Tensor spherical harmonics on S^2 and S^3 as eigenvalue problems[a)]

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Tensor spherical harmonics for the 2-sphere and 3-sphere are discussed as eigenfunction problems of the Laplace operators on these manifolds. The scalar, vector, and second-rank tensor harmonics are given explicitly in terms of known functions and their properties summarized.

The analysis of scalar, vector, and tensor wave equations on the manifolds S^2 and S^3 is greatly facilitated by having a set of basis functions that reflect the symmetries and are eigenfunctions of the Laplace operator. The use of scalar S^2 harmonics in multipole expansions of electrostatic fields is probably the most well known example;[1] but cosmological pertubation,[2] stellar pulsations,[3,4] and scattering problems also make use of multipole expansion using the vector and tensor harmonics as well. In this paper the S^2 and S^3 harmonics are approached as eigenfunction problems (based on an analogy with the discussion of S^2 harmonics by Thorne and Compolattaro[4] and the discussion of S^3 harmonics by Lifshitz and Khalatnikov[2]) with an emphasis on explicit solutions, summarized in Tables I and II. These harmonics will

TABLE I. S^2 tensor harmonics.

	$\gamma_{\theta\theta}=1,\quad \gamma_{\varphi\varphi}=\sin^2\theta$	$\epsilon_\theta{}^\varphi=\dfrac{1}{\sin\theta},\quad \epsilon_\varphi{}^\theta=-\sin\theta$
Scalar:	$Y^{(lm)}$	$\nabla^2 Y^{(lm)}=-l(l+1)\,Y^{(lm)}$
Vector:	$\psi_a^{(lm)}=Y^{(lm)}_{\|a}$	$\nabla^2\psi_a^{(lm)}=[1-l(l+1)]\psi_a{}^{(lm)}$
		$\psi^{(lm)}_{a\|b}\gamma^{ab}=-l(l+1)\,Y^{lm}$
	$\phi_a{}^{(lm)}=\epsilon_a{}^b\,Y^{(lm)}_{\|b}$	$\nabla^2\phi_a^{(lm)}=[1-l(l+1)]\phi_a{}^{(lm)}$
		$\phi^{(lm)}_{a\|b}\gamma^{ab}=0$
Tensor:	$\eta_{ab}^{(lm)}=Y^{(lm)}\gamma_{ab}$	$\nabla^2\eta_{ab}^{(lm)}=-l(l+1)\,\eta_{ab}^{(lm)}$
		$\eta^{(lm)}_{ab\|c}\gamma^{bc}=\psi_a^{(lm)}$
		$\eta_{ab}^{(lm)}\gamma^{ab}=2Y^{(lm)}$
	$\psi_{ab}^{(lm)}=Y^{(lm)}_{\|a\|b}+\frac{1}{2}l(l+1)\,Y^{(lm)}\gamma_{ab}$	
		$\nabla^2\psi_{ab}^{(lm)}=[4-l(l+1)]\psi_{ab}^{(lm)}$
		$\psi^{(lm)}_{ab\|c}\gamma^{bc}=\frac{1}{2}[2-l(l+1)]\psi_a^{(lm)}$
		$\psi_{ab}^{(lm)}\gamma^{ab}=0$
	$\chi_{ab}^{(lm)}=Y^{(lm)}\epsilon_{ab}$	$\nabla^2\chi_{ab}=-l(l+1)\chi_{ab}^{(lm)}$
		$\chi^{(lm)}_{ab\|c}\gamma^{bc}=\phi_a^{(lm)}$
		$\chi_{ab}^{(lm)}\gamma^{ab}=0$
	$\phi_{ab}^{(lm)}=\frac{1}{2}(\phi^{(lm)}_{a\|b}+\phi^{(lm)}_{b\|a})$	
		$\nabla^2\phi_{ab}^{(lm)}=[4-l(l+1)]\phi_{ab}^{(lm)}$
		$\phi^{(lm)}_{ab\|c}\gamma^{bc}=\frac{1}{2}[2-l(l+1)]\phi_a^{(lm)}$
		$\phi_{ab}^{(lm)}\gamma^{ab}=0$

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Orthogonality relations

$d^2\Omega = \sin\theta\, d\theta\, d\varphi \qquad 0<\theta<\pi, \quad 0<\varphi<2\pi$

$$\int d^2\Omega \;\; Y_{lm} Y^*_{l'm'} = \delta_{ll'}\delta_{mm'}$$

$$\int d^2\Omega \;\; \psi_a^{\,lm}\psi_b^{*l'm'}\gamma^{ab} = l(l+1)\,\delta_{ll'}\delta_{mm'}$$

$$\int d^2\Omega \;\; \phi_a^{\,lm}\phi_b^{*lm}\gamma^{ab} = l(l+1)\,\delta_{ll'}\delta_{mm'}$$

$$\int d^2\Omega \;\; \eta_{ab}^{\,lm}\eta_{cd}^{*l'm'}\gamma^{ac}\gamma^{bd} = -2\delta_{ll'}\delta_{mm'}$$

$$\int d^2\Omega \;\; \psi_{ab}^{\,lm}\psi_{cd}^{*l'm'}\gamma^{ac}\gamma^{bd} = l(l+1)\,[\tfrac{1}{2}l(l+1)-1]\,\delta_{ll'}\delta_{mm'}$$

$$\int d^2\Omega \;\; \chi_{ab}^{\,lm}\chi_{cd}^{*l'm'}\gamma^{ac}\gamma^{bd} = 2\delta_{ll'}\delta_{mm'}$$

$$\int d^2\Omega \;\; \phi_{ab}^{\,lm}\phi_{cd}^{*lm}\gamma^{ac}\gamma^{bd} = l(l+1)\,[\tfrac{1}{2}l(l+1)-1]\,\delta_{ll'}\delta_{mm'}$$

All other products vanish, e.g.,

$$\int d^2\Omega \;\; \psi^*_{ab}\chi^{ab} = 0, \text{ etc.}$$

The completeness of these functions follows from the completeness of the scalar harmonics

$$\sum_{l,m} Y^{(lm)}(\theta,\varphi)\, Y^{*(lm)}(\theta',\varphi') = \delta\;(\cos\theta-\cos\theta')\delta(\varphi-\varphi')$$

TABLE II. S^3 tensor harmonics.

$g_{\chi\chi}=1, \quad g_{\theta\theta}=\sin^2\chi, \qquad g_{\varphi\varphi}=\sin^2\chi\,\sin^2\theta, \qquad \epsilon_{\chi\theta\varphi}=\sin^2\chi\,\sin\theta$

Scalar: $Y^{(nlm)}(\chi,\theta,\varphi) \qquad \Delta Y^{(nlm)} = -n(n+1)\,Y^{(nlm)}$

Vector: $$A_\alpha^{(nlm)} = \Big(0, \sin^{l+1}\chi\, C_{n-l}^{(l+1)}(\cos\chi)\,\phi_i^{(lm)}(\theta,\varphi)\Big)$$

$$B_\alpha^{(nlm)} = \Big(-l(l+1)\sin^{l-1}\chi C_{n-l}^{(l+1)}(\cos\chi)\,Y^{(lm)}(\theta,\varphi) - \partial_\chi[\sin^{l+1}\chi\, C_{n-l}^{(l+1)}(\cos\chi)]\psi_i^{(lm)}(\theta,\varphi)\Big)$$

$$C_\alpha^{(nlm)} = \left(\frac{2^{2l+1}(n+1)(n-l)!(l!)^2}{\pi(n+l+1!)}\right)^{1/2} (\partial_\chi(\sin^l\chi\, C_{n-l}^{(l+1)}(\cos\chi))\,Y^{(lm)}(\theta,\varphi),\ \sin^l\chi\, C_{n-l}^{(l+1)}(\cos\chi)\,\psi_i^{(lm)}(\theta,\varphi)$$

$$B_\alpha^{(nlm)} = \epsilon_\alpha^{\ \mu\nu} A_{\mu;\nu}^{(nlm)}$$

$$C_\alpha^{(nlm)} = Y^{(nlm)}_{\ \ ;\alpha}$$

$$\Delta A_\alpha^{(nlm)} = [1-n(n+2)]A_\alpha^{(nlm)}$$

$$\Delta B_\alpha^{(nlm)} = [1-n(n+2)]B_\alpha^{(nlm)}$$

$$\Delta C_\alpha^{(nlm)} = [2-n(n+2)]C_\alpha^{(nlm)}$$

$$A_{\alpha;\beta}g^{\alpha\beta} = B_{\alpha;\beta}g^{\alpha\beta} = 0$$

$$C_{\alpha;\beta}g^{\alpha\beta} = -n(n+2)\,Y^{(nlm)}$$

Tensor: $A_{\alpha\beta}^{(nlm)} = \tfrac{1}{2}(A_{\alpha;\beta}^{(nlm)} + A_{\beta;\alpha}^{(nlm)}), \qquad \tilde{A}_{[\alpha\beta]}^{(nlm)} = \tfrac{1}{2}\epsilon^{\mu}_{\ \alpha\beta}B_\mu^{(nlm)}$

$B_{\alpha\beta}^{(nlm)} = \tfrac{1}{2}(B_{\alpha;\beta} + B_{\beta;\alpha}), \qquad \tilde{B}_{[\alpha\beta]}^{(nlm)} = -\tfrac{1}{2}[1+n(n+2)]\epsilon^{\mu}_{\ \alpha\beta}A_\mu^{(nlm)}$

$$C^{(nlm)}_{\alpha\beta}=Y^{(nlm)}_{;\alpha;\beta}+\tfrac{1}{3}n(n+2)\,Y^{(nlm)}g_{\alpha\beta}$$

$$D^{(nlm)}_{\alpha\beta}=Y^{(nlm)}g_{\alpha\beta}$$

$$G^{(nlm)}_{\alpha\beta}=Y^{(nlm)}_{;\mu}\epsilon^{\mu}{}_{\alpha\beta}$$

$$E^{(nlm)}_{\chi\chi}=0$$

$$E^{(nlm)}_{a\chi}=\sin^{l}\chi\,C^{(l+1)}_{n-1}(\cos\chi)\,\phi^{(lm)}_{a}(\theta,\varphi)$$

$$E^{(nlm)}_{ab}=-\frac{2}{[2-l(l+1)]}\frac{\partial}{\partial\chi}[\sin^{l+2}\chi\,C^{(l+1)}_{n-l}(\cos\chi)]\phi^{(lm)}_{ab}(\theta,\varphi)$$

$$F^{(nlm)}_{\chi\chi}=-l(l+1)\sin^{l+1}\chi\,C^{(l-1)}_{n-l}(\cos\chi)\,Y^{(lm)}(\theta,\varphi)$$

$$F^{(nlm)}_{a\chi}=-\csc\chi\frac{\partial}{\partial\chi}[\sin^{l+1}\chi\,C^{(l+1)}_{n-l}(\cos\chi)]\psi^{(nlm)}_{a}(\theta,\varphi)$$

$$F^{(nlm)}_{ab}=\tfrac{1}{2}(l+1)\sin^{l}\chi\,C^{(l-1)}_{n-l}(\cos\chi)\,\phi^{(lm)}_{ab}(\theta,\varphi)+\Big[\sin^{l}\chi\,C^{(l-1)}_{n-l}(\cos\chi)$$
$$+\frac{2}{[2-l(l+1)]}\Big(\frac{\partial^2}{\partial\chi^2}(\sin^{l+2}\chi\,C^{(l-1)}_{n-l}(\cos\chi)-\cot\chi\frac{\partial}{\partial\chi}(\sin^{l+2}\chi\,C^{(l-1)}_{n-l}(\cos\chi))\Big)\Big]\psi^{(lm)}_{ab}(\theta,\varphi)$$

$$F^{(nlm)}_{\alpha\beta}=\tfrac{1}{2}[E_{\alpha\mu;\nu}\epsilon_{\beta}{}^{\mu\nu}+E_{\beta\mu;\nu}\epsilon_{\alpha}{}^{\mu\nu}]$$

Eigenvalues:

$\Delta A_{\alpha\beta}=[5-n(n+2)]A_{\alpha\beta}$

$\Delta\widetilde{A}_{[\alpha\beta]}=[1-n(n+2)]\widetilde{A}_{[\alpha\beta]}$

$\Delta B_{\alpha\beta}=[5-n(n+2)]B_{\alpha\beta}$

$\Delta\widetilde{B}_{[\alpha\beta]}=[1-n(n+2)]\widetilde{B}_{[\alpha\beta]}$

$\Delta C_{\alpha\beta}=[6-n(n+2)]C_{\alpha\beta}$

$\Delta D_{\alpha\beta}=-n(n+2)D_{\alpha\beta}$

$\Delta E_{\alpha\beta}=[2-n(n+2)]E_{\alpha\beta}$

$\Delta F_{\alpha\beta}=[2-n(n+2)]F_{\alpha\beta}$

$\Delta G_{\alpha\beta}=[2-n(n+2)]G_{\alpha\beta}$

Divergence conditions:

$A_{\alpha\beta;\gamma}g^{\beta\gamma}=\tfrac{1}{2}(3-n(n+2))A_{\alpha}$

$B_{\alpha\beta;\gamma}g^{\beta\gamma}=\tfrac{1}{2}(3-n(n+2))B_{\alpha}$

$C_{\alpha\beta;\gamma}g^{\beta\gamma}=\tfrac{2}{3}(3-n(n+2))C_{\alpha}$

$D_{\alpha\beta;\gamma}g^{\beta\gamma}=C_{\alpha}$

$E_{\alpha\rho;\gamma}g^{\beta\gamma}=0$

$F_{\alpha\beta;\gamma}g^{\beta\gamma}=0$

Trace conditions:

$A_{\alpha\beta}g^{\alpha\beta}=B_{\alpha\beta}g^{\alpha\beta}=C_{\alpha\beta}g^{\alpha\beta}=E_{\alpha\beta}g^{\alpha\beta}=F_{\alpha\beta}g^{\alpha\beta}=0$

$D_{\alpha\beta}g^{\alpha\beta}=3Y^{(nlm)}$

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be used in a separate paper to discuss perturbations in space–times with these symmetries.

We use the conventions of Ref. 1 for the scalar S^2 harmonics and the conventions of Ref. 5 for the Gegenbauer polynomials. We denote three-dimensional covariant derivatives by a semicolon, two-sphere covariant derivatives by a vertical line, represent the two-sphere metric by γ_{ab}, the three-sphere metric by $g_{\mu\nu}$, and define the sign of the curvature tensor so that the Ricci identity is given by

$$V_{\alpha;\beta;\gamma}-V_{\alpha;\gamma;\beta}=V_{\mu}R^{\mu}{}_{\alpha\beta\gamma}.$$

Greek indices run from 1 to 3 and denote three-sphere indices, Latin indices run from 2 to 3 and denote two-sphere indices.

The manifold S^3 is characterized by its metric

$$ds^2=g_{\mu\nu}\,dx^{\mu}\,dx^{\nu}=d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\,d\varphi^2)\ ,$$

where the coordinates $x^{\mu}=(\chi,\theta,\varphi)$ have the domains $0<\chi<\pi$, $0\leqslant\theta<\pi$, and $0<\varphi<2\pi$ with the usual polar singularities at 0 and π. The sufaces $\chi=$const are conformal to S^2 [described as above with the coordinates $x^{a}=(\theta,\varphi)$]. The S^3 harmonics are the tensorial eigenfunction solutions

$T_{\alpha\beta\cdots\gamma}$ to the equations

$$\Delta T_{\alpha\beta\cdots\gamma} \equiv T_{\alpha\beta\cdots\gamma;\mu;\nu} g^{\mu\nu} = \lambda T_{\alpha\beta\cdots\gamma} \quad (1)$$

that are regular on S^3 and have eigenvalues λ. The S^2 harmonics satisfy the obvious, similar conditions to the above.

At this point it is convenient to restrict consideration to the S^2 harmonics and review the scalar, vector, and second rank tensor solutions of Ref. 4. The scalar harmonics are the well known $Y^{(lm)}(\theta,\varphi)$ listed in Ref. 1 and these form a complete basis for scalars on S^2. The tangent space to a point on S^2 is two-dimensional, to span it we need two linearly independent solutions to the vector form of Eq. (1). These can be obtained from the gradient of the scalar harmonics ($\psi_a^{(lm)}$) and the dual of the gradient ($\phi_a^{(lm)}$) (see Table I for definitions) (since the space is two-dimensional taking a vector's dual gives another vector). That the gradient ψ_a is a solution to the vector form of Eq. (1) follows from Ricci's identity

$$Y_{,a|b|c}\,\gamma^{bc} = (Y_{,b|c}\,\gamma^{bc})_{,a} + Y_{,d} R^{d}_{\ bac}\gamma^{bc}$$

$$= [1 - l(l+1)] Y_{,a},$$

where (for S^2)

$$R_{abcd} = \gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}.$$

That the dual vector ϕ_a satisfies the same equation with the same eigenvalues follows from the vanishing of the covariant derivative of the Levi–Civita tensor. Under the improper transformation $\theta' = \pi - \theta$, $\varphi' = \varphi + \pi$ which corresponds to a coordinate inversion ψ_a transform as a polar vector and ϕ_a transforms as an axial vector, hence they are called even and odd parity vector spherical harmonics, respectively. For second rank tensors the space is four-dimensional and can be spanned by a skew tensor χ_{ab} and three symmetric tensors η_{ab}, ψ_{ab}, and ϕ_{ab} defined in Table I. These satisfy the tensor form of Eq. (1) from arguments analogous to the vector case. The point to be made here is that all the S^2 harmonics can be constructed from a knowledge of the scalar S^2 harmonics, but this is not the case for the S^3 harmonics as will be shown below.

The dimensionalities of the tensor spaces over S^3 complicate the previous analysis as can be seen by a count of the number of independent solutions to Eq. (1) as a function of the rank of the tensor. For scalars there is one set of functions $Y^{(nlm)}(\chi,\theta,\varphi)$. For vectors there are three linearly independent harmonics. In three dimensions we cannot use the trick of using the dual of a vector harmonic as we did on S^2, but we can use a generalization of this idea and use the curl of a vector to generate a linearly independent vector. In three dimensions there are two main types of vectors: divergence and curl free. The latter is exemplified by the gradient of the scalar harmonic

$$C_\alpha^{(nlm)} = Y_{;\alpha}^{(nlm)}. \quad (2)$$

Two other vector harmonics $A_\alpha^{(nlm)}$ and $B_\alpha^{(nlm)}$ can be found by imposing the divergence condition

$$A_{\alpha;\beta} g^{\alpha\beta} = 0, \quad (3)$$

solving for A_α from

$$\Delta A_\alpha = \lambda A_\alpha, \quad (4)$$

and forming the third vector from $B_\alpha = -(\text{curl}A)_\alpha$ (obviously B_α is also divergenceless). The vectors $A_\alpha^{(nlm)}$, $B_\alpha^{(nlm)}$, and $C_\alpha^{(nlm)}$ form an harmonic basis for the three-dimensional space of vectors on S^3. The second rank tensors on S^3 for a nine-dimensional vector space so we need nine independent tensor harmonics to span it. Two candidates come from the scalar harmonics

$$D_{\alpha\beta}^{(nlm)} = Y_{(\chi,\theta,\varphi)}^{(nlm)} g_{\alpha\beta} \quad (5)$$

and

$$C_{\alpha\beta}^{(nlm)} = Y_{;\alpha;\beta}^{(nlm)} + \tfrac{1}{3} n(n+2) Y^{(nlm)} g_{\alpha\beta} \quad (6)$$

[n.b. these are symmetric tensors and $C_{\alpha\beta}^{(nlm)} g^{\alpha\beta} = 0$]. Two more come from the divergenceless vectors

$$\widetilde{A}_{\alpha\beta}^{(nlm)} = A_{\alpha;\beta}^{(nlm)}, \quad (7)$$

$$\widetilde{B}_{\alpha\beta}^{(nlm)} = B_{\alpha;\beta}^{(nlm)}. \quad (8)$$

These can be further decomposed into symmetric and antisymmetric tensors:

$$A_{\alpha\beta}^{(nlm)} = \tfrac{1}{2}(\widetilde{A}_{\alpha\beta}^{(nlm)} + \widetilde{A}_{\beta\alpha}^{(nlm)}), \quad (9)$$

$$B_{\alpha\beta}^{(nlm)} = \tfrac{1}{2}(\widetilde{B}_{\alpha\beta}^{(nlm)} + \widetilde{B}_{\beta\alpha}^{(nlm)}), \quad (10)$$

$$\widetilde{A}_{|\alpha\beta|}^{(nlm)} = \tfrac{1}{2} B_\gamma^{(nlm)} \epsilon^{\gamma}_{\ \alpha\beta}, \quad (11)$$

$$\widetilde{B}_{|\alpha\beta|}^{(nlm)} = -\tfrac{1}{2}[1 + n(n+2)] A_\gamma^{(nlm)} \epsilon^{\gamma}_{\ \alpha\beta}, \quad (12)$$

and

$$G_{\alpha\beta}^{(nlm)} = \epsilon_\beta^{\ \mu\nu} Y_{;\mu}^{(nlm)}, \quad (13)$$

where $\epsilon_{\alpha\beta\gamma}$ is completely antisymmetric,

with

$$\epsilon_{123} = \sin^2\chi \sin\theta \quad \text{and} \quad \epsilon_{1ab} = \sin^2\chi\,\epsilon_{ab}. \quad (14)$$

The antisymmetric tensors arises algebraically from the vectors while the symmetric tensors come from the covariant derivatives of the vectors. To find two more independent harmonic solutions we impose tracefree and divergenceless conditions on a symmetric tensor $E_{\alpha\beta}$ and solve Eq. (1) with these constraints. The last harmonic $F_{\alpha\beta}$ is then found from $E_{\alpha\beta}$ by taking the symmetrized curl

$$\widetilde{F}_{\alpha\beta} = E_{\alpha\mu;\nu}\epsilon_\beta^{\ \mu\nu}, \quad (15)$$

$$F_{\alpha\beta} = \tfrac{1}{2}(\widetilde{F}_{\alpha\beta} + \widetilde{F}_{\beta\alpha}), \quad (16)$$

(n.b. $\widetilde{F}_{[\alpha\beta]} = 0$ due to the trace and divergence conditions on $E_{\alpha\beta}$).

We now proceed to verify these statements. For the S^3 scalar harmonics we have

$$\csc^2\chi \left\{ \frac{\partial}{\partial\chi}\left(\sin^2\chi \frac{\partial Y}{\partial\chi}\right) + \csc\theta \left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + \csc\theta \frac{\partial^2 Y}{\partial\varphi^2} \right] \right\} = \lambda Y^{(nlm)}. \quad (17)$$

The solutions that are regular at the poles are

$$Y^{(nlm)}(\chi,\theta,\varphi) = \left(\frac{2^{2l+1}(n+1)(n-l)!(l!)^2}{\pi(n+l+1)!} \right)^{1/2}$$

$$\times \sin^l\chi\, C^{(l+1)}_{n-l}(\cos\chi)\, Y^{(lm)}(\theta,\varphi) \tag{18}$$

with eigenvalues given by

$$\lambda = -n(n+2), \quad |m| \leqslant l \leqslant n = 0,1,2,\ldots.$$

The $C^{(l+1)}_{n-l}(x)$ are Gegenbauer polynomials as defined in Ref. 5, the $Y^{(lm)}(\theta,\varphi)$ are the S^2 scalar harmonics, and the coefficient is chosen[6] to normalize the harmonics

$$\int d^3\Omega \;\; Y^{(nlm)}(\chi,\theta,\varphi)\; Y^{(n'l'm')}(\chi,\theta,\varphi)^* = \delta_{nn'}\delta_{ll'}\delta_{mm'}, \tag{19}$$

where the S^3 volume element is given by

$$d^3\Omega = \sin^2\chi d\chi \, \sin\theta\, d\theta\, d\varphi.$$

The vector $C^{(nlm)}_\alpha$ satisfies

$$\Delta C^{(nlm)}_\alpha = (\Delta Y^{(nlm)})_{;\alpha} + Y^{(nlm)}_{;\beta} R^{\beta}{}_{\mu\alpha\nu} g^{\mu\nu}. \tag{20}$$

On S^3 the curvature tensor is given by

$$R^{\beta}{}_{\mu\alpha\nu} = \delta^\beta_\alpha g_{\mu\nu} - g^\beta_\nu g_{\mu\alpha} \tag{21}$$

and hence

$$\Delta C^{(nlm)}_\alpha = [2-n(n+2)]\, C^{(nlm)}_\alpha \tag{22}$$

so $C^{(nlm)}_\alpha$ satisfies Eq. (1) for a vector. It is not divergenceless, but satisfies

$$C^{(nlm)}_{\alpha;\beta} g^{\alpha\beta} = -n(n+2)\, Y^{(nlm)}. \tag{23}$$

To solve Eq. (4) we consider the obviously divergenceless vector (motivated by considering an odd parity split of a divergenceless vector)

$$A^{(nlm)}_\alpha = (o, h(\chi)\, \phi^{(lm)}_a(\theta,\varphi)) \tag{24}$$

then using the properties of the S^2-harmonics Eq. (4) becomes

$$\frac{d^2h}{d\chi^2} + \{-\lambda + [2-l(l+1)]\csc^2\chi - 2\cot^2\chi\}\, h(\chi) = 0 \tag{25}$$

which has the regular solution

$$h^{(nl)}(\chi) = \sin\chi^{\,l+1}\, C^{(l+1)}_{n-l}(\cos\chi), \tag{26}$$

with eigenvalue $\lambda = [1-n(n+2)]$.

From Eq. (7) and Eq. (21) we note that the Laplace operator acting on $\widetilde{A}_{\alpha\beta}$ is given by

$$\Delta \widetilde{A}_{\alpha\beta} = [3-n(n+2)]\widetilde{A}_{\alpha\beta} + 2\widetilde{A}_{\beta\alpha}. \tag{27}$$

Therefore, the vector $B_\alpha = \epsilon_\alpha{}^{\mu\nu} A_{\mu\nu}$ satisfies the S^3 vector harmonic equation

$$\Delta B_\alpha = \epsilon_\alpha{}^{\mu\nu} \Delta A_{[\mu;\nu]} = [1-n(n+2)]\, B_\alpha \tag{28}$$

and obviously B_α is divergenceless. It has the components

$$B^{(nlm)}_1 = -l(l+1)\csc^2\chi\, h^{(nl)}(\chi)\, Y^{(lm)}(\theta,\varphi),$$

$$B^{(nlm)}_a = -\frac{dh^{(nl)}}{d\chi}\, \psi^{(lm)}_a(\theta,\varphi). \tag{29}$$

The tensor harmonics consist of three antisymmetric tensors and six symmetric tensors. It is easy to verify that $D^{(nlm)}_{\alpha\beta}$ and $C^{(nlm)}_{\alpha\beta}$ satisfy

$$\Delta D^{(nlm)}_{\alpha\beta} = -n(n+2)\, D^{(nlm)}_{\alpha\beta} \tag{30}$$

and

$$\Delta C^{(nlm)}_{\alpha\beta} = [6-n(n+2)]\, C^{(nlm)}_{\alpha\beta}. \tag{31}$$

Using Eqs. (11), (12), and (13), the vanishing of the covariant derivative of the $\epsilon_{\alpha\beta\gamma}$ tensor, and Eqs. (28), (27), and (22), it follows

$$\Delta \widetilde{A}^{(nlm)}_{[\alpha\beta]} = [1-n(n+2)]\widetilde{A}^{(nlm)}_{[\alpha\beta]}, \tag{32}$$

$$\Delta \widetilde{B}^{(nlm)}_{[\alpha\beta]} = [1-n(n+2)]\widetilde{B}^{(nlm)}_{[\alpha\beta]}, \tag{33}$$

and

$$\Delta G^{(nlm)}_{\alpha\beta} = [2-n(n+2)]\, G^{(nlm)}_{\alpha\beta}. \tag{34}$$

From Eq. (27) and the analogous equation for $\widetilde{B}_{\alpha\beta}$ we find

$$\Delta A^{(nlm)}_{\alpha\beta} = [5-n(n+2)]\, A^{(nlm)}_{\alpha\beta}, \tag{35}$$

$$\Delta B^{(nlm)}_{\alpha\beta} = [5-n(n+2)]\, B^{(nlm)}_{\alpha\beta}. \tag{36}$$

The two remaining tensor harmonics are found by solving Eq. (1) for a symmetric tracefree divergenceless tensor $E^{(nlm)}_{\alpha\beta}$.

The properties of the S^3-harmonics in Table I suggest as a candidate the odd parity traceless tensor

$$(E^{(nlm)}_{\alpha\beta}) = \begin{bmatrix} 0 & H(\chi)\phi^{(lm)}_a(\theta,\varphi) \\ H(\chi)\phi_a{}^{(lm)}(\theta,\varphi) & S(\chi)\phi^{(lm)}_{ab}(\theta,\varphi) \end{bmatrix}. \tag{37}$$

The conditions $E_{\alpha\beta;\gamma}\, g^{\beta\gamma} = 0$ impose the relation

$$\frac{dH}{d\chi} + 2\cot\chi H(\chi) + \tfrac{1}{2}[2-l(l+1)]\csc^2\chi S(\chi) = 0 \tag{38}$$

which we will use to determine S given H. (In what follows we assume $l > 1$. The $l = 1$ case will be treated later.) Using the divergence condition the E_{1a} equation

$$\Delta(E_{1a}) = \frac{\partial^2 E_{1a}}{\partial\chi^2} + \csc^2\chi\, E_{1a|b|c}\,\gamma^{bc}$$

$$+ [\csc^2\chi - 6\cot^2\chi] E_{1a}$$

$$- 2\cot\chi\csc^2\chi E_{ab|c}\,\gamma^{bc} = \lambda E_{1a} \tag{39}$$

decouples and we find

$$\frac{d^2H}{d\chi^2} + 2\cot\chi\frac{dH}{d\chi} + \{[2-l(l+1)]\csc^2\chi - 2\cot^2\chi\} H = \lambda H. \tag{40}$$

The solution regular at the poles for $l > 1$ is given by

$$H^{(nl)}(\chi) = \sin^l\chi\, C^{(l+1)}_{n-l}(\cos\chi) \tag{41}$$

with the eigenvalue given by

$$\lambda = [2-n(n+2)]. \tag{42}$$

The symmetric tensor $F_{\alpha\beta}$ defined by Eq. (16) is obviously traceless, divergenceless, and linearly independent of the eight previously defined tensor harmonics. It is straightforward to show from Eq. (1) and Eq. (15) that $F_{\alpha\beta}$ satisfies the same harmonic equation as does $E_{\alpha\beta}$. The properties of the S^3 harmonics are summarized in Table II. The antisymmetric tensor $\widetilde{F}_{[\alpha\beta]}$ is identically zero. This follows from Eq. (15) and the divergenceless and traceless properties of $E_{\alpha\beta}$,

$$\epsilon_{\mu}{}^{\alpha\beta}\widetilde{F}_{[\alpha\beta]}=2E^{\ \nu}_{\mu;\nu}-2E^{\ \nu}_{\nu\ ;\mu}=0.$$

For the case in which $l=1$, Eq. (38) implies

$$\frac{dH}{d\chi}+2\cot\chi H=0$$

which integrates to give $H=\csc^2\chi$ and implies $\Delta E_{\alpha\beta}|_{l=1}=2E_{\alpha\beta}|_{l=1}$. But this solution is not regular at the poles. If we consider Eq. (39) with $l=1$, we find it is already decoupled but it is not divergencefree. In fact it is proportional to the $A_{\chi a}|_{l=1}$ tensor harmonic. There are no regular $l=1$ divergenceless tracefree harmonics.

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