BRIEF COMMUNICATIONS

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Remark on a result of D. Dritschel

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A hypothesis put forward by D. Dritschel [J. Fluid Mech. **94**, 511 (1988)], namely that an isolated symmetrical disturbance on a uniform vortex patch will filament in time proportional to the inverse square of the disturbance amplitude, is subject to independent testing using a nonintrusive numerical method. The hypothesis that the trend is maintained to substantially smaller amplitudes than were originally considered by Dritschel is both supported and verified. The results may be interpreted as providing limited evidence that contour smoothness is maintained in filamentation and that corner formation does not occur up to the time of wave overturning.

Dritschel[1] (henceforth referred to as D1) studied the dynamical evolution of isolated disturbances to the perimeter of an isolated circular patch of uniform vorticity fluid in the two-dimensional flow of an inviscid incompressible fluid. Using a "contour surgery" numerical method based on the well-known method of contour dynamics[2] (CD), Dritschel reported overturning of the vorticity discontinuity and subsequent jet formation, a phenomenon known as filamentation, in a broad parameter space describing disturbances to isolated vortex patches on the surface of a sphere. The filamentation phenomenon observed in D1 is intriguing and shows kinematic behavior apparently different from filamentation following the linear instability of steady, uniform vortex equilibria.[3,4] From his results Dritschel hypothesized that the time to filamentation t_f for almost all disturbances to vortex equilibria with uniform vorticity (which may be defined as any such flow that can be reduced to a steady flow by a suitable choice of reference frame) scales as $(am)^{-2}$, where a is the disturbance wave height and m is a parameter [see Eq. (2)], which specifies the extent of the initial disturbance. The supporting evidence given in D1 consists of two parts: First, calculations with the fully nonlinear evolution equations show an approximately linear relation when t_f is plotted versus $(am)^{-2}$. Each of the two data sets shown in Fig. 8 of D1 consists of just three points in the range am of order unity. Second, D1 argues that breakdown in numerical solutions of a weakly nonlinear evolution equation can be identified with the onset of filamentation of the fully nonlinear system. Since the weakly nonlinear equations are scaled on the long time scale $\omega a^2 t$ (ω is the vorticity), this assumption gives $\omega a^2 t_f =$ constant to arbitrarily small a.

We consider the evidence given in D1 to be suggestive but not conclusive. In particular, we do not presently accept identification of the breakdown of the weakly nonlinear equations with the onset of interface overturning for the reason that the latter lies outside the range of validity of these equations. The arguments presented in D1 (p. 527) depend very strongly on this identification, yet while breakdown indicates the beginning of a strongly nonlinear interaction, this may or may not extend to overturning in the limit of vanishingly small a. Further, recent numerical studies of vortex patch evolution using a numerical method different from CD report the formation of a tangent slope discontinuity (corner) in a finite time,[5] and this raises the disturbing possibility that some instances of filamentation observed in CD calculations may be of purely numerical origin resulting from finite precision arithmetic. In this scenario the true dynamics may dictate corner formation with subsequent nonanalyticity of the velocity field at the singularity formation time. This would invalidate assumptions of spatial and temporal smoothness implicit in the CD numerical method, which, unable to reproduce the subtle dynamics of corner formation, would respond through the generation of spurious wave overturning. The results of Buttke have been recently subject to contrary findings by Dritschel and McIntyre[6] and by Dritschel and Zabusky,[7] who argue that the singularity is spurious and arises from artifacts of the numerical method used in Ref. 5. We believe that the current weight of numerical evidence is against the singularity hypothesis in the flows studied in Refs. 5 and 6, but note that contour smoothness for all time awaits formal proof. In view of the importance of this issue for contour dynamics, corner

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formation as a possible explanation of filamentation must remain an outside possibility.

In this Brief Communication we present detailed calculations of filamentation resulting from an isolated symmetrical disturbance to a circular patch of uniform vorticity. Having only limited available computing resources, we concentrate on a particular case considered in D1 and study the variation of a single parameter, the disturbance amplitude, to values $am \ll 1$ compared to $am = O(1)$ in D1. In order to provide conclusive evidence that filamentation is not an artifact of the numerical method, we employ an uncomplicated CD code consisting of Runge–Kutta timewise integration of the Lagrangian contour-particle positions with fourth-order space and time accuracy, and without node insertion or other node adjustment procedures of any kind. Our results support the scaling for t_f suggested in D1 and, by fine scale resolution of the contour near the point of initial filamentation, they provide firm evidence that the D1 filamentation is of dynamical and not of numerical origin.

We consider the evolution of an isolated, nearly circular vortex patch with uniform vorticity ω in an otherwise irrotational flow. The vortex contour is described parametrically by $z(e,t) = x(e,t) + iy(e,t)$, where (x,y) are Cartesian coordinates for a point on the contour, t is the time, and e, $-\pi < e < \pi$ is a Lagrangian parameter. We work in a reference frame that rotates with angular velocity $\omega/2$ equal to the solid body rotation speed of the undisturbed vortex. Material particles on the undisturbed perimeter are then stationary. The CD equation of motion for $z(e,t)$ is then

$$\frac{\partial z^*(e,t)}{\partial t} = \frac{\omega}{4\pi} \int_{-\pi}^{\pi} \frac{z^*(e,t) - z^*(e',t)}{z(e,t) - z(e',t)} \frac{\partial z(e',t)}{\partial e'} de' + i\frac{\omega}{2} z^*(e,t) . \quad (1)$$

We nondimensionalize by setting the unit radius for the circular vortex and by setting $\omega = 1$. We consider initial conditions corresponding to the symmetrical isolated disturbance used in D1. Thus we take

$$z(e,0) = [1 + \rho(\theta)]\exp(i\theta) , \quad (2a)$$

$$\rho(\theta) = a \exp[-\tfrac{1}{2}(\theta/\theta_0)^2] , \quad (2b)$$

$$\theta(e) = e - \alpha \sin(e - e_1) , \quad (2c)$$

where θ, $-\pi < \theta < \pi$, is measured clockwise from the x axis, a is the disturbance amplitude, and $\theta_0 = \pi/m$ measures the angular extent of the disturbance.

To discretize we define the contour by N points equally spaced in e. Thus we put $e_j = 2\pi j/N, j = -N/2 + 1,...,N/2$, $\theta_j = \theta(e_j)$ from (2c), and $z_j(t=0) = z(e_j,0), j = -N/2 + 1,...,N/2$. With $\alpha > 0$, Eq. (2c) concentrates node density near $\theta = e_1$. The nonsingular integral in (1) was evaluated using the N-point periodic trapezoidal rule and all $\partial/\partial e$ derivatives were calculated using five-point centered differences. Experience indicates that the CD equation (1) is not stiff so that we need require only that the time step Δt be small compared to the rotation period 2π. Presently, for t integration, we used a standard fourth-order Runge–Kutta method with fixed Δt and encountered no apparent instability in the timewise integration. This lack of stiffness may be interpreted as evidence that singularities of the Buttke[5] type have not been observed.

TABLE I. Summary of cases run: Symmetrical disturbances to the circular vortex patch. All cases have $m = 20$ and $\Delta t = 0.4$.

a	N	α	e_1	am	t_f
$\sqrt{2}/10$	128	0.8	−0.533	$2\sqrt{2}$	18.0
1/10	128	0.8	−0.533	2	32.8
$\sqrt{2}/20$	128	0.9	0	$\sqrt{2}$	71.2
$\sqrt{2}/20$	256	0	0	$\sqrt{2}$	71.2[a]
1/20	256	0.9	0	1	146.4
1/20	256	0	0	1	146.4
$\sqrt{2}/40$	256	0.9	0	$1/\sqrt{2}$	285.2
1/40	256	0.9	0	1/2	551.6
1/40	1024	0.975	−0.333	1/2	551.2
$\sqrt{2}/80$	256	0.9	−0.533	$\sqrt{2}/4$	1091.2
1/80	256	0.9	0	1/4	2125.2
1/80	512	0.95	−0.533	1/4	2126.4
$\sqrt{2}/160$	256	0.9	−0.533	$\sqrt{2}/8$	4210.2[a]
$\sqrt{2}/160$	256	0.9	0	$\sqrt{2}/8$	4197.2
1/160	512	0.975	−0.533	1/8	8346.4

[a] $\Delta t = 0.2$.

In Table I we give a summary of the parameters used in 15 calculations. All runs used $m = 20$ corresponding to cases 4 and 22 of D1, which have $a = \frac{1}{20}$ and $a = \sqrt{2}/20$, respectively. Several cases were repeated with different parameters within the numerical method to verify that the calculated t_f were not dependent on their particular values.

Figure 1 shows t_f plotted against $(am)^2$ on a log–log scale. Over nearly three decades in each quantity there is very good agreement with the $(am)^{-2}$ scaling suggested in D1, although we differ from D1 in some calculated values of t_f. When $a = \sqrt{2}/20$ we find $t_f = 71.2$ compared with $t_f = 82.9$ given in D1, while when $a = \frac{1}{20}$ we find $t_f = 146.4$ compared with D1's $t_f = 160.2$. In each case the discrepancy is nearly equal to the local period of the disturbance. We can provide no explanation for these differences other than to suggest that our code is detecting very small-scale filamentation, missed by the D1 calculations, at approximately

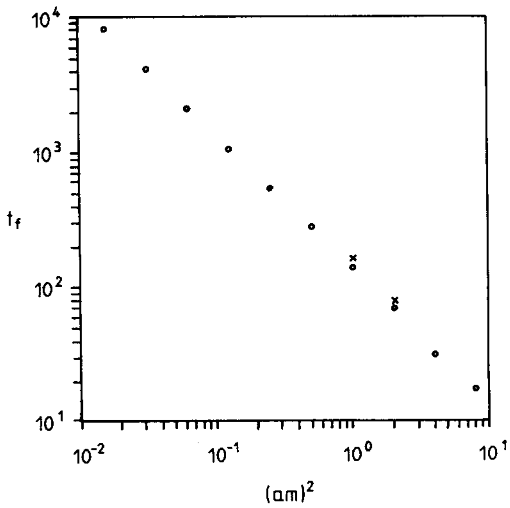

FIG. 1. Time t_f to filamentation versus $(am)^2$, $m = 20$. Present results: ○; Dritschel[1]: ×.

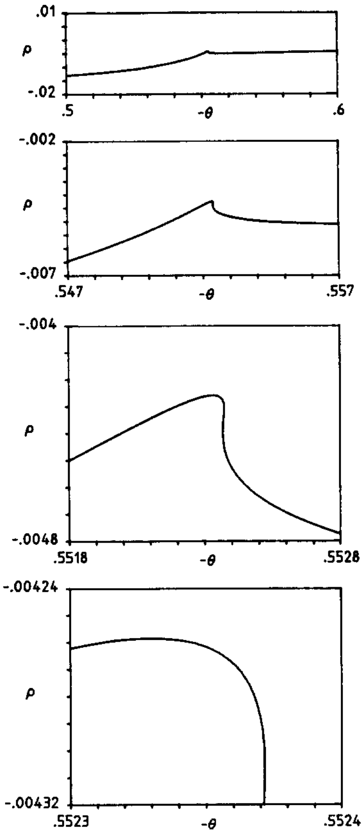

FIG. 2. Four views of the interface profile in (ρ-θ) coordinates at $t = t_f = 551.2$, $a = \frac{1}{40}$, $m = 20$, $N = 1024$, and $\alpha = 0.975$. From the top down, each window is a factor of 10 smaller in the θ direction.

one disturbance period earlier in time. With our limited computing resources we were restricted to $\min(am) = \frac{1}{8}$. This gives an initial slope maximum of 1.38°, which we believe to be within the linear regime. This case required about 4 h on a CRAY-XMP computer. By comparison, $\min(am) = \frac{3}{4}$ in D1.

Values of α and e_1 were chosen following some trial and error so as to concentrate nodes near the position on the contour at which overturning first occurs. In order to test that filament formation is not a spurious product of length scales introduced by node insertion, we calculated one case with $a = \frac{1}{40}$ at high resolution using $N = 1024$, $\alpha = 0.975$, and $e_1 = -0.3327$. This gives local node separation near the overturning interface equal to that obtained using about 4×10^4 nodes equally spaced in θ. Figure 2 shows a sequence of snapshots of the interface profile, each on a successively smaller scale, at the onset of wave overturning at $t = t_f = 551.2$. The region of very high curvature at the wave crest is clearly well resolved while contour smoothness is reasonably well maintained. The minimum node spacing is order 10^{-7} compared with the minimum contour radius of curvature of order 10^{-5}.

In conclusion, our results support the amplitude scaling put forward in D1 at disturbance amplitudes more within the dynamically linear regime and indicate that the contour remains smooth up to the time of wave overturning. Dritschel suggests that his results may be explained as a nonlinear cascade from the initial disturbance to smaller scales, and that his observed mode of filamentation has a universal character that is transparent to linear spectral stability of the underlying vortex equilibrium state. This appears to us as at best a partial explanation in terms of fundamental dynamics, and we consider that the basic dynamical mechanism underlying the D1 filamentation is yet to be elucidated.

ACKNOWLEDGMENT

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