 0\leqslant\eta\leqslant1. \tag{12}$$

Under this family of motions a particular point P in space-time sweeps out a cylindrical surface of length $Z=\lambda_0|\xi_{(z)}|$ and of area $A=Z|\xi_{(\varphi)}|$. The invariant cylinders swept out by different points in space-time will have different areas. Denote the magnitude of the space-time gradient of the area by $|A_{,i}|\equiv(g^{ij}A_{,i}A_{,j})^{1/2}$. Then, in terms of the geometrically defined quantities $|A_{,i}|$ and Z, the potential function for C energy is

$$E=-(1/4G)\ln(|A_{,i}|/2\pi Z). \tag{13}$$

Expressed directly in terms of the Killing vectors, E is a scalar field given by

$$E=-\frac{1}{8G}\ln\left\{\frac{g^{ij}(|\xi_{(z)}||\xi_{(\varphi)}|)_{,i}(|\xi_{(z)}||\xi_{(\varphi)}|)_{,j}}{4\pi^2|\xi_{(z)}|^2}\right\}. \tag{14}$$

To verify that (14) does, indeed, reduce to (9) in a standard coordinate system, note that

$$\xi_{(z)}{}^i=h_z\delta^{i2},\quad \xi_{(\varphi)}{}^i=2\pi\delta^{i3}\text{ in standard coordinates,} \tag{15}$$

where h_z is the z-coordinate interval associated with a translation of unit proper length on the axis of symmetry when there is no gravitational radiation there. Take expressions (15) and insert them into the machinery of Eq. (14) to obtain expression (9).

2. The C-Energy Flux Vector

In terms of the geometrically defined scalar field E, and the Killing vectors $\xi_{(z)}$, $\xi_{(\varphi)}$ for whole-cylinder symmetry, the C-energy flux vector P^i is

$$P^i=+[\epsilon^{ijkm}/(-g)^{1/2}]E_{,j}\times[\xi_{(z)k}/|\xi_{(z)}|^2][\xi_{(\varphi)m}/|\xi_{(\varphi)}|^2]. \tag{16}$$

In standard coordinates this becomes

$$P^{0*}=\frac{1}{h_z}\frac{e^{2(\psi^*-\gamma^*)}}{2\pi\alpha}\frac{\partial E}{\partial r^*},\qquad P^{r*}=-\frac{1}{h_z}\frac{e^{2(\psi^*-\gamma^*)}}{2\pi\alpha}\frac{\partial E}{\partial t^*},\quad P^z=P^\varphi=0. \tag{17}$$

The C-energy flux vector satisfies the differential conservation law

$$P^i{}_{;i}=[1/\surd(-g)][\surd(-g)\,P^i]_{,i} \tag{18}$$

for, in standard coordinates,

$$[\surd(-g)\,P^i]_{,i}=\frac{\partial}{\partial t^*}\left(\frac{+1}{2\pi h_z}\frac{\partial E}{\partial r^*}\right)+\frac{\partial}{\partial r^*}\left(\frac{-1}{2\pi h_z}\frac{\partial E}{\partial t^*}\right)=0.$$

Consequently, the integral of P^i over any closed

[12] We normalize $\xi_{(z)}{}^i$ on the axis of symmetry rather than at infinity because, for whole-cylinder-symmetric systems, space-time is *not* asymptotically flat at infinity, and $\xi_{(z)}{}^i\xi_{(z)i}$ will frequently be infinite there.

3-surface vanishes.

$$\int_{\text{(closed surface)}} P^i dS_i = 0. \tag{19}$$

3. The C Energy Measured by Observers

What is the physical significance of the C-energy flux vector P^i? Let v^i be the tangent vector to the world line of an observer at some point x^i of spacetime. Then the C-energy density measured by the observer at x^i in a local Lorentz frame is

$$\epsilon_{ce} = P^i v_i. \tag{20}$$

Similarly, if u^i is a space-like vector orthogonal to v^i, then the C-energy flux which the observer measures at x^i in the direction u^i is

$$F_{ce} = P^i u_i. \tag{21}$$

More generally, consider a family of observers whose world lines form a congruence of (locally parallel) curves filling up a 4-dimensional region of space-time. Let S be a segment of a space-like hypersurface orthogonal to the world lines, and let T be a segment of a time-like hypersurface parallel to the world lines near it. Then each observer can measure (see Sec. II.C) the C-energy density on S and the flux across T in his own neighborhood; and by combining their results the observers can obtain the total C energy on S and the total C energy transferred across T

(C energy on S)

$$= \int_S \epsilon_{ce} \sqrt{{}^{(3)}g}\; dx^1 dx^2 dx^3 = \int_S P^i dS_i, \tag{22}$$

(C energy transferred across T)

$$= \int_T F_{ce} \sqrt{{}^{(3)}g}\; dy^1 dy^2 dy^3 = \int_T P^i dS_i. \tag{23}$$

In particular, $E(r_o^*, t_o^*)$ [Eq. (9)] is the total C energy contained inside the disk $r^* < r_o^*$, $0 < z < h_z$ on the "standard hypersurface" $t^* = t_o^*$. This quantity will play a major role in subsequent sections of the paper. We call h_z [cf. Eq. (15)] the "standard (coordinate) length unit," and $E(r^*, t^*)$ the "C energy per unit standard length inside r^* at time t^*." By referring to the field equations (4), we see that $E(r^*, t^*) = E(r^*)$ can be written

$$E(r^*) = \int_0^{r^*} (g_{r^*r^*})^{1/2} dr^* \int_0^{2\pi} (g_{\varphi\varphi})^{1/2} d\varphi \times \int_0^{h_z} (g_{zz})^{1/2} dz\, \epsilon_{ce}, \tag{24}$$

where ϵ_{ce}, the C-energy density on the hypersurface of constant t^* is

$$\epsilon_{ce} = (1/h_z)(\alpha'^2 - \dot\alpha^2)^{-1}\{e^{(\gamma^* - \psi^*)}[\alpha' T_{0^*}{}^{0^*} + \dot\alpha T_{0^*}{}^{r^*}] + [e^{(\psi^* - \gamma^*)}/8\pi G][\alpha'(\dot\psi^{*2} + \psi^{*\prime 2}) - 2\dot\alpha \dot\psi^* \psi^{*\prime}]\}. \tag{25}$$

Note that on the symmetry axis, when the gravitational field is static there,

$$\epsilon_{ce} = T_{0^*}{}^{0^*}.$$

That is, under these conditions the C energy is due entirely to the material stress-energy present; the gravitational field makes no contribution.

When "pressure-energy equality" $T_{0^*}{}^{0^*} + T_{r^*}{}^{r^*} = 0$ is satisfied, we can set $\alpha = r$ (hyperbolic canonical coordinate system) and have

$$\epsilon_{ce} = (1/h_z)\{e^{(\gamma-\psi)} T_0{}^0 + [e^{(\psi-\gamma)}/8\pi G][\dot\psi^2 + \psi'^2]\}. \tag{26}$$

The physical interpretation of the integral conservation law (19) is now clear (see Fig. 1). It simply states that the increase in C energy on a segment of a space-like hypersurface as it moves in a time-like direction is the integral of the C-energy flux entering across its boundaries.

C. The Measurement of C Energy

The C-energy flux vector, P^i is uniquely defined in terms of the local geometry of space-time; and the unit tangent v^i to the world line of an observer is uniquely defined. Consequently, their scalar product $\epsilon_{ce} = P^i v_i$ is an intrinsic, local property of space-time and can be measured by the observer by means of local experiments. Consider for instance, a system satisfying "pressure-energy equality" $T_{0^*}{}^{0^*} + T_{r^*}{}^{r^*} = 0$, and an observer whose world line is orthogonal to the canonical hypersurfaces $t = \text{constant}$.[13] (We use the canonical coordinates of Sec. II.A.2.)

Let such an observer, situated at $r = r_1$, simultaneously drop two pebbles whose initial proper separation k_z, is in the z direction. These pebbles fall along paths $(\varphi, z) = \text{constant}$. Let a second observer situated a proper radial distance dl nearer the symmetry axis measure the proper separation, $k_z - dk_z$, of the pebbles as they pass him. Let the experiment be repeated with two pebbles whose initial separation k_φ is in the φ

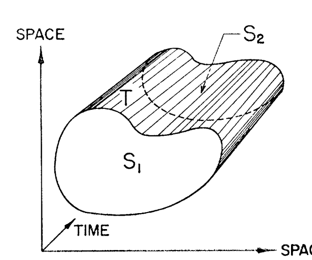

FIG. 1. Integral conservation law for C energy. The boundary of a space-like hypersurface S sweeps out T as the hypersurface moves in time from S_1 to S_2. The integral conservation law for C-energy states that (total C energy on S_2) $-$ (total C energy on S_1) $=$ (C energy which enters S across T); i.e., $\int_{S_2} P^i dS_i - \int_{S_1} P^i dS_i = -\int_T P^i dS_i$.

[13] Such an observer needs the support of a rocket engine to prevent him from falling toward the symmetry axis.

direction. Then, if $\mathfrak{C}$ is the proper circumference of a circle about the axis of symmetry and passing through the first observer's location, the C energy per unit standard length inside r_1 on the canonical hypersurface $t=$constant is

$$E(r_1)=-\left(\frac{1}{4G}\right)\ln\left[\frac{\mathfrak{C}}{2\pi}\left(\frac{1}{k_\varphi}\frac{dk_\varphi}{dl}+\frac{1}{k_z}\frac{dk_z}{dl}\right)\right]. \quad (27)$$

To measure the C-energy density in his local Lorentz frame, our observer can measure the C-energy per unit standard length at his position, $E(r_1)$, and at a distance dl nearer the symmetry axis, $E(r_1)-dE$; as well as $\Delta z=h_z e^{\psi}=|\xi_{(z)}|$, the proper distance corresponding to unit standard length in his neighborhood, and $\mathfrak{C}$, the proper circumference of a circle about the symmetry axis. Then he can apply the formula

$$\epsilon_{ce}=(dE/dl)/(\mathfrak{C}\Delta z). \quad (28)$$

The measurement of Δz can be made with a special longitudinal yardstick consisting of a frictionless wire on which two beads slide freely, and which is always kept oriented parallel to the axis of symmetry. The only longitudinal forces which will act to change the proper separation of the beads are gravitational forces due, for instance, to passing Einstein-Rosen gravitational waves or to a change in the location of the yardstick. Such gravitational forces do not alter the coordinate separation of the beads. Hence, once their coordinate separation has been fixed at one "standard length" unit, it will always remain so; and their proper separation will be the Δz which enters into Eq. (28) to give the local C-energy density.

If space-time is flat ($R^i{}_{jkl}=0$) in a neighborhood of $r=r_1$, the C energy per unit standard length on a canonical hypersurface ($t=$constant) has a very simple significance: The hypersurface near $r=r_1$ is a segment of a (3-dimensional) conical surface of half-angle arcsin $(e^{-4GE(r_1)})$. This was pointed out in 1957 by Fierz,[14] whose discussion of the geometry around an Einstein-Rosen gravitational wave moving on a flat-space background foreshadowed our definition of C energy.

D. Can C-Energy Density be Negative or Infinite?

One would hope that at any point in space-time where on physical singularities exist the C-energy density measured by an arbitrary observer in his local Lorentz frame would be non-negative and noninfinite. Unfortunately, this is not always the case. There exist solutions to the field equations (4) which, in a standard coordinate system (2), have nonsingular hypersurfaces on which $\alpha'^2-\dot{\alpha}^2=0$.[15] At each point on these hypersurfaces every observer measures an infinite C-energy density [cf. Eq. (25) and also Eq. (9)], but the geometry of space-time is perfectly smooth. On one side of such a hypersurface the C-energy density is positive, while on the other side it is negative.

Between the symmetry axis ($\alpha=0$) and the first such "critical" hypersurface one has $\alpha'>|\dot{\alpha}|$, for the symmetry axis must sweep out a time-like path in the r^*-t^* plane. In this "inner region" all observers measure a nonnegative, noninfinite C-energy density; equivalently, in this inner region the C-energy flux vector lies inside the future light cone. To see this, insert E from Eq. (9) into Eq. (17) and then use the field equations (4) to obtain

$$P^{0*}=\frac{1}{h_z}\frac{1}{\alpha'^2-\dot{\alpha}^2}\left\{\alpha' T_{0*}{}^{0*}+\dot{\alpha}T_{0*}{}^{r*}+\frac{e^{2(\psi^*-\gamma^*)}}{8\pi G}[\alpha'(\dot{\psi}^{*2}+\psi^{*\prime 2})-2\dot{\alpha}\dot{\psi}^*\psi^{*\prime}]\right\}, \quad (29)$$

$$P^{r*}=\frac{1}{h_z}\frac{1}{\alpha'^2-\dot{\alpha}^2}\left\{\alpha' T_{0*}{}^{r*}-\dot{\alpha}T_{r*}{}^{r*}+\frac{e^{2(\psi^*-\gamma^*)}}{8\pi G}[\dot{\alpha}(\dot{\psi}^{*2}+\psi^{*\prime 2})-2\alpha'\dot{\psi}^*\psi^{*\prime}]\right\}, \quad (30)$$

$$P^{0*}\pm P^{r*}=\frac{1}{h_z}\frac{1}{\alpha'^2-\dot{\alpha}^2}\times\left\{\alpha'(T_{0*}{}^{0*}\pm T_{0*}{}^{r*})+\dot{\alpha}(T_{0*}{}^{r*}\mp T_{r*}{}^{r*})+\frac{e^{2(\psi^*-\gamma^*)}}{8\pi G}(\alpha'\pm\dot{\alpha})(\psi^{*\prime}\mp\dot{\psi}^*)^2\right\}. \quad (31)$$

Now, for any physical source of stress-energy one has $T_{0*}{}^{0*}-|T_{r*}{}^{r*}|\geq 0$ and $T_{0*}{}^{0*}-|T_{r*}{}^{r*}|-2|T_{0*}{}^{r*}|\geq 0$. Consequently, in the inner region where $\alpha'>|\dot{\alpha}|$, expressions (29) and (31) are non-negative. This, along with $g_{0*0*}=-g_{r*r*}$, guarantees that P^i lies inside the future light cone, so that in the inner region all observers measure a nonnegative, noninfinite C energy density.

In particular, *in vacuo* and in electromagnetic systems, when the nonsingular hyperbolic canonical coordinate system (6) can be introduced,[10] the C-energy density as measured by an arbitrary observer is everywhere non-negative and noninfinite.

In systems where the C-energy density is not everywhere non-negative and noninfinite, one hesitates to identify C energy with classical energy. However, in such systems C energy remains a useful tool for the analysis of dynamics.

Note added in proof. One can redefine C energy so as to make the C-energy density measured by all observers *finite* at every point in space-time. This is done by taking

[14] M. Fierz, as quoted by J. Weber and J. A. Wheeler, Rev. Mod. Phys. **29**, 509 (1957).

[15] Specific examples will be discussed in later communications (see, e.g., K. S. Thorne, Ref. 10).

as the C-energy potential function (Sec. II-B)

$$E^{(\mathrm{new})}=(1/8G)(1-e^{-8GE})$$
$$=(1/8G)(1-[\alpha'^2-\dot{\alpha}^2]e^{-2\gamma^*})$$

instead of E. For a discussion of C energy from this slightly altered point of view, see the author's unpublished Ph.D. thesis (Princeton University, 1965).

III. THE *C* ENERGY OF STATIC CYLINDERS AND TORUSES

A. The *C* Energy of a Static Cylinder with Small Mass per Unit Length

The best test of the energy properties of C energy, which can be performed without considering toruses, is to check whether C energy reduces to Newtonian energy, or "proper mass," in the case of a static cylinder with small proper mass per unit proper length. We perform this test entirely within the framework of Einstein's theory; but we keep in mind the approximate flatness of space-time throughout the interior of the cylinder, which results from its small mass per unit length, and which permits the introduction of the concept of proper mass.

Consider a particular static cylinder with small mass per unit length. Throughout space-time introduce a standard line element (2) with respect to which the metric is static. Choose $\gamma^*=\psi^*=\alpha=0$ and $\alpha'=1$ on the symmetry axis so that the metric is essentially flat throughout the cylinder. Then, integration of the Einstein field equations (4) yields, to first order in the dimensionless quantity $G\times$(mass per unit length),[16]

$$\alpha=r^*-4G\int_0^{r^*}(M_{0^*}{}^{0^*}+M_{r^*}{}^{r^*})dr^*,$$
$$\psi^*=-2G\int_0^{r^*}(M_{0^*}{}^{0^*}+M_{r^*}{}^{r^*}+M_\varphi{}^\varphi-M_z{}^z)r^{*-1}dr^*,\quad(32)$$
$$\gamma^*=-4GM_{r^*}{}^{r^*},$$

where

$$M_i{}^i=\int_0^{r^*}T_i{}^i 2\pi r^*dr^*.\quad(33)$$

The C energy per unit standard length inside the cylinder and on the standard hypersurface $t^*=$ constant is then

$$E(r_s{}^*)=(\tfrac{1}{4}G)(\gamma^*-\ln\alpha')=\int_0^{r_s{}^*}T_{0^*}{}^{0^*}(2\pi r^*dr^*),\quad(34)$$

where $r_s{}^*$ is the radial coordinate of the surface of the cylinder. In this case (standard length) = (coordinate length) $\approx$ (proper length) inside the cylinder, so the total C energy per unit standard length is identical to the proper mass per unit proper length.

If we had used a different space-like hypersurface passing through ($t^*=$ constant, $r^*=r_s{}^*$) would we have obtained the same result? Yes. The integral conservation law (19) guarantees that segments of two space-like hypersurfaces with the same (2-dimensional) boundary contain the same total C energy.

B. Comparison of *C*-Energy with Levi-Civita's Definition of Mass

The metric outside any static cylinder is the same as the metric of a line mass, which was first derived by Levi-Civita[17]:

$$ds^2=\tilde{r}^{2c}dt^2-A^2\tilde{r}^{2c(c-1)}(d\tilde{r}^2+d\tilde{z}^2)-\tilde{r}^{2(1-c)}d\varphi^2.\quad(35)$$

When transformed to hyperbolic canonical coordinates [Eq. (6)] this line element becomes

$$ds^2=e^{2(\gamma-\psi)}(dt^2-dr^2)-e^{2\psi}dz^2-r^2e^{-2\psi}d\varphi^2,$$
$$\psi=\kappa\ln r+a,\quad(36)$$
$$\gamma=\kappa^2\ln r+b,$$

where

$$\kappa=-c/(1-c).\quad(37)$$

Levi-Civita[17] and others[18] have suggested that the constant $c/2G=-(\kappa/2G)/(1-\kappa)$ be called the mass per unit length of the cylinder generating the gravitational field, because for $0\leq c\ll 1$ several features[18] of the line element (35) resemble the Newtonian gravitational field for a cylinder with this mass per unit length.

By extending the interior solution (32) for a small-mass cylinder into the exterior region and then converting to hyperbolic canonical coordinates, we find that

$$(c/2G)\approx-(\kappa/2G)$$
$$=\int_0^{r_s{}^*}(T_{0^*}{}^{0^*}+T_{r^*}{}^{r^*}+T_\varphi{}^\varphi-T_z{}^z)(2\pi r^*dr^*).\quad(38)$$

As long as the internal pressures of the cylinder are much smaller than its energy density, then, to first order in the quantity $G\times$(mass per unit length), one has the equality (*Levi-Civita's mass*) = (*total C energy inside cylinder*) = (*proper mass*). But, regardless of how small the mass per unit length of the cylinder is, if the internal pressures are of the same order as the energy density, (*Levi-Civita's mass*) $\neq$ (*total C energy*) = (*proper mass*).

For example, consider a whole-cylinder-symmetric

[16] In conventional units this dimensionless quantity is $(G/c^2)\times$(*mass per unit length*).

[17] T. Levi-Civita, Rend. Acc. Lincei **28**, 3 and 101 (1919).

[18] T. Levi-Civita (Ref. 1) argued $m/l=c/2$ from the form of the metric for small c. W. Wilson, Phil. Mag. **40**, 710 (1920) made this association by examining the equations of motion for free particles when $c\ll 1$. L. Marder, Proc. Roy. Soc. (London) **244**, 524 (1958) showed that for small c the proper mass per unit proper length of certain solid cylinders is $c/2$.

bundle of magnetic field lines, all parallel to the symmetry axis. At a particular moment of (coordinate) time, $t^*=0$, let them be perfectly stationary (configuration of time-symmetry) and so distributed that

$$B=1\ \mathrm{G}, \quad 0<r^*<1\ \mathrm{m}$$
$$=0, \quad r^*>1\ \mathrm{m}.$$

One can verify that Eqs. (32) represent a solution to the initial value equations for the hypersurface of time-symmetry; and that for this solution Eqs. (34) and (38) are valid. At the moment of time-symmetry

$$T_{0^*}{}^{0^*}=-T_{r^*}{}^{r^*}=T_z{}^z=-T_\varphi{}^\varphi=B^2/8\pi.$$

Consequently,

(C energy per unit length on hypersurface $t^*=0$)

$$=\int_0^{1\mathrm{m}}(B^2/8\pi)(2\pi r^*dr^*)=1.4\times10^{-18}\mathrm{g/cm}, \quad (39)$$

but

(Levi-Civita's mass per unit length)

$$=-2\int_0^{1\mathrm{m}}(B^2/8\pi)(2\pi r^*dr^*)$$
$$=-2.8\times10^{-18}\mathrm{g/cm}. \quad (40)$$

This negative value for Levi-Civita's mass may not be as serious an indictment as it at first appears, since this configuration of magnetic fields actually repels particles outside it rather than attracting them.

C. The C Energy of a Light Torus

Consider a static torus with major radius $\mathcal{R}$ much larger than minor radius r_s^* (thin ring), and $G\times$(mass per unit length)$\ll1$. In the region[19] $0<r^*\lesssim0.01\mathcal{R}$ about the guiding line of the torus, we can introduce standard whole-cylinder-symmetric coordinates and calculate the C energy. Using the analysis of Sec. III-A for a static cylinder, we see that

$$(\text{total } C \text{ energy within torus})=(\text{proper mass within torus})=(\text{Schwarzschild mass of torus}). \quad (41)$$

(The second equality follows from Newtonian gravitational theory.)

If we include the C energy contained in the gravitational field outside the torus ($r_s^*<r^*\lesssim0.01\mathcal{R}$), will our result be altered? By extending the interior solution (36) for a static cylinder (or torus) into the exterior region, we find the following: If a photon of frequency ν_o is emitted radially by an atom sitting on the major circumference or "guiding line" of the torus, and if it is measured to have frequency $\nu_o-\Delta\nu$ when passing $r^*=0.01\ \mathcal{R}$ (gravitational red shift), then

(C energy in region $r_s^*<r^*\lesssim0.01\mathcal{R}$)/(proper mass within torus)

$$\sim\ln[g_{0^*0^*}(0.01\mathcal{R})/g_{0^*0^*}(0)]=\ln[\nu_o/(\nu_o-\Delta\nu)]. \quad (42)$$

Consequently, the C energy associated with the gravitational field outside the torus can be of the same order as the C energy or proper mass of the material stress-energy inside the torus only when the torus is sufficiently massive to cause a large gravitational red shift. In such a case Newtonian gravitational theory breaks down, and we must turn to the full Einstein theory in order to test the relationship between C energy and Schwarzschild mass. The full-Einstein-theory analysis has not yet been completed. Pending the completion of it, we know that the total C energy and the Schwarzschild mass of a light thin-ring torus are equal, but we do not know whether they are equal for a torus sufficiently massive that the Newtonian approximation breaks down.

IV. C ENERGY OF A PURE GRAVITATIONAL FIELD

In this section we will consider the C-energy properties of the gravitational field surrounding a cylinder. Unless specified otherwise, we do not require that the cylinder be static or have small mass per unit length.

A. C Energy of a Static Gravitational Field

The hyperbolic canonical line element for the gravitational field outside a static cylinder of arbitrary mass takes the form (36).[20] If r_s is the radial coordinate value at the surface of the cylinder, the total gravitational C energy per unit standard length on *any* hypersurface between ($r=r_s$, $t=t_o$) and ($r=R$, $t=t_o$) is

$$E(R)-E(r_s)=(1/4G)[\gamma(R)-\gamma(r_s)]$$
$$=\kappa^2\ln(R/r_s)=(\text{a positive definite quantity}). \quad (43)$$

[Cf. discussions following Eqs. (23) and (34).] Hence, so long as $\kappa\neq0$, the total C energy per unit standard length outside the cylinder, $E(\infty)-E(r_s)$, is infinite! This result is not so surprising when one recalls that the Newtonian gravitational potential of a cylinder also diverges at infinity.

[19] Throughout this section we use the standard coordinates of Sec. III. A, centered on the major circumference or "guiding line" of the torus. Since $g_{0^*0^*}\approx g_{r^*r^*}\approx g_{zz}\approx1$, $g_{\varphi\varphi}\approx r^{*2}$, these coordinates can be interpreted in the ordinary, everyday sense of Newtonian gravitational theory.

[20] This line element also holds for static regions of space-time outside a dynamical cylinder.

One should remember that nothing in principle prevents κ from vanishing even when a gravitating source is present. [Cf. Eq. (38) and discussion following it.] If κ vanishes, one has a situation like that discussed by Fierz[14] (see end of Sec. II-C): Space-time is locally flat outside the cylinder, but globally outside the cylinder the canonical hypersurface, $t=$constant, is a segment of a 3-dimensional conical surface of half-angle $\sin^{-1}\{\exp[-4G\times(C$ energy per unit standard length in source cylinder$)]\}$. The local flatness of space-time reflects the local absence of C energy, while the global conical geometry outside the cylinder reflects the presence of C energy in the neighborhood of the symmetry axis.

B. C-Energy Minimum Property of Static Gravitational Fields

We shall now prove a very useful C-energy property of pure gravitational fields, and in the next section we shall use it to clarify the nature of Einstein-Rosen gravitational radiation.

We have seen that the canonical line element (6) can be expressed in terms of two functions $\psi(r,t)$ and $\gamma(r,t)=4GE(r,t)$. In the region outside any source ($T_i{}^j=0$), ψ alone is needed to characterize the line element. Once ψ (the "gravitational potential function") is known, γ (the "gravitational C-energy factor") can be computed up to an additive constant from

$$\begin{aligned}\gamma'/4G&=E'=r(\dot\psi^2+\psi'^2)/4G,\\ \dot\gamma/4G&=\dot E=r\dot\psi\psi'/2G.\end{aligned}\tag{44}$$

[Cf. Eqs. (8) and (9).] Now choose two radii $r=r_1$, $r=r_2$; and specify $\psi(r_1)$, $\psi(r_2)$. *That (continuously differentiable) potential function $\psi(r)$ with these endpoints which gives an* absolute minimum *of $E(r_2)-E(r_1)$ is the* unique static solution *of the field Eqs. (8), which satisfies the specified boundary conditions* ("C-energy minimum principle").

To see why, let κ and a be constants so chosen that $\kappa\ln r_1+a=\psi(r_1)$ and $\kappa\ln r_2+a=\psi(r_2)$. Then the unique static solution, with these boundary conditions, $\psi_o(r)=\kappa\ln r+a$, clearly gives a stationary value of

$$E(r_2)-E(r_1)=(1/4G)\int_{r_1}^{r_2}r(\dot\psi^2+\psi'^2)dr;\tag{45}$$

for $\delta(E(r_2)-E(r_1))=0$ yields as Euler-Lagrange equations

$$r\dot\psi=0,\quad (r\psi')'=0.$$

To verify that $\psi_o(r)=\kappa\ln r+a$ gives an absolute minimum of (45), we set $u=\ln r$. Then, for any choice of $\psi(r,t)$ satisfying the boundary conditions,

$$4G[E(r_2)-E(r_1)]\geq\int_{r_1}^{r_2}r\psi'^2dr=\int_{u_1}^{u_2}(d\psi/du)^2du=(u_2-u_1)^{-1}\left[\int_{u_1}^{u_2}1^2du\right]\left[\int_{u_1}^{u_2}(d\psi/du)^2du\right]$$

$$\geq(u_2-u_1)^{-1}\left[\int_{u_1}^{u_2}1\cdot(d\psi/du)du\right]^2=[\psi(r_2)-\psi(r_1)]^2/\ln(r_2/r_1)=\kappa^2\ln(r_2/r_1)=(\text{the value of }4G\Delta E\text{ for }\psi(r)=\psi_o(r)).$$

Another property of the C energy of a pure gravitational field, which is closely related to the above minimum property, is the following: *The difference between the "kinetic" and "potential" parts of the C energy gives a variational principle for the dynamical field equation obeyed by the potential function ψ*; thus, the extremum requirement

$$\delta\int_{t_1}^{t_2}\int_{r_1}^{r_2}(1/4G)r(\psi'^2-\dot\psi^2)drdt=0$$

yields

$$\ddot\psi-(1/r)(r\psi')'=0.\tag{46}$$

In Sec. V we will generalize this property and the C-energy minimum principle to electromagnetic universes.

C. Einstein-Rosen Gravitational Waves

Einstein and Rosen[21] were the first to notice that the vacuum field equations for systems with whole-cylinder symmetry admit wave-like solutions. These solutions, termed "Einstein-Rosen gravitational waves," together with wave-like solutions to the linearized Einstein equations for asymptotically flat systems, provided the basis for a long disagreement over whether gravitational waves are "real" and carry energy, or can be removed by an appropriate choice of coordinates. That gravitational waves are, indeed, real has recently been demonstrated[22] by showing that the geometry of space-time is permanently affected by their passage; and that one can (in principle) build machines to extract energy from them.

By using the C-energy minimum principle of the last section we can clarify the exact nature of Einstein-Rosen waves and their energy properties. The wave

[21] A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937).

[22] On Einstein-Rosen waves see, for instance, N. Rosen in *Jubilee of Relativity Theory*, edited by A. Mercier and M. Kervaire (Birkhaüser Verlag, Basel, 1956); J. Weber and J. A. Wheeler, Rev. Mod. Phys. **29**, 509 (1957); and L. Marder (Ref. 18). On the more general problem, see the lectures of R. Sachs and R. Penrose in C. DeWitt and B. DeWitt, *Relativity, Groups and Topology* (Gordon and Breach Science Publishers, New York, 1964), and other papers cited there.

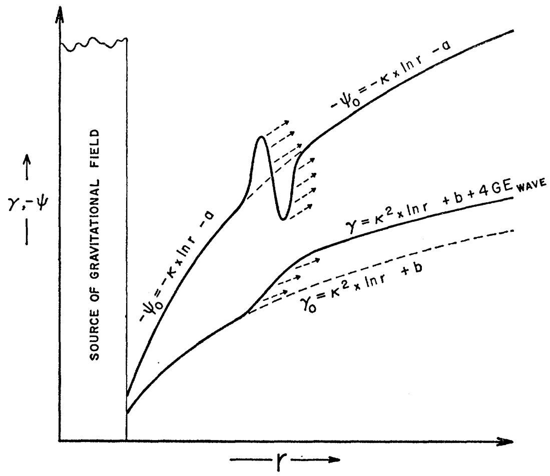

FIG. 2. The gravitational potential function ψ and the gravitational C-energy factor $\gamma = 4G\,E(r)$, which enter into the canonical metric

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - dr^2) - e^{2\psi}dz^2 - r^2 e^{-2\psi}d\varphi^2,$$

for a cylinder plus an Einstein-Rosen wave pulse. The disturbance in the potential function, ψ, propagates linearly on the back of the static potential function ψ_o [cf. Eq. (46)]; but according to the C-energy minimum principle, the C-energy factor γ is permanently decreased by an additive constant $4G\,E_{\text{wave}}$ when the wave pulse passes. E_{wave} is the C energy per unit standard length carried by the wave pulse.

equation (46) for the gravitational potential function ψ, being linear, permits the superposition of solutions. Consequently, a wave pulse in ψ, traveling on top of a static background potential $\psi_0 = \kappa \ln r + a$, will be totally oblivious of the existence or form of the static background metric. When the wave pulse has passed, $\psi(r)$ will return to its original static value $\psi_0 = \kappa \ln r + a$ (see Fig. 2). However, the gravitational C-energy factor $\gamma(r) = 4GE(r)$ will be permanently affected by the passage of the pulse: The static solution gives an absolute minimum of the C-energy $E(r_2) - E(r_1)$, subject to $\psi(r_2)$, $\psi(r_1)$ constant. Consequently, when the wave pulse is inside r, $\gamma(r) = 4GE(r)$ will be greater by an additive constant $4GE_{\text{wave}}$, than when the wave pulse is outside r. E_{wave} is just the C-energy per unit standard length carried by the wave pulse.

Marder[18] has calculated the change in proper mass per unit proper length of a particular solid cylinder, when it emits a gravitational wave pulse. By comparing *his* Eqs. (21) and (31) one readily sees that the decrease in proper mass per unit proper length of the cylinder is precisely E_{wave}, the C energy per unit standard length carried by the wave pulse which it emits!

D. Gravitational Waves with Two States of Polarization

Einstein-Rosen gravitational waves have only one polarization state, corresponding to the single "gravitational potential function" ψ. By relaxing the demand that space-time be invariant under reflections in planes containing the symmetry axis or perpendicular to it, Jordan and Ehlers, and independently Kompaneets[23] have obtained cylindrical gravitational waves with *two* polarization states. This relaxation of the symmetry does *not* destroy the Killing vector fields $\xi_{(\varphi)}{}^i$ and $\xi_{(z)}{}^i$ in terms of which C energy is defined, but it *does* destroy their uniqueness. (If there is no reflection invariance, there is no way to select out preferred z and φ directions on the z-φ surface.) Consequently, *there are infinitely many inequivalent ways to define C energy for 2-component cylindrical gravitational waves.*

Select a particular pair of Killing vectors $\xi_{(z)}{}^i$ and $\xi_{(\varphi)}{}^i$ and calculate the corresponding C-energy potential E from Eq. (13). (In this calculation the area A of the invariant surface will be

$$A = \lambda_o[|\xi_{(\varphi)}|^2|\xi_{(z)}|^2 - (\xi_{(z)}{}^i\xi_{(\varphi)i})^2]^{1/2}, \quad (47)$$

rather than $A = \lambda_o|\xi_{(\varphi)}||\xi_{(z)}|$, because $\xi_{(z)}{}^i$ and $\xi_{(\varphi)}{}^i$ will not necessarily be orthogonal.) Then use the appropriate generalization of Eq. (16),

$$P^i = \frac{\epsilon^{ijkm}}{(-g)^{1/2}} \frac{E_{,j}\xi_{(z)k}\xi_{(\varphi)m}}{|\xi_{(z)}|^2|\xi_{(\varphi)}|^2 - [\xi_{(z)}{}^n\xi_{(\varphi)n}]^2}, \quad (16')$$

to calculate the C-energy flux vector. In the coordinate system

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - dr^2) - e^{2\psi}(dz + \omega d\varphi)^2 - r^2 e^{-2\psi}d\varphi^2, \quad (48)$$

one will find

$$E(r,t) = (1/4G)\gamma \quad (49)$$

as in the whole-cylinder-symmetric case; and, as before, E will be not only the C-energy potential function, but also the C energy per unit standard length on the hypersurface $t = \text{constant}$.

The vacuum field equations corresponding to Eq. (48) are

$$\ddot{\psi} - (1/r)(r\psi')' = (1/2r^2)e^{4\psi}(\dot{\omega}^2 - \omega'^2), \quad (50a)$$

$$\ddot{\omega} - r(\omega'/r)' = 4(\omega'\psi' - \dot{\omega}\dot{\psi}), \quad (50b)$$

$$\gamma' = r(\dot{\psi}^2 + \psi'^2) + (1/4r)e^{4\psi}(\dot{\omega}^2 + \omega'^2), \quad (50c)$$

$$\dot{\gamma} = 2r\dot{\psi}\psi' + (1/2r)e^{4\psi}\dot{\omega}\omega'. \quad (50d)$$

The quantities ψ and ω are seen to be gravitational potential functions corresponding to the two polarization states. They are governed by the two coupled, nonlinear wave equations (50a) and (50b). Once these wave equations have been solved for ψ and ω, the C-energy factor γ can be obtained by quadrature from Eqs. (50c) and (50d).

As in the Einstein-Rosen case, the C energy per unit standard length $E(r,t)$ has a minimum property: Choose a hypersurface $t = \text{constant}$ and two radii $r = r_1$ and

[23] P. Jordan, J. Ehlers, and W. Kundt, Akad. Wiss. Mainz, Abh. Math.-Nat. Kl., Jahrg. 1960, Nr. 2; A. S. Kompaneets, Zh. Eksperim. i Teor. Fiz. 34, 953 (1958) [English transl.: Soviet Phys.—JETP 7, 659 (1958)]. The author thanks J. Stachel for pointing out these references and for suggesting the generalization of C energy to gravitational waves with two states of polarization.

$r=r_2$; and specify $\psi(r_1)$, $\omega(r_1)$, $\psi(r_2)$, $\omega(r_2)$. *Those (continuously differentiable) potential functions ψ and ω with these endpoints, which give an* absolute minimum of *$E(r_2,t)-E(r_1,t)$ correspond to the* unique static solution *of the field equations* (50), *which satisfies the specified boundary conditions.*[24] C energy also provides a Lagrangian for the field equations, just as in the Einstein-Rosen case: Decompose the C energy into a kinetic part K (involving $\dot\psi^2$ and $\dot\omega^2$) plus a potential part P (involving ψ'^2 and ω'^2), $E=K+P$. *The difference between the kinetic and potential parts of the C energy acts as a Lagrangian for the coupled wave equations* (50a) *and* (50b) *which govern ψ and ω*; thus, the extremum requirement

$$\delta\int_{t_1}^{t_2}\int_{r_1}^{r_2}\frac{1}{G}\left\{\frac{(\dot\psi^2-\psi'^2)}{8\pi}+\frac{e^{4\psi}}{4r^2}\frac{\dot\omega^2-\omega'^2}{8\pi}\right\}2\pi r dr dt=0\,, \quad (51)$$

yields as Euler-Lagrange equations Eqs. (50a) and (50b).

These two theorems hold for each of the infinitely many inequivalent definitions of C energy which are possible when the assumption of reflection invariance of space-time is dropped.

V. CYLINDRICAL ELECTROMAGNETIC UNIVERSES

We turn now to the application of C energy to cylindrical electromagnetic universes.

A. Brief Description of Cylindrical Electromagnetic Universes

A cylindrical electromagnetic universe (C.E.U.) is a whole-cylinder-symmetric system, constructed entirely from gravitational and electromagnetic fields. There has been considerable discussion of such universes recently,[4,5,7,25] much of it stimulated by the question of whether they exhibit gravitational collapse. No evidence for the gravitational collapse of any cylindrical electromagnetic universe has yet been found; C.E.U.'s seem to be extremely stable against perturbations. (See Refs. 4 and 7, and Sec. VI-B of this paper.)

Our discussion of cylindrical electromagnetic universes will be entirely within the framework of the hyperbolic canonical coordinate system

$$ds^2=e^{2(\gamma-\psi)}(dt^2-dr^2)-e^{2\psi}dz^2-r^2e^{-2\psi}d\varphi^2\,. \quad (6)$$

The most general C.E.U. has electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ which, on the canonical hypersurface $t=$ constant, lie in the $z-\varphi$ cylindrical surface. (A nonvanishing radial component leads to a singularity on the symmetry axis.) Consequently, the gauge can be chosen so that the only nonvanishing components of the electromagnetic vector potential function are $A_2(r,t)$ and $A_3(r,t)$, and the only nonvanishing components of the electromagnetic field tensor are[26]

$$\begin{aligned} f_{02}&=-f_{20}=\dot A_2/\sqrt{G}\,, & f_{12}&=-f_{21}=A_2'/\sqrt{G} \\ f_{03}&=-f_{30}=\dot A_3/\sqrt{G}\,, & f_{13}&=-f_{31}=A_3'/\sqrt{G}\,. \end{aligned} \quad (52)$$

The physical electric and magnetic fields, as measured by an observer with world line orthogonal to the canonical hypersurfaces $t=$ constant are

$$\begin{aligned} \mathbf{B}&=-[A_3'e^{2\psi-\gamma}/(r\sqrt{G})]\mathbf{e}_z+[A_2'e^{-\gamma}/\sqrt{G}]\mathbf{e}_\varphi\,, \\ \mathbf{E}&=-[\dot A_2e^{-\gamma}/\sqrt{G}]\mathbf{e}_z-[\dot A_3e^{2\psi-\gamma}/(r\sqrt{G})]\mathbf{e}_\varphi\,. \end{aligned} \quad (53)$$

The total magnetic flux parallel to the symmetry axis and inside r is

$$\Phi(r)=\int_0^r\int_0^{2\pi}B_z(g_{rr}g_{\varphi\varphi})^{1/2}d\varphi dr =2\pi[A_3(0)-A_3(r)]/\sqrt{G}\,, \quad (54)$$

and the combined Einstein-Maxwell field equations reduce to[27]

$$\ddot\psi-(1/r)(r\psi')'=-e^{-2\psi}(\dot A_2{}^2-A_2'^2) +(e^{2\psi}/r^2)(\dot A_3{}^2-A_3'^2)\,, \quad (55a)$$

$$\ddot A_2-(1/r)(rA_2')'=2(\dot\psi\dot A_2-\psi'A_2')\,, \quad (55b)$$

$$\ddot A_3-r(A_3'/r)'=-2(\dot\psi\dot A_3-\psi'A_3')\,, \quad (55c)$$

$$\dot A_2\dot A_3=A_2'A_3'\,, \quad (55d)$$

$$\gamma'=r(\dot\psi^2+\psi'^2)+re^{-2\psi}(\dot A_2{}^2+A_2'^2) +(e^{2\psi}/r)(\dot A_3{}^2+A_3'^2)\,, \quad (55e)$$

$$\dot\gamma=2r\dot\psi\psi'+2re^{-2\psi}\dot A_2A_2'+2(e^{2\psi}/r)\dot A_3A_3'\,, \quad (55f)$$

plus other equations which can be derived from these. The last two of these equations can be integrated to obtain the C-energy factor γ once the first four have been solved for the gravitational and electromagnetic potential functions ψ and A_j.

The most *general static solution* to these field equations with $A_2=0$ describes the most general static magnetic C.E.U. It is

$$\begin{aligned} &\psi=\ln(1+\beta^2r^{2q})+(1-q)\ln r+\ln\lambda\,, \\ &A_2=0\,, \\ &A_3=1/[\beta\lambda(1+\beta^2r^{2q})]+\delta\,, \\ &\gamma=2\ln(1+\beta^2r^{2q})+(1-q)^2\ln r+\ln\eta\,, \\ &\mathbf{B}=\frac{2}{\sqrt{G}}\frac{q\beta\lambda}{\eta}\frac{1}{r^{(1-q)^2}(1+\beta^2r^{2q})^2}\mathbf{e}_z\,,\quad \mathbf{E}=\mathbf{0}\,, \end{aligned} \quad (56)$$

[24] *Proof*: Make the change of variables $\hat\psi=2\psi$, $\hat\gamma=4\gamma$, $\hat A_3=\omega$. Then the field Eqs. (50) become identical to the Einstein-Maxwell Eqs. (55) for a cylindrical electromagnetic universe with $\hat A_2=0$. Consequently, the corresponding proof in the electromagnetic case (see Appendix) goes through here.

[25] M. Misra and L. Radhakrishna, Proc. Natl. Inst. Sci. India **28A**, 632 (1962); B. K. Harrison, Phys. Rev. (to be published).

[26] Note that $1/\sqrt{G}=3.48\times10^{24}$ cm G$=1.044\times10^{27}$ V.

[27] These can be obtained from the Einstein field equations (8) plus the Maxwell equation $f^{ij}{}_{;j}=0$. They are also derived by Misra and Rhadakrishna and by Harrison (Ref. 25).

where β, q, λ, δ, η are constants.[28] This solution was first given by Melvin,[5] but in a different coordinate system. Elementary flatness of the metric at $r=0$ requires $q=0$, $\ln\eta=0$. We can transform $\ln\lambda$ to 0 without leaving the canonical coordinate system by the transformation (7); and δ, having no physical significance, may be set to zero. Consequently, the general static magnetic solution regular at the origin is

$$\begin{aligned}&\psi=\ln(1+\beta^2r^2)\,,\quad \gamma=2\ln(1+\beta^2r^2)\,,\\&A_2=0\,,\quad A_3=1/[\beta(1+\beta^2r^2)]\,,\\&\mathbf{B}=(2\beta/\sqrt{G})/(1+\beta^2r^2)^2\mathbf{e}_z\,,\quad \mathbf{E}=\mathbf{0}\,,\\&ds^2=(1+\beta^2r^2)^2(dt^2-dr^2-dz^2)-r^2/(1+\beta^2r^2)^2d\varphi^2\,.\end{aligned}\tag{57}$$

This cylindrical electromagnetic universe will be called Melvin's magnetic universe, after Melvin,[5] who first discussed it. Melvin's magnetic universe is not the only physically acceptable, static magnetic solution of the field equations (55). The general solution (56), although irregular on the symmetry axis, is acceptable as the exterior solution of a static, solid cylinder surrounded by and containing a magnetic field. In fact, for $\beta=0$, (55) reduces to the vacuum gravitational field outside a solid cylinder, which was discussed extensively in Secs. III and IV.

In addition to static magnetic solutions to the field equations (55), there are also static solutions formed by combining longitudinal electric and magnetic fields in any way such that (1) at every point the electromagnetic energy density $(\mathbf{E}^2+\mathbf{B}^2)/8\pi$ is the same as in Melvin's universe, and (2) the ratio of electric field strength to magnetic field strength is the same everywhere. All such static solutions have the same metric (gravitational field).

B. The *C* Energy of Cylindrical Electromagnetic Universes

1. Expressions for the C Energy

We turn now from our general description of cylindrical electromagnetic universes to a discussion of their C-energy properties. A C.E.U. has a C-energy density on the canonical hypersurface, $t=$ constant, of

$$\begin{aligned}\epsilon_{ce}&=(1/h_z)[e^{\psi-\gamma}(\psi'^2+\dot\psi^2)/8\pi G+e^{\gamma-\psi}(\mathbf{B}^2+\mathbf{E}^2)/8\pi]\\&=(8\pi Gh_z)^{-1}[e^{\psi-\gamma}(\psi'^2+\dot\psi^2)\\&\quad+e^{-\psi-\gamma}(A_2'^2+\dot A_2^2)+(e^{3\psi-\gamma}/r^2)(A_3'^2+\dot A_3^2)]\,.\end{aligned}\tag{58}$$

[Cf. Eqs. (26) and (53).] In Melvin's magnetic universe (the only static C.E.U. of this type) ϵ_{ce} reduces to the ordinary electromagnetic energy density on the symmetry axis

$$\epsilon_{ce}(\text{Melvin's universe at } r=0)=(\mathbf{B}^2+\mathbf{E}^2)/8\pi\,.$$

It is convenient to divide the C-energy density into a kinetic part involving $\dot\psi^2$ and $\dot A_j^2$, and a potential part involving ψ'^2 and $A_j'^2$; so that the kinetic C-energy K, and the potential C-energy P, per unit standard length contained inside a radius r on the hypersurface $t=$ constant are

$$\begin{aligned}K(r)&=(1/4G)\int_0^r[r\dot\psi^2+re^{-2\psi}\dot A_2^2+(e^{2\psi}/r)\dot A_3^2]dr\,,\\P(r)&=(1/4G)\int_0^r[r\psi'^2+re^{-2\psi}A_2'^2+(e^{2\psi}/r)A_3'^2]dr\,.\end{aligned}\tag{59}$$

The total C energy per unit standard length inside r is $E(r)=K(r)+P(r)$.

As in the vacuum case (cf. Sec. IV-B and IV-D), the difference between potential and kinetic C-energies is a Lagrangian for the coupled Einstein-Maxwell equations, which govern the dynamical interaction of the potential functions ψ and A_j:

$$\delta\int_{t_1}^{t_2}(P\text{-}K)dt=0\tag{60}$$

has, as Euler-Lagrange equations, Eqs. (55a), (55b), and (55c).

Note that Eq. (55d) is *not* obtained from the variational principle (60); it must be imposed independently. Equation (55d) results from our demand that space-time be invariant under reflection in planes containing the symmetry axis or perpendicular to it ($R_{23}=8\pi GT_{23}=0$). If that demand is relaxed, e.g., by the use of the line element (48) rather than (6), then six independent field equations analogous to (55a)–(55f) result. The first four of these new equations are coupled nonlinear partial differential equations for the four gravitational and electromagnetic potential functions ψ, ω, A_2, A_3; the last two are expressions for $\dot\gamma$ and γ'. As usual, $E=\gamma/4G$ is the C energy per unit standard length. In this case the difference between the kinetic and potential C energies provides a Lagrangian for the four coupled equations governing ψ, ω, A_2, A_3. If ω is set to zero in these four equations, Eqs. (55a)–(55d) are recovered. Hence, Eq. (55d) need not be imposed independently of the variational principle (60) if we wait until after the variation is performed to set $\omega=0$ (i.e., to impose reflection symmetry).

2. C-Energy Minimum Properties of Cylindrical Electromagnetic Universes

The C energy of a cylindrical electromagnetic universe has an absolute minimum property similar to the C-energy minimum principle of the vacuum case. Let the gravitational potential function ψ be specified at some radius $r=r_1$, and let the total longitudinal magnetic field inside r_1, $\Phi(r_1)$ [Eq. (54)], be specified. [Equivalently, since an additive constant in the electromagnetic potential A_j has no physical significance, specify $A_3(0)$, $A_3(r_1)$, and $\psi(r_1)$.] *Then, that C.E.U. which gives an absolute minimum of both the potential C energy and the total C energy per unit standard length inside r_1 on the canonical hypersurface $t=$ constant, sub-*

[28] This general solution is derived in the same way as Melvin (Ref. 5) derives the corresponding solution in the Weyl canonical coordinates.

ject to these constraints, is Melvin's magnetic universe.[29] (Recall that Melvin's magnetic universe is the only static magnetic C.E.U.)

Similarly, consider a system composed of a solid cylinder, together with whole-cylinder-symmetric electromagnetic fields. Specify the gravitational potential function ψ at two radii, r_1 and r_2, outside the cylinder, and specify the total longitudinal magnetic flux, $\Phi(r_2)-\Phi(r_1)$, in the shell between r_1 and r_2. Equivalently, specify

$$\begin{aligned} \psi(r_1)&=\psi(1)\,, & \psi(r_2)&=\psi(2)\,, \\ A_3(r_1)&=A_3(1)\,, & A_3(r_2)&=A_3(2)\,. \end{aligned} \tag{61}$$

Those configurations of gravitational and electromagnetic fields which make the potential and total C energy per unit standard length in the shell $r_1<r<r_2$ *of the hypersurface* $t=constant$ *an* absolute minimum, *subject to constraints* (61), *are all identical in the region* $r_1<r<r_2$. *There they coincide with that* unique static system (56) *which satisfies the boundary conditions* (61).[30] Both of these C-energy minimum properties are proved in the Appendix.

VI. *C* ENERGY AS A TOOL FOR ANALYZING DYNAMICS

A. Examples Considered Elsewhere

As a result of its conservations laws and minimum properties, C energy can be a very powerful tool in the analysis of the dynamics of whole-cylinder-symmetric systems. For instance, in Sec. VI.C we used it to clarify the nature of Einstein-Rosen gravitational radiation; and in a separate communication[4] we use it to show that Melvin's magnetic universe (Sec. V) is stable against arbitrarily large radial perturbations which are confined to a finite region about the axis of symmetry. More particularly, if the gravitational and electromagnetic fields of Melvin's magnetic universe are strongly distorted inside a radius $r=R$ and then released, they will oscillate turbulently, emitting gravitational and electromagnetic waves, until all the C energy associated with the perturbation has been radiated away toward radial infinity. Then they will settle down into the configuration of minimum C energy, Melvin's unperturbed, static universe.

B. Resistance of Magnetic Flux to Gravitational Collapse

1. In Cylindrical Electromagnetic Universes

As a third example of the usefulness of the C-energy concept, we shall prove the following theorem about the dynamics of all cylindrical electromagnetic universes with longitudinal magnetic field and azimuthal electric field: *Magnetic field lines can never become so concentrated near the symmetry axis of such a C.E.U. that they undergo gravitational collapse before a finite canonical coordinate time has elapsed.*[31] The proof is quite simple. The first step is to prove the lemma: *as magnetic field lines become compressed into a smaller and smaller region about the axis of symmetry, the C energy in that region gets larger and larger, tending finally toward infinity.* In particular, suppose that a total magnetic flux Φ_o is to be lost in gravitational collapse. Consider the universe at a canonical coordinate time when the flux Φ_o has been squeezed into a region of proper radius s. The total C energy inside this region is [cf. Eq. (59)]

$$E(r_s)\geq(1/4G)\int_0^{r_s}[r\psi'^2+(e^{2\psi}/r)A_3'^2]dr\,,$$

where r_s is the value of the radial coordinate corresponding to proper radial distance s. Make the change of variables $dl=e^{-\psi}dr$, $y=re^{-\psi}$. Then

$$E(r_s)\geq(1/4G)\int_0^{l_s}(1/y)[(1-dy/dl)^2+(dA_3/dl)^2]dl\,.$$

Let y_m be the maximum value taken on by y in the interval $0<l<l_s$, and let it be taken on at $l=l_m$. Then

$$E(r_s)\geq(1/4G)(1/y_m)\left\{\int_0^{l_m}(1-dy/dl)^2 + \int_0^{l_s}(dA_3/dl)^2dl\right\}.$$

Applying Schwarz's inequality we then find

$$E(r_s)\geq(1/4G)\{[l_m-y_m]^2/[l_my_m] + [A_3(r_s)-A_3(0)]^2/[l_sy_m]\}\,. \tag{62}$$

The actual values of l_s, l_m, and y_m will depend upon the particular C.E.U. being considered. Whatever those values may be, we have

$$E(r_s)\geq\frac{1}{16\pi^2}\frac{\Phi_o^2}{l_sl_m[1+(G/4\pi^2)\Phi_0^2/l_sl_m]^{1/2}} \tag{63}$$

as is seen by minimizing expression (62) with respect to y_m and then dropping the first term, and by using

$$A_3(r_s)-A_3(0)=-[(\sqrt{G})/2\pi]\Phi_o\,.$$

Now, the proper radius of the region containing the C energy (63) is

$$s=\int_0^{r_s}e^{\gamma-\psi}dr=\int_0^{l_s}e^{\gamma}dl\geq l_s\geq l_m\,.$$

[29] Melvin's magnetic universe involves three arbitrary constants corresponding to the three constraints we impose. However, two of these constants can be transformed away; only one has physical significance. Cf. the discussion preceding Eq. (57).

[30] The one-to-one correspondence between the 4-parameter class of solutions (56) and the 4-parameter class of boundary conditions (61) is proved in the appendix.

[31] Our result is confined to C.E.U.'s for which the hyperbolic canonical line element (6) is nonsingular throughout some initial space-like hypersurface, except possibly at infinity. The case of a C. E. U. whose initial space-like hypersurface cannot be covered by hyperbolic canonical coordinates in a nonsingular manner will be dealt with in a later communication. (Cf. footnote 10.)

Consequently, when the magnetic field lines have been squeezed so far that $s \ll \sqrt{G}\Phi_o$, the C energy associated with them has risen to

$$E(r_s) \gtrsim (8\pi\sqrt{G})^{-1}|\Phi_o|/s; \tag{64}$$

and as the magnetic field is squeezed more and more tightly beyond this point, its C energy rises toward infinity.

Now consider a C.E.U. at some arbitrary moment of coordinate time when the electromagnetic and gravitational field are well behaved everywhere.[31] At that moment, inside any finite radius about the symmetry axis there is only a finite C energy per unit standard length. Consequently, the infinite C energy needed to induce the collapse of the magnetic field near the symmetry axis must be supplied from radial infinity. But C energy is locally measurable and is thus governed by the fundamental principles of relativity: It cannot be propagated with a speed exceeding that of light. In the canonical coordinate system, the coordinate speed of light is $dr/dt=1$; consequently, the infinite C energy needed to induce gravitational collapse can be supplied from radial infinity only after the lapse of an infinite coordinate time. Our theorem is proved.

2. *In the Physical Universe*

This theorem has implications for the behavior of magnetic fields in the physical universe as well as in cylindrical electromagnetic model universes. Consider a locally whole-cylinder-symmetric bundle of magnetic field lines which may be part of some larger, noncylindrically-symmetric configuration of magnetic fields and matter.[32] No matter how tightly this bundle of field lines is squeezed, it cannot be induced to undergo cylindrical gravitational collapse. For collapse to occur, an infinite C energy per unit length would have to be supplied from outside the whole-cylinder-symmetric region. It could be brought in only by very strong, imploding cylindrical gravitational and electromagnetic waves. But it would seem natural to rule out such incoming cylindrical radiation from a noncylindrical region as physically unreasonable; and it is not even clear that such strong incoming radiation is allowed in principle.

We can extrapolate still further: We have seen that the factor impeding collapse is the divergent increase in the C energy as a fixed amount of flux is compressed into a smaller and smaller region. If magnetic flux resists compression when it is distributed whole-cylinder-symmetrically, then it most probably does so regardless of how it is distributed. Hence, the following conjecture (*principle of flux resistance to gravitational collapse*): *In a configuration of electromagnetic fields gravitationally collapsing to a singularity, the total electric and magnetic flux across a 2-surface through the collapsing region must vanish as the singularity is reached—a nonzero flux will stop the collapse.*[33] More precisely, let S_2 be an arbitrary space-like 2-surface passing through the singularity (or through the point at which the singularity is evolving, just before it is reached); and let $\bar{S}_2$ be that segment of S_2 located in (or very near) the singularity. Then, the principle of flux resistance to gravitational collapse states

$$\int_{\bar{S}_2} f_{ij}d\tau^{ij} = \int_{\bar{S}_2} {}^*f_{ij}d\tau^{ij} = 0. \tag{65}$$

[Here ${}^*f_{ij}=\sqrt{(-g)}\epsilon_{ijkl}f^{kl}$ is the dual of the electromagnetic field tensor f^{ij}.] An alternative statement of the principle is: *As collapse proceeds toward a singularity, the electric and magnetic field lines separate into those which are to be completely destroyed in the singularity and those which are to be left completely free.* (See Fig. 3.)

Comments on the conjecture: (1) In particular, a cloud of electromagnetic radiation can undergo collapse; but the center of a dipole magnetic field cannot (there would be field lines left protruding from the collapsed region). Also, a toroidal bundle of magnetic field lines can collapse to its center (major and minor radii going to zero simultaneously), but it cannot collapse to its guiding line (minor radius vanishing, major radius finite).[34] (2) This conjecture is meant to apply only when electromagnetic fields alone are present; we exclude from attention the case where particles or neutrinos or other fields contribute to the stress-energy. (3) We suggest that electric flux, as well as magnetic flux, resists collapse because, in the absence of electric charge, the electric and magnetic fields are dynamically equivalent[35]; interchanging them in a solution to the Einstein-Maxwell equations leads to another solution.

In addition to the C-energy analyses presented here, and the proof[4] that Melvin's magnetic universe can never undergo collapse no matter how turbulently it is perturbed, there is other support for the principle of flux resistance to gravitational collapse: The Kruskal extension[36] of the Schwarzschild solution is a prototype for spherical gravitational collapse[37,38]; Fuller and Wheeler[37] have shown that the throat of the Einstein-

[32] We assume that in the whole-cylinder-symmetric region of the initial hypersurface the hyperbolic canonical line element is nonsingular. (Cf. Refs. 10 and 31.)

[33] For a more detailed discussion of this conjecture, see the chapter by K. S. Thorne, in the *Proceedings of the Second Texas Symposium on Relativistic Astrophysics* (The University of Chicago Press, Chicago, 1965).

[34] For independent evidence that a toroidal magnetic field may not collapse to its guiding line, see the chapter by K. S. Thorne in I. Robinson, A. Schild, and E. Schücking, *Quasistellar Sources and Gravitational Collapse* (The University of Chicago Press, Chicago, 1965); for arguments that collapse to the center *should* occur for sufficiently massive toruses, see the above, as well as J. A. Wheeler in C. DeWitt and B. DeWitt, *Relativity, Groups and Topology* (Gordon and Breach Science Publishers, New York, 1964).

[35] Although the electric and magnetic fields are dynamically equivalent according to classical theory, they differ when vacuum polarization effects are taken into account. Vacuum polarization becomes important when the fields reach the order of magnitude $E\sim B\sim F_{\rm crit}=mc^2/[e(\hbar/mc)]=4.4\times10^{13}$ G$=1.3\times10^{18}$ V/m. The quite new considerations which enter in this region are not dealt with here.

[36] M. D. Kruskal, Phys. Rev. **119**, 1743 (1960).

[37] R. W. Fuller and J. A. Wheeler, Phys. Rev. **128**, 919 (1962).

[38] J. A. Wheeler, Ref. 26, Sec. 9.

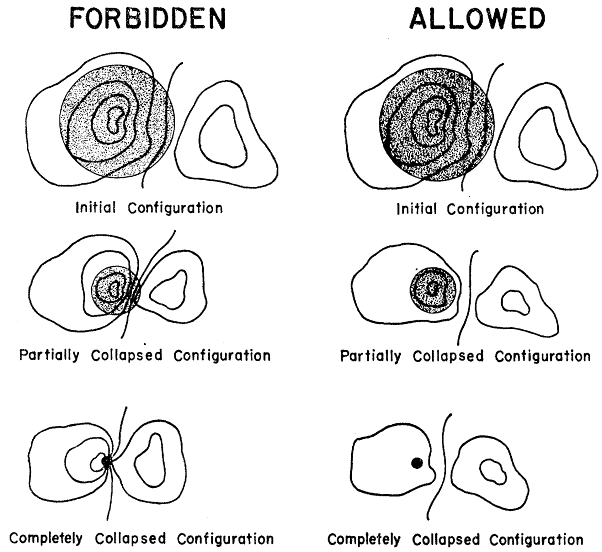

FIG. 3. The principle of flux resistance to gravitational collapse. Two modes of collapse for a magnetic field configuration are shown, one forbidden and the other allowed by the principle of flux resistance to gravitational collapse. The solid lines represent magnetic field lines, and the shaded regions represent the regions of space in which gravitational collapse toward a singularity is occurring. The forbidden and allowed collapse modes differ in that there is a magnetic flux threading the collapsed region in the forbidden mode, but not in the allowed mode. In the allowed mode every field line is either completely swallowed or left completely free.

Rosen bridge[39] of the Schwarzschild solution collapses, as it is followed in time in the Kruskal extension. However, if the bridge is threaded by electric or magnetic field lines, the flux acts as a cushion, causing the throat to bounce rather than collapse to a singularity.[40] Even a very minute flux is capable of preventing collapse.

VII. CONCLUSIONS

In this paper we have introduced the concept of C energy for systems with whole-cylinder symmetry, we have discussed its many energy-like properties (for a summary of them see Sec. I), and we have demonstrated the usefulness of C energy in the analysis of the dynamics of whole-cylinder-symmetric systems. Three major questions remain to be investigated: (1) What is the relationship between the C energy of a massive, thin-ring torus and its Schwarzschild mass? (2) What is the relationship between C energy and other energy concepts of general relativity, especially the localizable energies of Bergmann, Komar, and Møller?[2] (3) Can energy-like concepts similar to C energy be formulated for other nonasymptotically flat systems?

With regard to the last question: Our approach to the problem of energy in the whole-cylinder-symmetric case might prove fruitful in other cases. For a given nonasymptotically flat situation, one might seek a canonical coordinate system which most completely embodies the symmetries of the problem. In this canonical coordinate system the proper definition of an energy-like quantity might be quite evident; and once formulated there, it might be generalizable into a covariant form.

ACKNOWLEDGMENTS

The author is indebted to Professor J. A. Wheeler for many discussions in which he emphasized the role that an energy-like quantity could play in the stability analysis of Melvin's magnetic universe. It was out of these discussions that the work presented here grew. Professor Wheeler and Professor M. A. Melvin both provided valuable criticism of this presentation of C energy.

APPENDIX: PROOF OF C-ENERGY MINIMUM PRINCIPLES FOR C.E.U.'s

Our proof of the C-energy minimum principles for cylindrical electromagnetic universes will be based on the following Lemma:

Lemma: Let $r_1\neq 0$, $r_2>r_1$, $\psi(1)$, $\psi(2)$, $A_3(1)$, $A_3(2)$, *be given numbers; and let* $\mathcal{C}$ *be the class of all continuous, piecewise smooth, vector-valued functions* $(\psi(r),A_3(r))$ *on the interval* (r_1,r_2), *such that*

$$\left.\begin{aligned}\psi(r_i)&=\psi(i)\\ A_3(r_i)&=A_3(i)\end{aligned}\right\} i=1,2. \tag{A1}$$

Then

$$\bar{P}\equiv\int_{r_1}^{r_2}[r\psi'^2+(e^{2\psi}/r)A_3'^2]dr \tag{A2}$$

is made an absolute minimum over $\mathcal{C}$ *by that unique solution to the Euler-Lagrange equations for* $\delta\bar{P}=0$, *which is in* $\mathcal{C}$

$$\begin{aligned}\psi_0&=\ln(1+\beta^2r^{2q})+(1-q)\ln r+\ln\lambda,\\ A_{30}&=(1/\beta\lambda)[1/(1+\beta^2r^{2q})]+\delta.\end{aligned} \tag{A3}$$

This lemma follows directly from a very powerful theorem on absolute minima in the calculus of variations, due to Graves.[41] However, rather than relying on the theorem of Graves, we shall here present a simple proof due to Bargmann.[42] Appreciation is expressed to Professor Bargmann for permission to quote his proof here.

Introduce the change of variables

$$x=A_3,\quad y=re^{-\psi},\quad \sigma=\ln r.$$

Then $\bar{P}$ becomes

$$\bar{P}=2[\psi(2)-\psi(1)]-\ln(r_2/r_1)+J,$$

where

$$J=\int_{\sigma_1}^{\sigma_2}\left[\frac{(dx/d\sigma)^2+(dy/d\sigma)^2}{y^2}\right]d\sigma.$$

The general solution (A3) to the Euler-Lagrange equa-

[39] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).
[40] J. C. Graves and D. R. Brill, Phys. Rev. **120**, 1507 (1960); G. Brigman (to be published).

[41] L. M. Graves, Ann. Math. **28**, 153 (1926).
[42] V. Bargmann (private communication).

tions for $\delta\bar{P}=\delta J=0$ is then

$$x_0=a\tanh(b\sigma+c)+f;\quad y_0=a/\cosh(b\sigma+c)\,,\qquad (A3')$$

or

$$x_0=g=\text{constant};\quad y_0=\exp(h\sigma+k)\,,\qquad (A3'')$$

where a, b, c, f, or g, h, k are constants related to the β, q, λ, δ of Eq. (A3). The solution (A3′) is a semicircle in the x-y plane

$$y_0^2+(x_0-f)^2=a^2\,,\qquad (A4)$$

while the solution (A3″) is a line.

The one-to-one correspondence between boundary conditions (A1) and solutions (A3) of the Euler-Lagrange equations is seen as follows: Let the boundary conditions (A1) be given, and set $Z=x+iy$. If $x_1=x_2$, then (A3″) is the form of the solution, and

$$g=x(1)$$

$$\left.\begin{aligned}h\sigma_1+k&=\ln y(1)\\ h\sigma_2+k&=\ln y(2)\end{aligned}\right\}\to h,\,k\,.$$

If $x(1)\neq x(2)$, we get from double application of (A4)

$$f=(|Z(1)|^2-|Z(2)|^2)/2[x(1)-x(2)]\,.$$

Since $y=re^{-\psi}>0$, we have $a>0$. Consequently, by (A4)

$$a=\{y(1)^2+[x(1)-f]^2\}^{1/2}\,.$$

Having determined f and a uniquely, we get b and c from (A3′):

$$b\sigma_1+c=\tanh^{-1}\{[x(1)-f]/a\}\,,$$

$$b\sigma_2+c=\tanh^{-1}\{[x(2)-f]/a\}\,.$$

Let the constants in (A3′) or (A3″) be so computed. If the solution is of the form (A3′), we change it to the form (A3″) by the transformation

$$\omega=-(Z-f+a)/(Z-f-a)=u+iv\,,\qquad (A5)$$

which gives

$$u_0=0\,,\quad v_0=\exp(-b\sigma-c)\,.\qquad (A3''')$$

This transformation leaves the form of J invariant:

$$J=\int_{\sigma_1}^{\sigma_2}\left[\frac{(du/d\sigma)^2+(dv/d\sigma)^2}{v^2}\right]d\sigma\,.\qquad (A6)$$

Now, let $(\psi(r),A_3(r))$ be an arbitrary function in the class $\mathcal{C}$ mentioned in the lemma. If $(u(\sigma),v(\sigma))$ is this function transformed to the new variables, an application of Schwarz's inequality gives

$$J[\psi,A_3]=\int_{\sigma_1}^{\sigma_2}\frac{1}{v^2}\left[\left(\frac{du}{d\sigma}\right)^2+\left(\frac{dv}{d\sigma}\right)^2\right]d\sigma$$

$$\geq\int_{\sigma_1}^{\sigma_2}\frac{1}{v^2}\left(\frac{dv}{d\sigma}\right)^2d\sigma\geq\frac{1}{(\sigma_2-\sigma_1)}\left[\int_{\sigma_1}^{\sigma_2}\frac{1}{v}\left(\frac{dv}{d\sigma}\right)d\sigma\right]^2$$

$$=\frac{\ln(v_2/v_1)}{\sigma_2-\sigma_1}=b^2(\sigma_2-\sigma_1)=J[\psi_0,A_{30}]\,.$$

[The last equality follows from inserting (A3‴) into (A6).] Since $\bar{P}$ and J differ by a constant depending only on the endpoints of the integration, the above inequality is equivalent to

$$\bar{P}[\psi,A_3]\geq\bar{P}[\psi_0,A_{30}]\,,\quad\text{for all }(\psi,A_3)\in\mathcal{C}\,.$$

Q.E.D.

Now, consider an arbitrary cylindrical electromagnetic universe which, at a particular moment of canonical coordinate time satisfies the boundary conditions (A1).

Since the potential C energy per unit standard length contained in the shell between r_1 and r_2 is just

$$P(r_2)-P(r_1)=\bar{P}/4G+(1/4G)\int_{r_1}^{r_2}re^{-2\psi}A_2'^2dr\,,$$

our lemma tells us that $P(r_2)-P(r_1)$ is made an absolute minimum by cylindrical electromagnetic universes which, in the shell $r_1<r<r_2$, are identical to the static magnetic C.E.U. (ψ_0, $A_{20}=0$, A_{30}). Since these C.E.U.'s have $K(r_2)-K(r_1)=0$, they also give an absolute minimum of the total C energy $E(r_2)-E(r_1)$. This was one of our C-energy minimum principles.

The above C-energy minimum principle is restricted to the case $r_1>0$. Suppose $r_1=0$ and

$$A_3(0)=A_3(0)\,,\quad A_3(r_2)=A_3(2)\,,\quad \psi(r_2)=\psi(2)\qquad (A7)$$

are specified. Let that C.E.U. which minimizes the potential C energy inside $r=r_2$ have the form

$$A_2(r)=0,\quad A_3(r)={}^mA_3(r)\,,\quad \psi(r)={}^m\psi(r)\,,$$

and let it give $P(r_2)=\mathcal{P}[m]$. Then for all $\epsilon>0$, $({}^mA_3,{}^m\psi)$ also gives an absolute minimum of $P(r_2)-P(\epsilon)$ subject to the end-point constraints $(A_3(\epsilon),\psi(\epsilon))=({}^mA_3(\epsilon),{}^m\psi(\epsilon))$, $(A_3(r_2),\psi(r_2))=({}^mA_3(r_2),{}^m\psi(r_2))$. (Otherwise $\mathcal{P}[m]$ could be made smaller by changing the configuration on $\epsilon<r<r_2$.) Consequently, $({}^mA_3,{}^m\psi)$ must have the form (A3) on the entire interval $(0,r_2)$. Since $q\neq1$ leads to a divergent $P(r_2)$, $({}^mA_3,{}^m\psi)$ must have $q=1$; i.e., it must correspond to that Melvin universe which fits the boundary conditions on ψ and A_3.

Because the kinetic C energy of Melvin's magnetic universe is zero, it minimizes the total C energy per unit standard length inside r_2, as well as the potential C energy.

These minimum C-energy properties of Melvin's magnetic universe complete the set of C-energy minimum principles for cylindrical electromagnetic universes.