Gapless Hamiltonians for the Toric Code Using the Projected Entangled Pair State Formalism
Supplementary Material

Carlos Fernández-González,[1,2] Norbert Schuch,[3,4] Michael M. Wolf,[5] J. Ignacio Cirac,[6] and David Pérez-García[2]

[1]*Departamento de Física de los Materiales, Universidad Nacional de Educación a Distancia (UNED), 28040 Madrid, Spain*

[2]*Departamento de Análisis Matemático & IMI, Universidad Complutense de Madrid, 28040 Madrid, Spain*

[3]*Institute for Quantum Information, California Institute of Technology, MC 305-16, Pasadena, Calfornia 91125, USA*

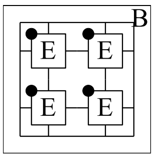
[4]*Institut für Quanteninformation, RWTH Aachen, 52056 Aachen, Germany*

[5]*Department of Mathematics, Technische Universität München, 85748 Garching, Germany*

[6]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Strasse 1, 85748 Garching, Germany*

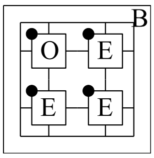
APPENDIX A: GROUND SPACE OF THE UNCLE HAMILTONIAN FOR THE TORIC CODE

In this appendix, we derive the structure of the ground space of the toric code uncle Hamiltonian. Recall first how parent and uncle Hamiltonians are constructed. Being E (O) the orthogonal projection on $(\mathbb{C}^2)^{\otimes 4}$ onto the subspace of even (odd) parity spin configurations, we can consider the spaces

$E_{22} = \Big\{$ 

, B boundary condition$\Big\}$

and

$O_{22} = \Big\{ \sum_{\text{pos } O}$ 

, B boundary condition$\Big\}$.

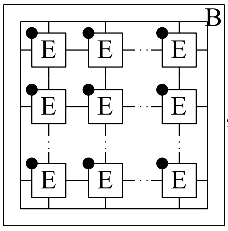
The parent Hamiltonian is then constructed as the sum over every 2×2 square sublattice of the projection $h_{\rm loc}$ onto the orthogonal complement of E_{22} (that is, $\ker h_{\rm loc} = E_{22}$). The uncle Hamiltonian is constructed in the same way, but its local Hamiltonian $h'_{\rm loc}$ has as kernel the space $E_{22} + O_{22}$.

The structure of the ground space of the parent Hamiltonian can be found in [2]. We will follow here the same steps in deriving the ground state subspace of the uncle Hamiltonian, allowing the reader interested in further details to find them in [2].

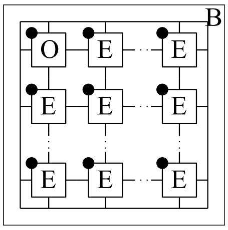
We will prove in Proposition 1 that the intersection of the kernels of a family of local Hamiltonians $h'_{\rm loc}$ effectively acting on a given sublattice with dimension $n \times m$, which we will call S_{nm}, keeps having the same structure. It is the vector space:

$$S_{nm} = E_{nm} + O_{nm}$$

where

$$E_{nm} = \mathrm{span}\Big\{$$ 

$$,\ B \text{ boundary condition}\Big\},$$

$$O_{nm} = \mathrm{span}\Big\{ \sum_{\mathrm{pos}\,O}$$ 

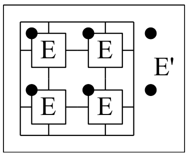
$$,\ B \text{ boundary condition} \Big\}.$$

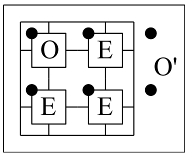
Let us note that in E_{nm} only even parity boundary conditions give rise to non-zero vectors, and in O_{nm} only odd boundary conditions do so. However, as we show in Proposition 2, the O summand disappears when imposing periodic boundary conditions to the full $N \times M$ lattice, and the ground space of the uncle Hamiltonian is exactly the same as the ground space of the parent Hamiltonian.

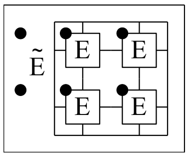
Let us first prove that the intersection of the kernels of the h'_{loc} is indeed described by S_{nm}. The following proposition serves as the first step in an induction over n and m.

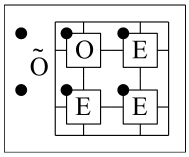
Proposition 1 (Intersection property) *Given a* 2×3 *lattice,* $S_{22} \otimes \mathbb{C}^{2^8} \cap \mathbb{C}^{2^8} \otimes S_{22} = S_{23}$.

Proof. Let $|\phi\rangle$ be an unnormalized vector in $S_{22} \otimes \mathbb{C}^{2^8} \cap \mathbb{C}^{2^8} \otimes S_{22}$. This vector can be written in two different ways:

$$|\phi\rangle =$$ 

$$+ \sum_{\mathrm{pos}\,O}$$ 

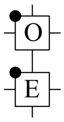
$$=$$ 

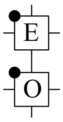
$$+ \sum_{\mathrm{pos}\,O}$$ 

(1)

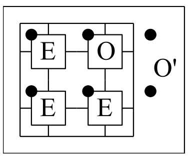
W.l.o.g. we can assume that the boundary conditions given by E' and $\tilde{E}$ have always even parity, and those given by O' and $\tilde{O}$ have always odd parity.

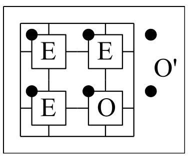
We will now perform the projection

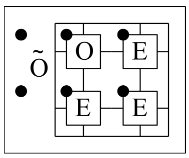


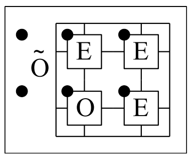
$$+$$


on the physical levels in the second column. As different configurations of E's and O's are orthogonal, this exactly selects this pattern in the second column, and we obtain the equality



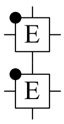
$$+$$ 

$$=$$ 

$$+$$ 

. (2)

In order to infer the structure of O' and $\tilde{O}$, we will now project either the first or the third column onto



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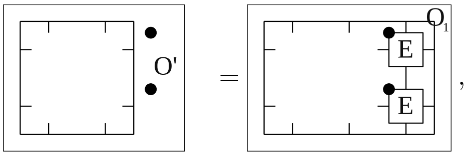
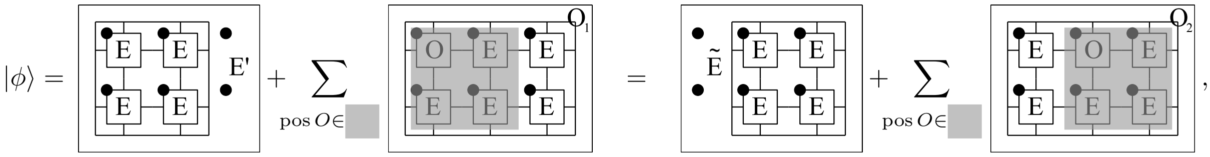
and use the fact that *i)* O' and $\tilde{O}$ have odd parity and *ii)* the resulting tensor network of E's and O's is equivalent to a projection onto the odd parity subspace. By projecting the first row, we find that



(3)

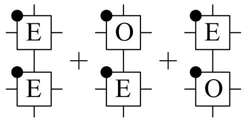
and by projecting the third row, we obtain a corresponding equation for $\tilde{O}$ with a boundary O_2. Re-substituting in (2), we find that $O_1 = O_2$; moreover, the new boundary condition has odd parity. Substituting Eq. (3) and its analog for $\tilde{O}$ back into Eq. (1), we obtain



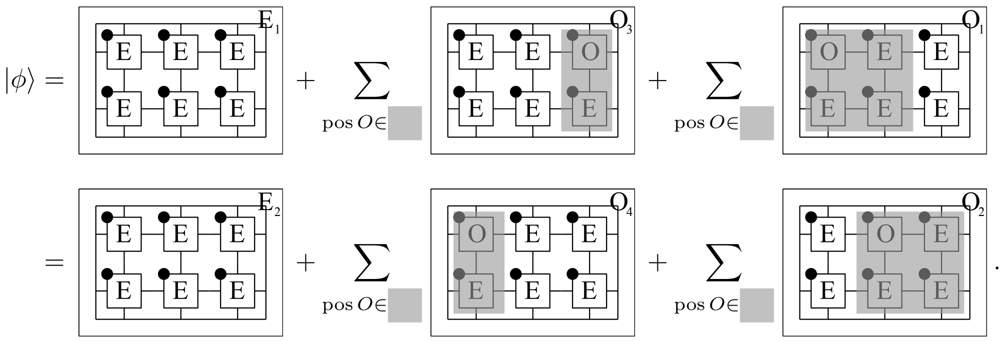
(4)

where the sums run over all positions of the O tensor inside the gray regions.

We now use the same trick to also infer the structure of E' and $\tilde{E}$: We apply the projection



in either the first or the third column of Eq. (4); after re-substituting the resulting conditions, we find that



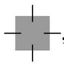
By matching equal patterns of E's and O's, we can easily check that $E_1 = E_2$, $O_1 = O_4$, and $O_2 = O_3$. Thus, there exist unique even and odd spin parity boundary conditions $B_E = E_1 = E_2$ and $B_O = O_1 = O_2 = O_3 = O_4$ which describe the state $|\phi\rangle$ as an element from S_{23}. □

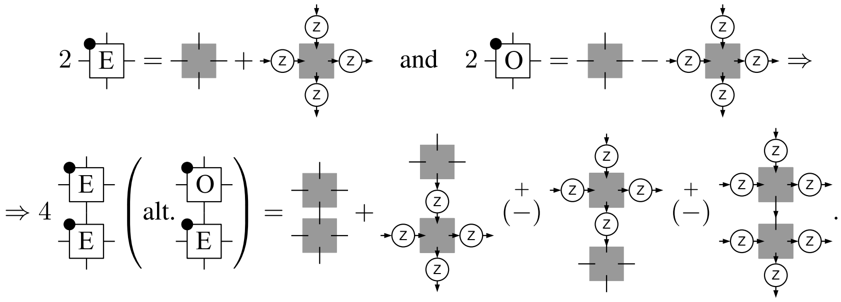
Using this argument inductively, we can indeed prove for any contractible rectangle of size $n \times m$ (or in fact any contractible region) that S_{nm} is equal to the intersection of the kernels of the local Hamiltonians h'_{loc} which act inside the region.

Proposition 2 (Closure property) *The ground space of the uncle Hamiltonian coincides with the ground space of the parent Hamiltonian.*

Proof. Exploiting the σ_z symmetry of E and O tensors, we can prove that for a state to lie in the kernel of every h'_{loc}, and therefore in the kernel of H', it should remain invariant under the projection at any two sites connected by any bond onto $\text{span}\{|00\rangle + |11\rangle, |0\rangle\sigma_z(|0\rangle) + |1\rangle\sigma_z(|1\rangle)\} = \text{span}\{|00\rangle, |11\rangle\}$.

Let us show why.

If we denote the identity by , we have



The first and last summands remain invariant under projection onto $\mathrm{span}\{|00\rangle + |11\rangle\}$ at the sites conected by the bond, and second and third summands under projection onto $\mathrm{span}\{|0\rangle\sigma_z(|0\rangle) + |1\rangle\sigma_z(|1\rangle)\}$. Therefore, if we project onto the sum of these two spaces the tensors remains unchanged.

Thus only linear combinations of the identity and σ_z may appear in the closure bonds when imposing periodic boundary conditions, and all periodic boundary conditions are necessarily even. Hence, given the full lattice and periodic boundary conditions, the elements in S_{NM} which came from O_{NM} need to vanish.

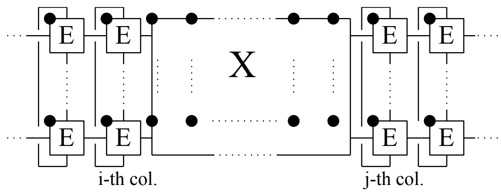
Consequently S_{final}, the ground space of the uncle Hamiltonian, is constructed by imposing periodic boundary conditions to E_{NM}, and therefore coincides with the ground state subspace of the toric code parent Hamiltonian H_{TC}, whose detailed construction can be found in [2]. □

APPENDIX B: SPECTRUM OF THE UNCLE HAMILTONIAN IN THE THERMODYNAMIC LIMIT

In this section we prove that, once we fix one of the two dimensions of the lattice, the spectrum of the uncle Hamiltonian H' in the thermodynamic limit is $\mathbb{R}^+$. The proof follows essentially the same steps as the one from [3] for the uncle Hamiltonian of non-injective matrix product states. We sketch the main steps adapted to the toric code case.

The tensor E appearing in this appendix is the previous one multiplied by a constant so that the MPS corresponding to a column of E tensors is in its normal form [1]. This constant depends on the column size.

The thermodynamic limit of H' can be studied as acting on the closure of the space $S = \cup_{i<j} S_{i,j}$, where

$S_{i,j} = \{\phi_{i,j}(X) =$  $, X\},$

and X runs over all the possible tensors.

We will usually omit the location of X whenever this does not matter due to translational invariance of the Hamiltonian.

Inside S we can find the space S^2 spanned by vectors with the tensor E everywhere but two places in which the tensor O is located. In the case the tensors O are located in places (i,j) and (k,l) of the lattice, we call this state $|\phi_{i,j}^{k,l}\rangle$.

For each of these vectors, $H'(|\phi_{i,j}^{k,l}\rangle) \in \mathrm{span}\{|\phi_{i+\delta_i,j+\delta_j}^{k+\delta_k,l+\delta_l}\rangle,\ \delta_i,\delta_j,\delta_k,\delta_l \in \{-1,0,1\}\}$. Therefore, $H'(S^2) \subseteq S^2$. Moreover, $H'|_{S^2}$ is bounded, and consequently it can be uniquely extended to $\overline{S^2}$, coinciding on this space with the self-adjoint extension of H' to $\overline{S}$, also called H'. Further study of self-adjoint extensions of unbounded symmetric operators can be found in [4].

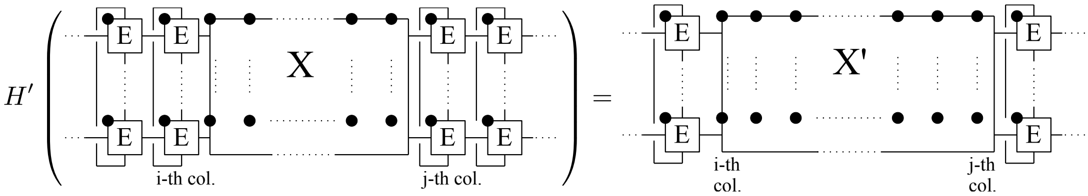
The unnormalized states $|\phi_{r,N}\rangle$, constructed as those from equation (4) for rectangular $r \times N$ regions, lie in S^2, and let us determine that $H'|_{\overline{S^2}}$ is gapless and there exists a sequence of elements in the spectrum $\{\lambda_i\}_i$ tending to 0. And one can find Weyl sequences in $\overline{S^2}$ associated to these values :

$$\frac{\|H(|\varphi_{\lambda_i,j}\rangle) - \lambda_i|\varphi_{\lambda_i,j}\rangle\|}{\||\varphi_{\lambda_i,j}\rangle\|} \overset{j\to\infty}{\longrightarrow} 0.$$

Using density arguments one can find these Weyl sequences lying in S^2. For any given λ_i and any $\delta > 0$ there exists a state $|\phi_{i,\delta}\rangle$ which is almost an eigenvector of H' for the value λ_i with an error at most δ , which means $\|(H' - \lambda_i \mathbb{I})|\phi_{i,\delta}\rangle\| \leq \delta \||\phi_{i,\delta}\rangle\|$.

If we write two –or more– of these states as $|\phi_{i_1,\delta_1}\rangle = |\phi(X_1)\rangle$ and $|\phi_{i_2,\delta_2}\rangle = |\phi(X_2)\rangle$, we can construct a new X by concatenating X_1 and X_2 separated by at least two columns of E tensors. We can call $|\phi_2(X_1, X_2)\rangle$ such a vector –the subindex indicates how many columns with E tensors are between X_1 and X_2. This vector is an approximated eigenvector of H' for $\lambda_{i_1} + \lambda_{i_2}$ with an error at most $\delta_1 + \delta_2$. Let us prove that.

The first thing we need to note is that for any $|\phi(X)\rangle$ there exists a tensor X' such that $H'(|\phi_{i,j}(X)\rangle) = |\phi_{i-1,j+1}(X')\rangle$



Due to the locality of H', we have that $H'(|\phi_2(X_1, X_2)\rangle) = |\phi_1(X_1', X_2)\rangle + |\phi_1(X_1, X_2')\rangle \overset{\delta_1+\delta_2}{\sim} \lambda_{i_1}|\phi_2(X_1, X_2)\rangle + \lambda_{i_2}|\phi_2(X_1, X_2)\rangle = (\lambda_{i_1} + \lambda_{i_2})|\phi_2(X_1, X_2)\rangle$.

This family of vectors let us see that any finite sum of λ_i lies in the spectrum of H'. The set of finite sums of a sequence of elements tending to 0 is dense in the positive real line, and the spectrum is closed, hence $\sigma(H') = \mathbb{R}^+$.

The same tensors X_i and X_i' can be used in vectors in big enough finite dimensional lattices to show that the spectra of the uncle Hamiltonians H' on finite dimensional lattices tend to be dense in the positive real line. A similar treatment is detailed in [3].

[1] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, Quant. Inf. Comput. **7**, 401 (2007), quant-ph/0608197.
[2] N. Schuch, I. Cirac, and D. Pérez-García, Annals of Physics **325**, 2153 (2010), arXiv:1001.3807.
[3] C. Fernández-González, N. Schuch, M. M. Wolf, J. I. Cirac, and D. Pérez-García, arXiv:1210.6613.
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