Simple quantum systems in spacetimes with closed timelike curves

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Three simple examples illustrate properties of path-integral amplitudes in fixed background spacetimes with closed timelike curves: nonrelativistic potential scattering in the Born approximation is nonunitary, but both an example with hard spheres and the exact solution of a totally discrete model are unitary.

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Path integral or sum-over-histories quantum mechanics has been proposed as a possible means of generating a self-consistent dynamics in the presence of closed timelike curves (CTC's) [1]. It has been argued [2–5], however, that for interacting systems such evolution (e.g., from before a compact region with CTC's to after) is necessarily nonunitary. (Such nonunitarity would present dire though perhaps not insuperable [2,3,6] obstacles to the interpretation of the predictions of such mechanics.) The purpose of this paper is to offer three very simple examples to illuminate the issue of unitarity.

The first example, considered in Sec. II, is nonrelativistic particles that scatter via a real potential in the Born approximation as one particular particle traverses a simply specified time machine that defines the compact CTC region. This is just a variant of Boulware's calculation [4] of relativistic particles in a Gott spacetime, but it has two virtues: (1) One can carry the calculation to the end and close a logical loophole left open in Ref. [4]; (2) the example is so simple that there is no mystery or subtlety as to how the nonunitarity arises. This calculation is also an example of the general analysis of perturbation theory given in Ref. [3] and agrees with those arguments.

Inspired by Thorne and Klinkhammer [5], I consider in Sec. III WKB hard-sphere quantum mechanics with the same simple time machine as defined in Sec. II. If one includes (a) all numbers of encounters, and (b) an excluded volume effect on the "disconnected" graphs, the amplitudes are Galilean covariant (otherwise they would not be) and unitary. In fact, they are equal to the noninteracting amplitudes for particles traversing the time machine. Hence there is unitarity but no net interaction with the potentially dangerous time travelers.

In Sec. IV, I turn to a minimal discrete model that can be solved by enumeration of configurations. It is intended to be a cartoon of a general, nonlinear quantum field theory with a specified compact region of CTC's. The local-field variable is reduced to two possible values, the spatial positions inside the time machine are reduced to one location, all spatial positions outside are likewise one location, and time is discrete. The model is presumably no more a free field theory than the general, noncritical Ising model. A nearest-neighbor action that gives unitary time evolution on the normal, flat spacetime lattice generates a different but unitary evolution from before the CTC to after.

If the nonunitarity of perturbation theory [3,4] is, indeed, generic, then it behooves us to understand what is special about the nonperturbative systems discussed in Secs. III and IV. Regrettably, no such explanation is offered at present.

The philosophy and motivation of current investigations of time machines is to seek out whether some fundamental principle forbids their existence or whether they are physically realizable. Even failing that thus far, these questions offer a challenging context to test and stretch our understanding of gravity and quantum mechanics. For now, we begin with a little background.

I. BACKGROUND

It is not known at present whether a compact region containing CTC's can arise in the context of classical gravitation [7]. Microscopic versions may exist as quantum fluctuations of spacetime. Alternatively, large CTC regions may exist as relics of the quantum gravity epoch of the big bang.

Entertaining the existence of time machines as worthy of consideration, one is faced with two paradoxes of classical particle mechanics: a collision may render an "earlier" portion of a trajectory as inconsistent with the collision itself, and given initial conditions may correspond to several trajectories that satisfy the classical equations of motion [1]. An action formulation allows one to consider only those trajectories that are globally self-consistent. A quantum action principle gives an interpretation to the multiple classically allowed trajectories. Each is a stationary point of the action, but all paths are added coherently with the appropriate phase. (An inherently quantum-mechanical singularity in the stress tensor does apparently develop just before the first CTC's [8]. This is thought by some to signal a back reaction, which, handled consistently, may forbid the formation of CTC's. However, the strength of the singularity is sufficiently weak that the relevant distance scales are so small as to require a quantum gravity analysis, and the sign of the effect, opposite for fermions and bosons, is not understood.)

Without a globally definable time sequence or foliation, there is no Hamiltonian evolution in the presence of

CTC's and hence no obvious reason for unitarity of evolution from before to after the CTC region. Nevertheless, free particle systems yield unitary evolution [9]. Viewed in terms of particle trajectories, this unitarity relies on cooperation between the different numbers of windings through the time machine. (See Appendix B for a sketch of a proof in this language.)

It has been argued that interacting systems are generally not unitary (even if the same local action on a foliable spacetime yields unitary Hamiltonian evolution). This paper considers three examples. The background spacetimes are chosen by fiat, there being no known "realistic" models with compactly generated CTC's. For simplicity, the spacetimes are locally flat, with all the curvature located at singular points.

Nonunitary amplitudes may still be used to generate relative probabilities for sequences of events or observations [3,6]. However, there is a consequent acausality in that construction because the geometry of all future CTC's have an effect, in principle, on current observations.

II. BORN APPROXIMATION

The background spacetime is defined as follows. Figure 1 illustrates the construction in $1+1$ dimensions. In the flat space, whose points are labeled (z,t), the heavy lines centered at $z=y_0$ and of length Y at $t=0$ and $t=T$ are identified so that along them the region immediately before $t=0$ connects smoothly to that after $t=T$, while the region immediately before $t=T$ connects smoothly to that after $t=0$. In the new spacetime, we preserve the original local direction of time and can use the old coordinates to label points. The handle thus formed contains the CTC's.

The spacetime is flat except for the two singular points at the handle ends that have excess angles of 2π. The topological theorems regarding compactly generated CTC regions [10] are satisfied, albeit somewhat singularly. For example, the Cauchy or chronology horizon (i.e., the onset of the CTC region) loops through the singular points. (If one were to smooth out the curvature over a finite region, the Cauchy horizon would be the first lightlike curves that circle the handle.) Also, there exist the isolated geodesics that enter but do not exit the CTC region (or vice versa); these are the limiting case of trajectories that enter (or exit) the CTC region at velocity v as $v \to 0$.

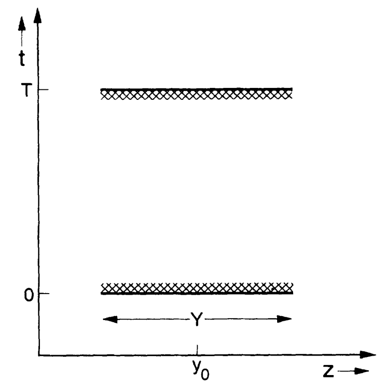

FIG. 1. Time machine in $1+1$ dimensions. The heavy lines are identified such that the cross-hatched regions join smoothly to each other and are disconnected from the likewise joined $t<0$ and $t>T$ regions.

As the Born approximation always has zero radius of convergence in $1+1$ dimensions (because all potentials have at least one bound state), I will use $3+1$ dimensions for an explicit example. A generalization of the spacetime defined above is clear: a compact region of space (e.g., sphere) centered on $\mathbf{y}_0$ and of volume Y^3 is identified at $t=0$ and $t=T$ in flat spacetime analogously to the $(1+1)$-dimensional case to define the time machine.

The dynamics is that of nonrelativistic bosons ($\hbar=m=1$). Their flat space, free propagator is

$$K_f(\mathbf{z}_2,t_2;\mathbf{z}_1,t_1)=\begin{cases}[2\pi i(t_2-t_1)]^{-3/2}\exp\{i(\mathbf{z}_2-\mathbf{z}_1)^2/[2(t_2-t_1)]\}\ , & t_2>t_1,\\ \delta^3(\mathbf{z}_2-\mathbf{z}_1)\ , & t_2=t_1\ ,\\ 0\ , & t_2<t_1\ .\end{cases} \tag{2.1}$$

The particles interact via a real two-particle potential $\lambda V(r)$, chosen (for simplicity) to depend on the magnitude of the two-particle separation r.

Let $K(\mathbf{z}_2,t_2;\mathbf{z}_1,t_1)$ be the amplitude corresponding to all paths that begin at $(\mathbf{z}_1,t_1)$ and end at $(\mathbf{z}_2,t_2)$ including all windings of the machine, self-scatterings, and scatterings off closed loops within the machine. And define the coefficients of the λ expansion of K by $K=K_0+\lambda K_1+\lambda^2 K_2+\cdots$. There are disconnected paths (winding around the machine) that contribute a common factor to K, to the before-to-after vacuum-to-vacuum amplitude, and to all other amplitudes. Hence, they are to be divided out and, in practice for the present context, ignored.

Important notation convention: I adopt the following convention to indicate the allowed domains for spatial position coordinates: Points restricted to lie *within* the identified volume Y^3 will be labeled $\mathbf{y}$ (with subscripts and primes). Points restricted to be *outside* will be labeled $\mathbf{x}$. Unrestricted positions are $\mathbf{z}$. Position integrals are to be taken over the thus implicitly defined ranges.

Unitarity of single-particle evolution from before to after the time machine would require

$$\int d\mathbf{x}\, K(\mathbf{x},T;\mathbf{x}_1,0)K^*(\mathbf{x},T;\mathbf{x}_1',0)=\delta^3(\mathbf{x}_1-\mathbf{x}_1')\ . \tag{2.2}$$

The free particle amplitude K_0 (which includes all windings of the time machine) is unitary in this sense. So next consider the $O(\lambda)$ contribution to Eq. (2.2). Is

$$\int d\mathbf{x}[K_1(\mathbf{x},T;\mathbf{x}_1,0)K_0^*(\mathbf{x},T;\mathbf{x}_1',0) + K_0(\mathbf{x},T;\mathbf{x}_1,0)K_1^*(\mathbf{x},T;\mathbf{x}_1',0)]=0? \quad (2.3)$$

To evaluate Eq. (2.3) in closed form, I make the further simplification that $T \gg Y^2$ and consider points $\mathbf{x}_1$ and $\mathbf{x}_1'$ such that $T \gg (\mathbf{x}_1^{(\prime)} - \mathbf{y}_0)^2$. For such a large T machine, the leading contribution comes from the minimal number of windings (as discussed in Appendix B).

The minimal, i.e., one, winding contribution to $\int d\mathbf{x}\, K_1(\mathbf{x},T;\mathbf{x}_1,0)K_0^*(\mathbf{x},T;\mathbf{x}_1',0)$, defined to be $A(\mathbf{x}_1,\mathbf{x}_1')$, is illustrated in Fig. 2. Lines with arrows pointing up (down) are factors of $K_f^{(*)}$, the dashed line is a factor of $V(|\mathbf{z}_2-\mathbf{z}_1|)$, the heavy horizontal lines define the time machine, and the factors must be integrated over $0<t<T$ and all $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{y}$, and $\mathbf{x}$.

Boulware [4] notes that for a field-theoretic local $\lambda\phi^4$ interaction, the $O(\lambda)$ contribution is of the form of a particle scattering off an effective potential given by $\lambda K(\mathbf{z},t;\mathbf{z},t)$ (which, naively, is the density of particles looping the machine). However, this is clearly a complex, oscillatory function of $\mathbf{z}$ and t. Since unitarity of potential scattering requires a real potential, Boulware concludes that unitarity is violated.

In the present case, the analysis can be carried a bit further to address the following two issues: Since the question of unitarity cannot be posed without integrating over all $\mathbf{z}_1$, $\mathbf{z}_2$, and t between 0 and T, is it possible that the net integrated effect is, in fact, unitary? And if not, is the nonunitarity trivial, e.g., is the amplitude unitary up to an overall factor, which could then be reabsorbed into the measure? (This latter possibility is realized in the example of Sec. III.) The answers here are no as found in Appendix A. For simplicity, use the time machine as the origin of coordinates, i.e., take $\mathbf{y}_0=0$. Then the $T \gg \mathbf{x}_1^{(\prime)2} \gg Y^2$ amplitude is

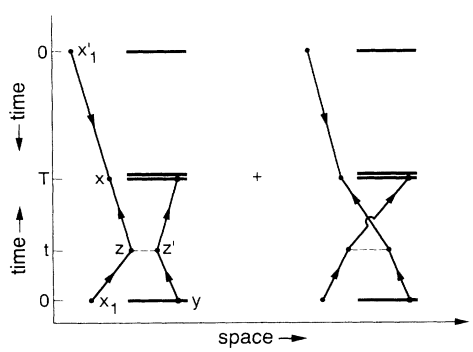

FIG. 2. $O(\lambda)$ propagator unitarity testing amplitude $A(\mathbf{x}_1,\mathbf{x}_1')$. The lines pointing upward denote factors of K_f, downward K_f^*, and dashed horizontal V; the solid horizontal lines denote the time machine as in Fig. 1.

$$A(\mathbf{x}_1,\mathbf{x}_1') \simeq \frac{Y^3}{8\pi}(2\pi i T)^{-3/2} e^{i(\mathbf{x}_1'-\mathbf{x}_1)^2/2T} \times \left[\frac{\mathbf{x}_1'\cdot(\mathbf{x}_1-\mathbf{x}_1')}{|\mathbf{x}_1-\mathbf{x}_1'|^3} W\left[\frac{\mathbf{x}_1'\cdot(\mathbf{x}_1-\mathbf{x}_1')}{|\mathbf{x}_1-\mathbf{x}_1'|}\right] + \frac{\mathbf{x}_1'\cdot(\mathbf{x}_1-\mathbf{x}_1')}{x_1'^3} W\left[\frac{\mathbf{x}_1'\cdot(\mathbf{x}_1-\mathbf{x}_1')}{x_1'}\right]\right], \quad (2.4)$$

where $W(r)$ is a complex linear functional of $V(r)$ defined in Eq. (A4), which satisfies $\mathrm{Re}W(r)=V(r)$ and $W(-r)=W(r)^*$. Quite generally, then, $A(\mathbf{x}_1,\mathbf{x}_1')+A^*(\mathbf{x}_1',\mathbf{x}_1)\neq 0$.

III. HARD SPHERES

Quantum billiards or impenetrable spheres can be treated in a WKB approximation because their interaction, rather than being smooth on the scale of a wavelength, can be treated as a boundary condition [5]. (The "approximation" is thus exact, in the sense it is exact for free particles.) I consider here a single particle's traversal of the same time machine as described in Sec. II. In particular, I consider paths of initially compact wave packets whose spread is small compared to the hard-sphere diameter. The packets traverse the time machine in a proper time sufficiently small that wave packet spreading can be ignored; this can be guaranteed for all numbers of windings and self-collisions by suitable choice of initial conditions and ratio of the hard-sphere size to the size of the time machine.

The amplitudes are given by a factor of $i\exp\{iS_{\text{classical}}\}$, where $S_{\text{classical}}=\frac{1}{2}\Delta\mathbf{z}^2/\Delta t$ for each collisionless leg of the journey and a $(-i)$ for each collision.

It is instructive to go to a moving frame rather than the machine rest frame. Let the center of the identified region at $t=T$ be $\mathbf{y}_0'$ and that at $t=0$ be $\mathbf{y}_0$ such that $\mathbf{y}_0'-\mathbf{y}_0=\mathbf{d}$. Paths with 0, 1, and 2 windings are shown in Fig. 3.

The action corresponding to a single winding for a particular collision point is $\frac{1}{2}\mathbf{d}^2/T$. For one winding, the integral over the possible locations of the collision gives a factor of $L/T^{1/2}$, where L is the length of the projection of the no-collision path onto the identified volumes. (In

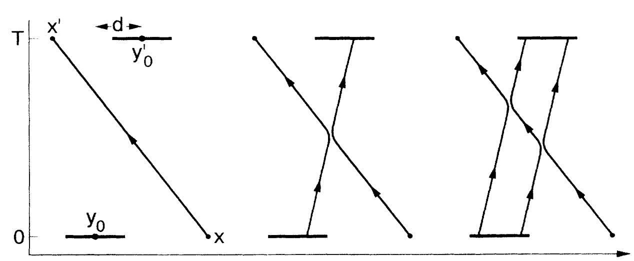

FIG. 3. Hard-sphere WKB trajectories through the CTC region with 0, 1, and 2 self-collisions.

1+1 dimensions, L is simply Y.) For n windings, the factor is $(L/T^{1/2})^n/n!$. (This ignores an excluded volume effect; the L^n should actually be $L(L-D)(L-2D)\cdots[L-(n-1)D]$, where D is the billiard ball diameter. For simplicity, we assume $L \gg D$.) Hence, the sum of amplitudes for such paths that go from $(\mathbf{x},0)$ to $(\mathbf{x}',T)$ with windings $n=0,1,2,\ldots$ is

$$\sum_{n=0}^{\infty} \exp\left[\frac{i}{2}\frac{(\mathbf{x}'-\mathbf{x})^2}{T}\right]\frac{i^{n+1}}{n!}\left[\frac{L}{T^{1/2}}\right]^n\left\{\exp\left[\frac{i}{2}\left[\frac{\mathbf{d}^2}{T}\right]^2\right]\right\}^n = i\exp\left[\frac{i}{2}\frac{(\mathbf{x}'-\mathbf{x})^2}{T}\right]\exp\left\{i\frac{L}{T^{1/2}}\exp\left[\frac{i}{2}\left[\frac{\mathbf{d}^2}{T}\right]^2\right]\right\}. \tag{3.1}$$

This itself is not unitary for $\mathbf{d}\neq 0$. Nor, however, is it Galilean covariant. What is missing is a correct account of the disconnected paths.

When the initial collisionless path from $\mathbf{x}$ to $\mathbf{x}'$ does not traverse the positions in the time machine, the machine is threaded by closed loops. The amplitude for these loops is the coherent sum over all numbers of loops, integrated over their allowed trajectories as restricted by the excluded volume effect of the impenetrable spheres. The sum of these closed-loop amplitudes is also the before-to-after vacuum-to-vacuum amplitude.

When the initial collisionless path does traverse the machine, each of the paths included in Eq. (3.1) excludes a volume for possible closed-loop disconnected paths, i.e., the velocity $\mathbf{d}/T$ paths that pass through the collisionless path. However, the factor thus lost from the completely disconnected volume is precisely the factor acquired by summing the possible collisions as in Eq. (3.1). Hence, the product of the amplitude in Eq. (3.1) with the allowed disconnected loops is equal to the product of the unitary collisionless amplitude with the hard-sphere vacuum-to-vacuum amplitude. All of the apparent nonunitarity and frame (or $\mathbf{d}$) dependence resides in a common factor of all amplitudes, which is the naive before-to-after vacuum-to-vacuum amplitude. This factor, however, is unobservable and is properly divided out everywhere.

IV. AN ISING MODEL

Finally, I consider a totally discrete model that can be solved by enumeration of the finite number of configurations. The analogue flat spacetime, depicted in Fig. 4(a), consists of a $2\times m$ lattice, i.e., with two spatial positions and m times. At each lattice site $\mathbf{n}$, there is a two-valued field $s(\mathbf{n})=\pm 1$. The "path integral" is

$$Z=\sum_{s(\mathbf{n})}\exp\left[i\alpha\sum_{\mathbf{n},\mu}[s(\mathbf{n}+\mu)-s(\mathbf{n})]^2\right], \tag{4.1}$$

where μ runs over the two positive unit vectors. The choice $\alpha=\pi/8$ yields a unitary 4×4 transfer matrix that relates the four possible s configurations at a given time to those at the next time. The time machine is defined by reidentifying two of the timelike links as indicated in Fig. 4(b). The amplitude of interest is the 4×4 matrix for the sum over all intermediate time configurations with a particular initial configuration (immediately preceding the CTC) and a particular final configuration (immediately following the CTC). I have done the sums for systems with $m=3$, 4, and 5. In each of these cases, the amplitude differs from the analogous flat spacetime system but is, nevertheless, unitary.

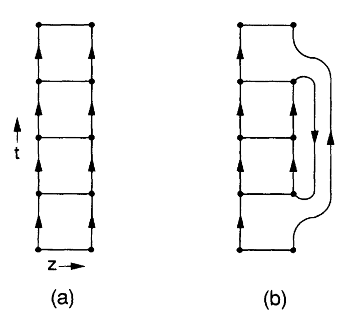

FIG. 4. (a) Flat 2×5 lattice spacetime, with arrows indicating the positive sense of the timelike links. (b) 2×5 spacetime with a closed timelike curve.

V. CONCLUSION

The nonunitarity of interacting particle propagation across a compact region of spacetime with closed timelike curves (CTC's) is demonstrated with an exceedingly simple, nonrelativistic example. In a particular limit of the parameters, all integrals can be evaluated for arbitrary incoming states. This was not done in previous analyses. An analogous issue arises in the general, relativistic perturbation-theory analysis of Ref. [3]. There the nonunitarity is demonstrated by identifying combinations of propagator functions that are nonzero in the presence of CTC's but which would have to integrate to zero against general state functions were the theory unitary. It is not immediately obvious that the state functions form a complete set with respect to the relevant integrals.

The nonunitarity of perturbation theory is presumably generic. However, analogous calculations in two nonperturbative examples do not exhibit nonunitarity at the same level. This challenges us to better understand the issues. Free theories are unitary, presumably, because particles that do go back in time still cannot influence anything in the past, and they themselves eventually propagate into the future because quantum diffusion prevents a particle from time cycling indefinitely with a nonvanishing probability. In the hard-sphere example considered here, there certainly are interactions, e.g., two incoming wave packets could scatter off each other, and there certainly is time travel, i.e., of the type performed by free particles, which definitely alters their trajectories even after the CTC region. But the particular time machine considered here appears to have the property that the multiple classical alternatives allowed by having both interactions and CTC's sum to something equivalent to having no interactions.

The discrete model considered here is not a free field theory in that it is not linearly coupled harmonic oscillators because of the restriction $s=\pm 1$. Its simplicity allows an exact solution but precludes much in the way of interpretation.

A better understanding of each of these examples would be very welcome.

ACKNOWLEDGMENTS

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APPENDIX A

The leading large T behavior of the $O(\lambda)$ unitarity violation discussed in Sec. II can be evaluated in three steps. First consider the unitarity of the Born approximation in flat spacetime for a single-particle scattering off a potential. Then generalize to two-to-two particle scattering. And, finally, modify the latter to fit the time machine boundary conditions.

Let $B(\mathbf{z}_0,\mathbf{z}_0')$ be the amplitude depicted in Fig. 5, i.e., including integrals over $\mathbf{z}$, $\mathbf{z}'$, and t:

$$B(\mathbf{z}_0,\mathbf{z}_0')=-i\int d\mathbf{z}\,d\mathbf{z}'\int_0^T dt\,K_f(\mathbf{z},t;\mathbf{z}_0,0)V(\mathbf{z})\times K_f(\mathbf{z}',T;\mathbf{z},t)\times K_f^*(\mathbf{z}',T;\mathbf{z}_0',0)\ . \tag{A1}$$

Free particle unitarity reduces this to

$$\begin{aligned}B(\mathbf{z}_0,\mathbf{z}_0')&=-i\int d\mathbf{z}\int_0^T dt\,K_f(\mathbf{z},t;\mathbf{z}_0,0)V(\mathbf{z})K_f^*(\mathbf{z},t;\mathbf{z}_0',0)\\&=\frac{-i}{(2\pi)^3}\int_0^T\frac{dt}{t^3}\exp\{i(z_0^2-z_0'^2)/(2t)\}\times\tilde{V}\left[\frac{\mathbf{z}_0'-\mathbf{z}_0}{t}\right]\ .\end{aligned} \tag{A2}$$

The second form uses the explicit form for K_f, and $\tilde{V}$ is the Fourier transform of V. Note that if V is real, $\tilde{V}(\mathbf{k})=\tilde{V}^*(-\mathbf{k})$, and then $B(\mathbf{z}_0,\mathbf{z}_0')+B^*(\mathbf{z}_0',\mathbf{z}_0)=0$, which is the statement of $O(\lambda)$ unitarity of potential scattering.

The t integral in Eq. (A2) can be expanded about the large T limit (changing variables to $k=1/t$)

$$B(\mathbf{z},\mathbf{z}')=\frac{-i}{(2\pi)^3}\int_0^\infty k\,dk\,\exp\{i(z^2-z'^2)k/2\}\tilde{V}((\mathbf{z}'-\mathbf{z})k)+\frac{i}{16\pi^3}\frac{\tilde{V}(0)}{T^2}+\cdots\ . \tag{A3}$$

The k integral is reminiscent of the radial part of a spherically symmetric Fourier transform. In particular, if we restrict $V(\mathbf{r})$ to real functions that depend only on r, the magnitude of $\mathbf{r}$, and define the function $W(r)$ by

$$W(r)\equiv-\frac{i}{2\pi^2 r}\int_0^\infty k\,dk\,e^{ikr}\tilde{V}(k)\ , \tag{A4}$$

then $\mathrm{Re}\,W(r)=V(r)$ (which is symmetric under $r\to -r$), while $\mathrm{Im}\,W(r)$ is antisymmetric in r. In terms of W,

$$B(\mathbf{z},\mathbf{z}')=\frac{1}{8\pi}\frac{z^2-z'^2}{|\mathbf{z}-\mathbf{z}'|^3}W\left[\frac{z^2-z'^2}{2|\mathbf{z}-\mathbf{z}'|}\right]+\frac{i\tilde{V}(0)}{16\pi^3T^2}+\cdots\ . \tag{A5}$$

For two-to-two scattering, the analogous amplitude $B(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_1',\mathbf{z}_2')$, depicted in Fig. 6, is given by

$$B(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_1',\mathbf{z}_2')=\delta^3(\mathbf{Z}-\mathbf{Z}')[B(\mathbf{z},\mathbf{z}')+B(\mathbf{z},-\mathbf{z}')]\ , \tag{A6}$$

where

$$\mathbf{Z}=(\mathbf{z}_1+\mathbf{z}_2)/2\ ,\quad \mathbf{z}=\mathbf{z}_1-\mathbf{z}_2\ , \tag{A7}$$

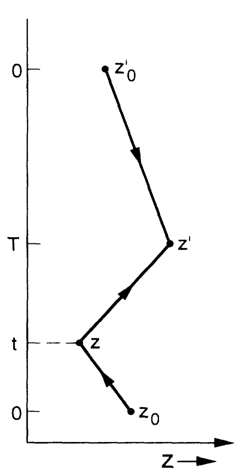

FIG. 5. $O(\lambda)$ unitarity amplitude for potential scattering $B(\mathbf{z}_0,\mathbf{z}_0')$.

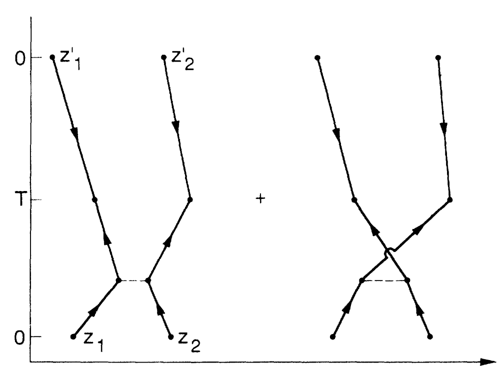

FIG. 6. $O(\lambda)$ unitarity amplitude for two-particle potential scattering $B(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_1',\mathbf{z}_2')$.

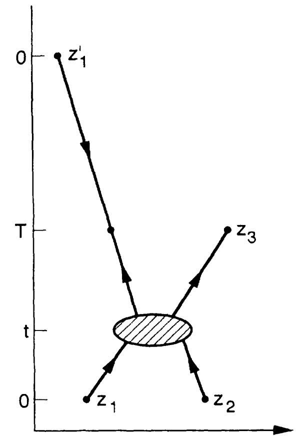

FIG. 7. Truncated two-particle amplitude $A(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_1',\mathbf{z}_3)$.

and the same definitions hold for the primed coordinates. Equation (A6) reflects that the problem is separable into free center-of-mass motion and scattering in the relative coordinate. Unitarity to $O(\lambda)$ is again clearly satisfied.

The amplitude we need to test unitarity to $O(\lambda)$ in the time machine is $A(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_1',\mathbf{z}_3)$, illustrated in Fig. 7. It is simply related to the B's using the unitarity of K_f, which implies that K_f is the inverse of K_f^*. Hence

$$A(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_1',\mathbf{z}_3)=\int d\mathbf{z}_2'\delta^3(\mathbf{Z}-\mathbf{Z}')[B(\mathbf{z},\mathbf{z}')+B(\mathbf{z},-\mathbf{z}')]\times K_f(\mathbf{z}_3,T;\mathbf{z}_2',0)\ . \tag{A8}$$

This final integral over $\mathbf{z}_2'$ is trivial because of the δ function.

The one-particle amplitude relevant to the time machine is (recalling the implicit ranges of coordinates $\mathbf{x}$, $\mathbf{y}$, or $\mathbf{z}$ defined in Sec. II)

$$A(\mathbf{x}_1,\mathbf{x}_1')=\int d\mathbf{y}\,A(\mathbf{x}_1,\mathbf{y},\mathbf{x}_1',\mathbf{y})\simeq Y^3A(\mathbf{x}_1,\mathbf{y}_0,\mathbf{x}_1',\mathbf{y}_0)\ , \tag{A9}$$

where the second form is the leading term for small Y. $A(\mathbf{x}_1,\mathbf{x}_1')+A^*(\mathbf{x}_1',\mathbf{x}_1)\neq 0$ [the actual expression is given in Eq. (2.4) using Eq. (A7) and the δ function], which implies there is no unitarity.

APPENDIX B

I address briefly free particle unitarity in the $T\gg Y^3$ time machine and for general T and discuss some aspects of windings in general and small Y. The expansion of the free particle amplitude $K_0(\mathbf{x}',T;\mathbf{x},0)$ from before to after the time machine in terms of numbers of windings and the flat-space free propagator K_f looks like (remembering $\mathbf{x}$'s are outside and $\mathbf{y}$'s are inside the Y^3 volume)

$$K_0(\mathbf{x}',T;\mathbf{x},0)=K_f(\mathbf{x}',T;\mathbf{x},0)+\int d\mathbf{y}\,K_f(\mathbf{x}',T;\mathbf{y},0)K_f(\mathbf{y},T;\mathbf{x},0)+\int d\mathbf{y}\,d\mathbf{y}'K_f(\mathbf{x}',T;\mathbf{y},0)K_f(\mathbf{y},T;\mathbf{y}',0)K_f(\mathbf{y}',T;\mathbf{x},0)+\cdots\ . \tag{B1}$$

To check unitarity, we replace the $\mathbf{x}''$ integral in $\int d\mathbf{x}''K_0(\mathbf{x}'',T;\mathbf{x},0)K_0^*(\mathbf{x}'',T;\mathbf{x}',0)$ with an integral $d\mathbf{z}''$, i.e., as if $\mathbf{x}''$ ran over the full range, minus an integral $d\mathbf{y}''$. The integral $d\mathbf{z}''$ always yields a δ function because of K_f unitarity. To leading order for $T\gg Y^3$, the nonunitarity of K_f when restricted to end points outside the time machine is canceled by a contribution from the one-winding term of K_0. (The evaluation is straightforward.) All effects of higher windings (and a residual nonunitarity of the one-winding term) are down by $Y^3/T^{3/2}$.

For arbitrary T, the unitarity of K_0 with the same time machine can be demonstrated using the same expansion [11]. Each successively higher winding restores the unitarity of the one fewer winding amplitude but introduces its own nonunitarity. Hence, one must sum all windings to recover unitarity.

A general amplitude written in terms of K_f's, i.e., before integrating over any spacetime coordinates, will have various $\mathbf{y}_i$ arguments. The integrals $\int d\mathbf{y}_i f(\mathbf{y}_i)$ can be replaced by $Y^3 f(\mathbf{y}_0)$ in the small Y limit as long as all the $\mathbf{y}$'s are independent. If, however, some $\int d\mathbf{z}$ yields a $\delta^3(\mathbf{y}_i-\mathbf{y}_j)$, then there is one fewer factor of Y^3 than given by counting the $\mathbf{y}$'s. This is essential to the free particle case discussed above. The only such $\mathbf{z}$ integral that occurs in the generalization of the calculation of Appendix A to higher winding numbers comes from the factor $\int d\mathbf{x}\,K_0^*(\mathbf{x},T;\mathbf{x}_1'0)K_0(\mathbf{x},T,\mathbf{z},t)$. For this integral, the following identity holds:

$$\int d\mathbf{x}\,K_0^*(\mathbf{x},T;\mathbf{x}',0)K_0(\mathbf{x},T,\mathbf{z},t)=K_f^*(\mathbf{z},t;\mathbf{x}',0)\ , \tag{B2}$$

which follows from free particle unitarity. The same identity with the K_0's replaced by K_f's was used to get Eq. (A2). Hence, adding all possible windings to the paths for this portion of the calculation has no net effect. Finally, adding higher windings to the other segments of the paths of the calculation of Appendix A indeed gives extra factors of Y^3. Hence, the leading small Y behavior is given by the minimal winding amplitude.

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