Supplementary Material: Exact decoupling of the XX spin chain

In this supplementary material we show that the XX quantum spin chain can be exactly decoupled into two (critical) quantum Ising spin chains by applying local disentanglers – provided that fermionic degrees of freedom are used to separate the two systems. This derivation follows from the answer to exercise 12.1 in page 480 of Ref. [16].

The XX quantum spin chain Hamiltonian without magnetic field is

$$H = \sum_r X_r X_{r+1} + Y_r Y_{r+1} \tag{A.1}$$

Let us introduce Majorana fermion operators c_r and d_r, given by

$$c_r \equiv \left(\prod_{l<r} Z_l\right) X_r, \quad d_r \equiv \left(\prod_{l<r} Z_l\right) Y_r. \tag{A.2}$$

Notice that all c_r's and d_r's anticommute pairwise, except with themselves (since they fulfill $(c_r)^2 = (d_r)^2 = I$). In terms of these Majorana operators the Hamiltonian reads

$$H = \sum_r i\,(c_r d_{r+1} - d_r c_{r+1})\,. \tag{A.3}$$

Further, for each odd r, we apply a unitary gate u on the pair of sites $(r, r+1)$, such that

$$\begin{aligned} u\ c_r\ u^\dagger &= c_{r+1}, \quad u\ d_r\ u^\dagger = c_r, \\ u\ c_{r+1}\ u^\dagger &= d_r, \quad u\ d_{r+1}\ u^\dagger = -d_{r+1}. \end{aligned} \tag{A.4}$$

[Notice that u preserves the anticommutation relations and it is therefore a canonical transformation of the fermionic variables.] Then the transformed Hamiltonian reads

$$H = \sum_r h_{r,r+2}, \qquad h_{r,r+2} \equiv i\,(d_r c_{r+2} - c_r d_r)\,, \tag{A.5}$$

or $H = H_A + H_B$, where Hamiltonians H_A and H_B,

$$H_A \equiv \sum_{\text{odd } r} h_{r,r+2}, \quad H_B \equiv \sum_{\text{even } r} h_{r,r+2} \tag{A.6}$$

commute with each other, since they are made of terms that are quadratic in Majorana fermion operators and they act on two different sets of sites. Our decoupling transformation V in Fig. 1 will then consist of a row of disentanglers u as in Eq. A.4, followed by trivial disentanglers v (i.e. $v = I$) since the system has already been decoupled. To group together all sites of sublattice $\mathcal{L}_A$ to the left, and all sites of sublattice $\mathcal{L}_B$ to the right, we will use fermionic swap gates, Eq. 5. Notice that these swaps are trivial in fermionic variables, simply interchanging (c_r, d_r) with (c_{r+1}, d_{r+1}). However, they are non-trivial when expressed in spin variables as in Eq. 5.

Once the two lattices have been separated, we can define independent Jordan Wigner transformations on each lattice $\mathcal{L}^{(\alpha)}$, $\alpha = A, B$, We can now define odd and even spin variables by

$$c_r^{(\alpha)} = \left(\prod_{l<r} Z_l^{(\alpha)}\right) X_r^{(\alpha)}, \quad d_r^{(\alpha)} = \left(\prod_{l<r} Z_l^{(\alpha)}\right) Y_r^{(\alpha)}, \tag{A.7}$$

in terms of which we obtain Hamiltonians

$$H^{(\alpha)} = \sum_r \left(X_r^{(\alpha)} X_{r+1}^{(\alpha)} + Z_r^{(\alpha)}\right), \quad \alpha = A, B, \tag{A.8}$$

which correspond to two copies of the quantum Ising model with critical transverse magnetic field.

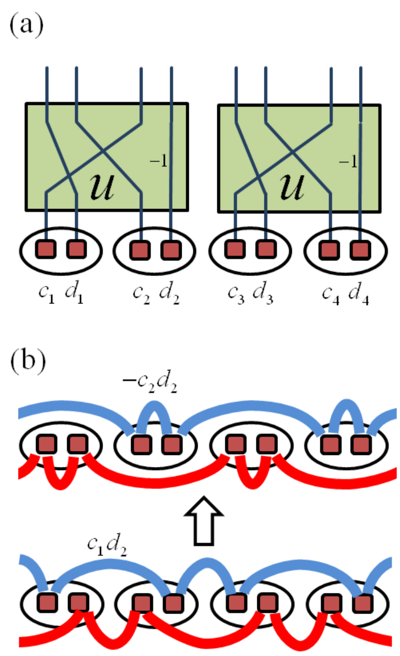

FIG. A.1. (Colour online) (a) Representation of the exact disentangler u of Eq. A.4 used to decouple Hamiltonian H in Eq. A.1. Each site, represented by an oval, contains two Majorana fermions, depicted as brown boxes. The disentangler u permutes the Majorana modes and adds a minus sign to one of them. (b) Hamiltonian terms before (below) and after (above) the disentangler u is applied on pairs of sites. Each Hamiltonian term, e.g. $c_1 d_2$, is depicted with a line connecting two Majorana modes. Notice that after applying the disentangler, which e.g. maps $c_1 d_2$ to $-c_2 d_2$, the two sublattices are no longer connected by interactions.