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Flow of vapour in a liquid enclosure

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A solution is developed for the flow of a vapour in a liquid enclosure in which different portions of the liquid wall have different temperatures. It is shown that the vapour pressure is very nearly uniform in the enclosure, and an expression for the net vapour flux is deduced. This pressure and the net vapour flux are readily expressed in terms of the temperatures on the liquid boundary. Explicit results are given for simple liquid boundaries: two plane parallel walls at different temperatures and concentric spheres and cylinders at different temperatures. Some comments are also made regarding the effects of unsteady liquid temperatures and of motions of the boundaries. The hemispherical vapour cavity is also discussed because of its applicability to the nucleate boiling problem.

1. Introduction

We consider here the problem of the steady flow of vapour in a cavity enclosed by a liquid surface on which there is a given temperature distribution. This problem arises in the boiling heat transfer from a heated solid to a liquid which is superheated at the solid surface and subcooled at a distance from the solid. In this situation vapour bubbles grow and collapse in a temperature field which varies rapidly over the bubble surface. It has been proposed (Snyder 1956; Bankoff 1962; Snyder & Robin 1969) that the enhanced heat transfer from the solid arises from the latent heat of the vapour which flows away from the hotter liquid at the bubble base and is deposited at the colder portions of the bubble boundary. Until now explicit expressions for this vapour flux and for the corresponding transport of latent heat have not been available; these are given here in terms of the temperature distribution on the surface of the cavity. These results are not limited to the case of a bubble, but apply to any cavity in a liquid with arbitrary time-dependent shape provided that the velocity of the cavity wall and the velocity of the vapour flow are small compared with the speed of sound in the vapour. General application of our results is then possible because with this limitation on these velocities the vapour flow can be taken to be quasi-steady. Further, the net vapour flux leaving an element of surface of the cavity wall

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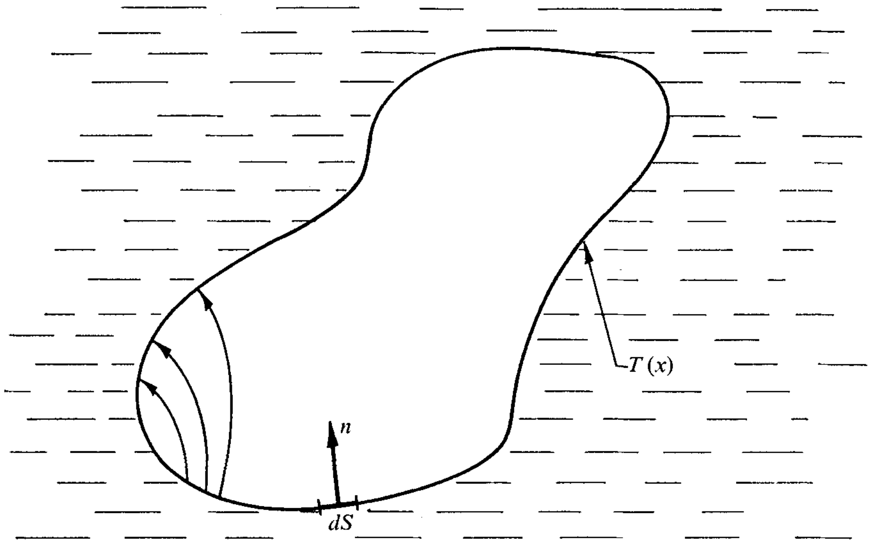

FIGURE 1. Vapour cavity in a liquid with varying temperature.

can be evaluated without carrying out a complete solution of the dynamical equations.

In the next section the steady problem is solved with the approximation of uniform vapour pressure and in the following section detailed solutions are carried out for some simple geometries for which the dynamical problem is straightforward. These explicit results facilitate the justification of the extension of the steady solution to unsteady situations, which require a re-examination of the assumption of uniform vapour pressure in the cavity.

The analysis throughout this paper supposes that the vapour behaves like an inviscid fluid and that its equation of state is that of a perfect gas.

2. The steady problem

We suppose that a cavity is enclosed in a liquid by a surface S and that a steady flow of vapour is maintained by temperature differences between points on the surface (see figure 1). The temperature distribution $T(\mathbf{x})$ on S is taken to be prescribed and is supposed to be kept constant in time by suitable addition or removal of heat. The vapour in the cavity is assumed to be a compressible inviscid fluid with the equation of state

$$p = R\rho T, \tag{1}$$

where p, ρ and T are the pressure, density and temperature respectively, and R is the ratio of the universal gas constant to the molecular weight of the vapour. The vapour flux density in the cavity is

$$\mathbf{J}(\mathbf{x}) = \rho(\mathbf{x})\,\mathbf{u}(\mathbf{x}), \tag{2}$$

where $\mathbf{u}(\mathbf{x})$ is the local velocity vector. The component of $\mathbf{J}$ normal to a surface element dS of the cavity can be connected with reasonable accuracy to the local values of the temperature and vapour density by means of elementary kinetic theory. If we take the vapour molecules to have a Maxwellian velocity distribution, then the normal mass flux into an element of boundary surface is

$$J_{-} = \beta(RT/2\pi)^{\frac{1}{2}}\rho, \tag{3}$$

where β is the accommodation coefficient for condensation, i.e. the probability that a vapour molecule striking the liquid surface sticks to the surface. The mass flux leaving an element of surface may similarly be written as

$$J_{+} = \alpha(RT/2\pi)^{\frac{1}{2}}\rho^{e}(T). \tag{4}$$

In (4), $\rho^{e}(T)$ is the equilibrium vapour density for the temperature T and α is the accommodation coefficient for evaporation. In equilibrium one has $\alpha = \beta$ but without equilibrium it is possible that $\alpha \neq \beta$. The net flux leaving the liquid surface is clearly

$$\mathbf{J}.\mathbf{n} = (RT/2\pi)^{\frac{1}{2}}[\alpha\rho^{e}(T) - \beta\rho], \tag{5}$$

where $\mathbf{n}$ is the unit normal into the cavity (figure 1).

By means of (1), (2) and (5) we find the following expression for the normal component of the vapour velocity at the boundary:

$$\mathbf{u}.\mathbf{n} = \left(\frac{RT}{2\pi}\right)^{\frac{1}{2}}\left[\alpha\frac{p^{e}(T)}{p} - \beta\right], \tag{6}$$

where $p^{e}(T)$ is the equilibrium vapour pressure corresponding to the local temperature T and p is the local value of the pressure acting on the vapour side of the surface element under consideration. This expression allows one to illustrate some unique characteristics of the flow under consideration. In the first place, it should be remarked that the component $\mathbf{u}.\mathbf{n}$ of the (macroscopic) flow velocity that the vapour has in the vicinity of the surface is not acquired through a (macroscopic) dynamical effect such as the acceleration caused by a (macroscopic) pressure gradient. Equation (6) is purely the macroscopic manifestation of the microscopic imbalance between the number of molecules that are emitted and the number of molecules that are absorbed by the surface element, which is thus seen to act as an external source or sink of mass. Further, it is apparent from (6) that the strength of this source or sink is determined by the *absolute* level of the pressure. This circumstance is a marked contrast to the usual situation of compressible fluid mechanics, in which the flow is dominated by pressure *gradients*, while the absolute level of the pressure is irrelevant.

By use of the relation $c^2 = \gamma RT$, which determines the speed of sound c in a perfect gas in terms of the temperature T and the ratio of the specific heats γ, (6) can be put in the form

$$\frac{\mathbf{u}.\mathbf{n}}{c} = \frac{\alpha}{(2\pi\gamma)^{\frac{1}{2}}}\left[\frac{p^{e}(T)}{p} - \frac{\beta}{\alpha}\right]. \tag{7}$$

Since the left-hand side of (7) is of the order of M, the Mach number for the vapour flow, it is evident that it is sufficient to specify p up to order M^2 to obtain an expression for the vapour flux correct to lowest order in the Mach number.

It will be shown below that, up to terms of order M^2, one can take the pressure to be uniform in the enclosure. Just as an indication of how this procedure may be expected to succeed, we consider here the case in which viscous effects are unimportant and the flow is steady. Under these conditions, the Navier–Stokes equations reduce to the Euler equations, which can be written as

$$\nabla . (p\mathbf{I} + \rho\mathbf{uu}) = 0,$$

where $\mathbf{I}$ is the unit second-rank tensor, with components δ_{ij}, and $\mathbf{uu}$ the tensor with components $u_i u_j$. From this equation we obtain

$$\nabla . [p(\mathbf{I} + \gamma\mathbf{uu}/c^2)] = 0,$$

which shows that, when the velocity field is smooth, one may take

$$p \simeq \text{constant}, \tag{8}$$

to second order in the Mach number. This relation justifies the assumption of the approximate uniformity of the pressure within the cavity so long as the flow velocity is small compared with the speed of sound. It follows that temperature variations are compensated locally by density variations as given by (1) with fixed pressure.

The determination of the value of the pressure in the cavity is now a straightforward matter if one observes that the net addition of vapour to the cavity must vanish in the steady state, so that

$$\int_S \mathbf{J}.\mathbf{n}\, dS = 0. \tag{9}$$

If we write $\rho^e(T) = p^e(T)/RT$, where $p^e(T)$ is the equilibrium vapour pressure, and if we write $\rho = p/RT$, then substitution of $\mathbf{J}.\mathbf{n}$ from (5) into (9) determines the pressure in the cavity as

$$p = \frac{\alpha}{\beta}\int_S T^{-\frac{1}{2}} p^e(T)\, dS \Big/ \int_S T^{-\frac{1}{2}}\, dS. \tag{10}$$

The coefficients α and β have been taken to be constants independent of the temperature. We see from (10) that the pressure in the cavity may be found when the temperature is specified over the surface of the cavity. When the pressure has been so determined, the net flux at any given point on the cavity surface is given directly from (5) as

$$\mathbf{J}.\mathbf{n} = (2\pi RT)^{-\frac{1}{2}}[\alpha p^e(T) - \beta p]. \tag{11}$$

It may be remarked that a rough approximation to the pressure in the cavity may be found from the observation that $p^e(T)$ varies with T much more strongly than $T^{-\frac{1}{2}}$. We may then write, with the simplification that $\alpha = \beta$,

$$p \cong \frac{1}{S}\int_S p^e(T)\, dS, \tag{12}$$

which says that the pressure in the cavity is given approximately by the average of the equilibrium vapour pressure over the cavity surface.

The general results contained in (5) and (10) will be applied in the next section to some specific examples with special symmetries. The effect of the energy equation will also be noted for these cases.

3. Examples with simple geometries

The general equations which describe the steady problem introduced above are the mass conservation equation

$$\nabla . (\rho \mathbf{u}) = 0, \tag{13}$$

the momentum conservation equation

$$\rho(\mathbf{u} . \nabla) \mathbf{u} = -\nabla p \tag{14}$$

and the energy conservation equation

$$\rho c_v(\mathbf{u} . \nabla) T = k\nabla^2 T - p\nabla . \mathbf{u}. \tag{15}$$

In the energy equation c_v is the specific heat of the vapour at constant volume and k is its thermal conductivity; these quantities will be assumed to be constant. Equation (15) may be written alternatively in the form

$$\rho c_p(\mathbf{u} . \nabla) T - (\mathbf{u} . \nabla) p = k\nabla^2 T, \tag{16}$$

where c_p is the specific heat at constant pressure. A direct solution of these equations may readily be carried out for symmetric cases for which the stream-lines are explicit.

Two plane parallel liquid walls at different temperatures

We suppose that the surfaces of the liquid planes are maintained at the temperatures T_0 and T_1 respectively with $T_0 > T_1$. Equation (10) then gives for the vapour pressure in the region between the liquid walls

$$p = \frac{\alpha}{\beta} \frac{p^e(T_0)\, T_0^{-\frac{1}{2}} + p^e(T_1)\, T_1^{-\frac{1}{2}}}{T_0^{-\frac{1}{2}} + T_1^{-\frac{1}{2}}}. \tag{17}$$

For this constant value of the vapour pressure, the equation of state gives ρT = constant, so that, for example, in the neighbourhood of the hot wall the vapour density is

$$\rho_0 = \frac{\alpha}{\beta} \rho^e(T_0) \left(\frac{T_1}{T_0}\right)^{\frac{1}{2}} \frac{1 + [\rho^e(T_1)/\rho^e(T_0)]\,(T_1/T_0)^{\frac{1}{2}}}{1 + (T_1/T_0)^{\frac{1}{2}}}. \tag{18}$$

The vapour flux is, from (5),

$$J = \alpha \rho^e(T_0) \left(\frac{RT_0}{2\pi}\right)^{\frac{1}{2}} \frac{1 - [\rho^e(T_1)/\rho^e(T_0)]\,(T_1/T_0)^{\frac{1}{2}}}{1 + (T_1/T_0)^{\frac{1}{2}}}; \tag{19}$$

alternatively, one may write

$$J = \frac{\alpha}{1 + (T_1/T_0)^{\frac{1}{2}}} \frac{p^e(T_0) - p^e(T_1)}{(2\pi R T_0)^{\frac{1}{2}}}.$$

From (18) and (19) we can evaluate the vapour velocity at the hot wall:

$$u_0 = \beta \left(\frac{RT_0}{2\pi}\right)^{\frac{1}{2}} \left(\frac{T_0}{T_1}\right)^{\frac{1}{2}} \frac{1 - [\rho^e(T_1)/\rho^e(T_0)]\,(T_1/T_0)^{\frac{1}{2}}}{1 + [\rho^e(T_1)/\rho^e(T_0)]\,(T_1/T_0)^{\frac{1}{2}}}. \tag{20}$$

The velocity of the vapour where its temperature is T would be given by

$$u = (T/T_0)\,u_0.$$

It is also of interest to determine the temperature distribution in the region between the walls. This calculation may be carried out quite easily from (16) for constant pressure and for this one-dimensional situation. We introduce the spatial co-ordinate x, measured from the hot wall with the cold wall at $x = a$. The equation to be solved is

$$c_p \rho u\, dT/dx = k d^2T/dx^2, \tag{21}$$

with the boundary conditions $T(x = 0) = T_0$ and $T(x = a) = T_1$. The flux $J = \rho u$ is constant, and in terms of this constant a characteristic length δ may be introduced:

$$\delta = k/c_p J. \tag{22}$$

Then the appropriate solution of (21) is

$$T(x) = \frac{T_0 - T_1 e^{-a/\delta}}{1 - e^{-a/\delta}} - \frac{T_0 - T_1}{1 - e^{-a/\delta}} \exp\left[-(a - x)/\delta\right]. \tag{23}$$

The characteristic length δ may also be written as

$$\delta = D_0/\gamma u_0,$$

where $D_0 = k/\rho_0 c_v$ is the thermal diffusivity of the vapour. Under ordinary circumstances δ is extremely small, so that to a very good approximation one has

$$T(x) \approx T_0 - (T_0 - T_1)\, e^{-(a-x)/\delta}, \tag{24}$$

from which it is evident that the temperature in the vapour retains the value T_0 as x increases from zero until a distance from the cold wall of the order of δ is reached. The temperature then drops rapidly from T_0 to its value T_1 at the cold wall.

The results for this plane case have been presented elsewhere (Plesset 1952) and that presentation also showed for this case that the corrections to the pressure lead to a correction to the uniform pressure in (17):

$$p' = p[1 + (\gamma u_0^2/c_0^2)(1 - u/u_0)], \tag{25}$$

where c_0 is the sound velocity near the hot wall. The corrections to the uniform pressure are seen explicitly to be of second order in the Mach number.

Concentric cylinders and spheres at different temperatures

These cases are respectively the two- and three-dimensional analogues of the plane problem just described. We let the inner surface have radius R_0 and uniform temperature T_0, and the outer surface have radius R_1 and temperature T_1. We use the area ratio of the boundary surfaces, which is $\sigma_1 = R_1/R_0$ for the cylindrical case and $\sigma_2 = R_1^2/R_2^2$ for the spherical case; we study both examples together with

$$\sigma_n = (R_1/R_0)^n, \tag{26}$$

where $n = 1$ or 2. The pressure between the surfaces is, from (10),

$$p = \frac{\alpha}{\beta}\frac{p^e(T_0)\,T_0^{-\frac{1}{2}} + \sigma_n p^e(T_1)\,T_1^{-\frac{1}{2}}}{T_0^{-\frac{1}{2}} + \sigma_n T_1^{-\frac{1}{2}}}. \tag{27}$$

By symmetry the streamlines issue from the common centre of the two surfaces, so that the continuity equation (13) leads to

$$r^n J(r) = R_0^n J_0 = R_1^n J_1, \tag{28}$$

where $J_0 = J(R_0)$ and $J_1 = J(R_1)$. In particular we have

$$J_0 = \sigma_n J_1. \tag{29}$$

The energy equation (16) with $p =$ constant may be easily integrated. For the spherical case one finds

$$T(r) = \frac{T_0 - \eta T_1}{1-\eta} - \frac{T_0 - T_1}{1-\eta}\exp\left[-\frac{R_1(R_1 - r)}{r\delta}\right], \tag{30}$$

where δ is $k/c_p J_1$ and the quantity η is given by

$$\eta = \exp\left[-R_1(R_1 - R_0)/R_0\,\delta\right],$$

which is usually small. The cylindrical case has the solution

$$T(r) = \frac{T_0 - (R_0/R_1)^{R_1/\delta} T_1}{1-(R_0/R_1)^{R_1/\delta}} - \frac{T_0 - T_1}{1-(R_0/R_1)^{R_1/\delta}}\left(\frac{r}{R_1}\right)^{R_1/\delta}, \tag{31}$$

where again $\delta = k/c_p J_1$. The one-dimensional results may be recovered in a straightforward manner from these expressions.

Although an explicit expression for the correction to the uniform pressure field analogous to (25) is not as easily available for this more complicated geometry, one can still estimate the error introduced by this approximation by investigating to what extent the momentum equation (14) is satisfied by the results of (27), (28) and (30). Integrating (14) between R_0 and R_1 we find

$$p_1 - p_0 = -\int_{R_0}^{R_1} \rho u \frac{du}{dr}\,dr$$

(where $p_i = p(R_i)$, $i = 0, 1$). For the spherical case one obtains

$$\frac{p_1 - p_0}{p} = \frac{\rho_1}{p} u_1^2 \left\{\frac{\rho_1}{\rho_0}\left(\frac{R_1}{R_0}\right)^4 \left[1 - \frac{1}{4}\left(1 - \frac{R_0^4}{R_1^4}\right)\right] - 1 + O(\eta)\right\}. \tag{32}$$

Similarly, for the cylindrical case one finds

$$\frac{p_1 - p_0}{p} = \frac{\rho_1}{p} u_1^2 \left\{\frac{1}{2}\left(\frac{R_1}{R_0}\right)^2 \frac{\rho_1}{\rho_0}\left[1 + \frac{R_0^2}{R_1^2} + O\left(\frac{\delta}{R_1}\right)\right] - 1\right\}. \tag{33}$$

In either case, the velocity u_1 has the form

$$u_1 = \beta\left(\frac{RT_1}{2\pi}\right)^{\frac{1}{2}} \frac{p^e(T_0) - p^e(T_1)}{p^e(T_0) + \sigma_n (T_0/T_1)^{\frac{1}{2}} p^e(T_1)}.$$

It is evident that both in the cylindrical and in the spherical geometry, the pressure drop needed to maintain the flow computed with the assumption of uniform pressure represents a correction to the value of p which is quadratic in the Mach number of the flow provided that the area ratio σ_n does not become too large.

Some considerations of more general geometries

The results obtained in the previous subsections for some special cases illustrate both the physical consequences and the accuracy of the assumption of uniform pressure within the cavity, and they allow one to draw some qualitative conclusions concerning more general geometries.

In the first place, upon comparison of the cylindrical and spherical cases with the plane one, it is seen that similar results are obtained for the mass flux from the hot surface and for the correction to the uniform value of the pressure provided that proper allowance is made for area effects. It is to be expected that a similar behaviour would be found also in more complex configurations, so that our results will apply whenever there are stream tubes that do not undergo extreme variations in their cross-section along their axis. Clearly, such pathological situations can be ruled out if the shape of the cavity and the temperature distribution on its surface are sufficiently regular.

The temperature distributions in all three cases considered are qualitatively similar, with the temperature nearly equal to its value at the hotter boundary and uniform except for a very thin layer adjacent to the colder liquid surface. This behaviour is of course a consequence of the small thermal diffusivities of vapours, and the fact that heat conduction effects cannot compete with even moderate convection of thermal energy unless very steep gradients occur. A similar behaviour would be expected in general, with the temperature along any streamline nearly equal to the value at its hot initial point until a sharp transition near its cold end point. The only regions of the flow in which rapid changes in velocity could take place are therefore the temperature boundary layers in the vicinity of the colder portions of the surface. Although one might expect considerable pressure gradients there, the thickness of the layer is so small that their overall contribution to the pressure drop is insignificant. Clearly, viscous effects could also be considerable only in such layers, and their contributions to the pressure drop would also be negligible for the same reasons.

4. Comments on the unsteady problem

In view of the comments of the previous section on the steady case, which would apply to an unsteady situation, we limit our considerations here for the most part to the unsteady problem for two plane parallel liquid surfaces at different temperatures $T_0(t)$ and $T_1(t)$. The hotter surface will be taken to be at rest, while the colder one is at a distance $a(t)$ from it. Again, one can estimate the error caused by the neglect of the momentum equation in attributing to the pressure its quasi-steady value given by (17), where the surface temperatures are now prescribed functions of time. To this end we consider the overall momentum balance for the flow between the two planes:

$$p_1 - p_0 = \rho_0 u_0^2 - \rho_1 u_1^2 - \frac{d}{dt}\int_0^a \rho u \, dx.$$

Clearly, the first two terms on the right-hand side give rise to a relative correction to the absolute pressure which is of the order of the square of the Mach number and hence can be neglected. Further, for purposes of an estimate, we may use a quasi-steady approximation in the last term, by using (19) to compute the mass flux. In this way, if P is the average pressure between the planes, we obtain

$$\frac{p_1-p_0}{P} = -\frac{\rho_1 u_1}{P}\frac{da}{dt} - \frac{a}{P}\frac{dJ}{dt} + O(M^2).$$

The first term is again negligible provided that da/dt is at most of the order of u (i.e. that the velocity of the liquid surface is small compared with the speed of sound in the vapour). From the second term we may obtain a limit on the values which the time scales for the variation of T_0 and T_1 must have in order for $(p_1-p_0)/P \ll 1$. If time derivatives are approximated by finite increments, we readily obtain from (19) the estimate

$$\Delta t \gtrsim \frac{\alpha a}{C}\frac{L}{RT}\frac{\Delta T}{T}\frac{P}{p_1-p_0}, \tag{34}$$

where U, T and C are appropriate scales for the vapour velocity, temperature and sound speed, L is the latent heat, and ΔT is of the order of the increment in T over the time scale Δt. For water at 100 °C, with $\Delta T/T \sim 10^{-1}$, $\alpha \sim 1$ and $p_1 - p_0 \sim 10^{-2}P$, (34) gives $\Delta t \gtrsim 10^{-3}a$, where a is expressed in centimetres and Δt in seconds. It may be seen that the constraint on the validity of the approximation of uniform pressure is not very stringent. This circumstance was to be expected in view of the fact that, as was noted above, the mass flux is determined by the magnitude of the pressure, and pressure perturbations propagate with the speed of sound.

While the above considerations show that the assumption of uniform pressure in the cavity is justified for the unsteady case also, the problem of computing this uniform pressure remains. To this end we consider the overall mass conservation equation for the unsteady situation:

$$\int_S \mathbf{J}.\mathbf{n}\,dS = \frac{d}{dt}\int_V \rho\,dV. \tag{35}$$

In the first place it should be noted that, if the surface element dS moves with a velocity v, then the expression for the mass flux from it becomes

$$\mathbf{J}.\mathbf{n} = [(RT/2\pi)^{\frac{1}{2}} + \mathbf{v}.\mathbf{n}]\,[\alpha\rho^e(T) - \beta\rho]. \tag{36}$$

This relation may readily be derived by considering the kinematic correction to the fluxes $J_\pm$ used to obtain (5). An adaptation of the reasoning that leads to (7) shows, however, that the difference between (36) and (5) is of second order in the Mach number whenever $|\mathbf{v}| \lesssim |\mathbf{u}|$, so that the expression for the flux appropriate for a stationary surface element can be employed in (35). In this way one readily finds

$$p = p_0\left\{1 - \frac{(2\pi R)^{\frac{1}{2}}}{\beta p_0 \int_S T^{-\frac{1}{2}}\,dS}\frac{d}{dt}\int_V \rho\,dV + O(M^2)\right\}, \tag{37}$$

where p_0 is the quasi-steady approximation to the pressure given by (10), with the surface temperature depending now not only on position but also, in general, on time.

It may readily be seen that the second term in (37) can introduce a correction which is of first order in the Mach number. To show this we consider again the case of two parallel planes and we observe that, since we are computing a term which is at most of the order of the Mach number, we can use expressions accurate to zero order (i.e. those computed in the quasi-steady approximation) for the various quantities. We then obtain, per unit area,

$$\int_V \rho\, dV = \rho_0 a[1+O(\delta/a)],$$

where δ is the very small quantity defined by (22). Equation (37) then gives

$$p = p_0\left\{1 - \frac{(2\pi RT_0)^{\frac{1}{2}}}{\beta[1+(T_0/T_1)^{\frac{1}{2}}]p_0}\left(a\frac{d\rho_0}{dt}+\rho_0\frac{da}{dt}\right)\right\}. \tag{38}$$

From the term containing $a\,d\rho_0/dt$ one can compute a time scale Δt such that temperature changes occurring in times of the order of Δt would result in a contribution of order $\epsilon \ll 1$ from this term. From (18) one readily finds

$$\Delta t \gtrsim \frac{\alpha}{\beta}\frac{a}{\beta\epsilon C}\frac{L}{RT}\frac{\Delta T}{T}, \tag{39}$$

which is essentially the same as (34). The last term in (38), however, clearly gives a contribution to p of first order in the Mach number unless the following very stringent relation is satisfied:

$$\frac{da}{dt} \lesssim \frac{U}{C}U.$$

5. Application to the nucleate boiling bubble

When boiling takes place at the surface of a heated solid immersed in a liquid, large increases in the heat transfer from the solid to the liquid are observed. In the situation of concern the liquid is superheated in the neighbourhood of the solid, and the liquid temperature decreases rapidly away from the surface. These large temperature gradients can be maintained even by natural convection or by moderate liquid flow velocities. It has been observed (Gunther 1951; Johnson, de la Pena & Mesler 1966) that the boiling bubbles grow and collapse in approximately hemispherical form (see figure 2), and that a thin liquid layer, the 'microlayer', lies between the bubble base and the solid (Cooper & Lloyd 1966, 1969; Judd & Shoukri 1975). It has been proposed that a significant portion of the increased heat transfer which is measured arises from the latent heat which is transported from the relatively hotter microlayer surface at the bubble base to the colder hemispherical bubble cap.

In order to get some idea of the effectiveness of the latent-heat transfer through a vapour bubble, it is necessary to explain the bubble dynamics; in particular, the radius–time behaviour of the bubble in a temperature field which

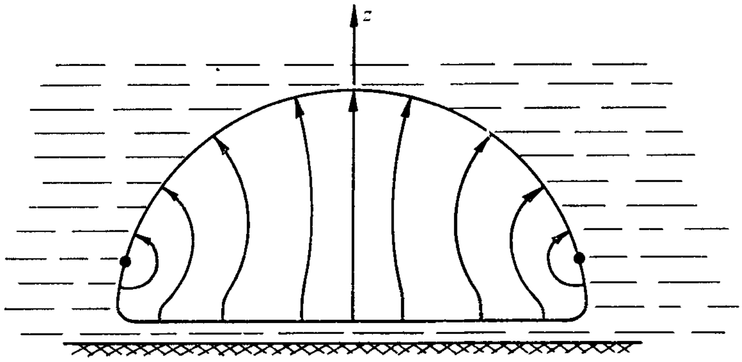

FIGURE 2. Schematic representation of the nearly hemispherical nucleate boiling bubble. The vapour-transport streamlines have only qualitative significance; the dots on the boundary mark the level at which the vapour flux $J = 0$. For simplicity the microlayer is shown with constant thickness.

varies appreciably over distances of the order of the bubble dimensions. According to experimental observations (Gunther & Kreith 1950) in highly subcooled nucleate boiling, the lifetime of the bubbles is of the order of 5×10^{-4} s and their maximum radius of the order of 10^{-2} cm. From the estimate (34) of the preceding section it is clear that one can then take the pressure to be uniform within the bubble. Further, from (38) it may be seen that the correction to the quasi-steady result (10) is at most of the order of the Mach number, and hence its effect on the bubble dynamics can be neglected. Therefore, if for simplicity the temperature T_0 over the base of the bubble is taken to be uniform, with $\alpha = \beta$ one obtains

$$p(t) = \frac{p^e(T_0) + 2T_0 \langle T^{-\frac{1}{2}} p^e(T) \rangle}{1 + 2T_0^{\frac{1}{2}} \langle T^{-\frac{1}{2}} \rangle}, \tag{40}$$

where
$$\langle T^{-\frac{1}{2}} p^e(T) \rangle = \frac{1}{R_b} \int_0^{R_b} T^{-\frac{1}{2}}(z)\, p^e(z)\, dz,$$

$$\langle T^{-\frac{1}{2}} \rangle = \frac{1}{R_b} \int_0^{R_b} T^{-\frac{1}{2}}(z)\, dz.$$

Here R_b denotes the radius of the bubble at the time instant considered, and $T(z)$ is the temperature distribution on the hemispherical surface as a function of the distance from the heated wall. The factor of 2 in (40) is the ratio of the area of the hemispherical cap to that of the base.

Although negligible for the computation of the pressure, the corrections of first order in the Mach number can *a priori* be important for the computation of the net vapour transport from the bubble base. Experiment shows, however, that the curves $R_b(t)$ are very nearly symmetric about the maximum radius (see, for example, Bankoff & Mikesell 1959), and it is clear that the correction in (38) changes sign upon a reversal of the velocity. Therefore one is led to the conclusion that, although the instantaneous mass flux will be somewhat in error if determined from the quasi-steady relations, the overall mass transport during a complete bubble lifetime will be estimated correctly to second order in the

Mach number. The streamlines for the vapour transport in the bubble are rather complex, but their qualitative features are readily understood and are illustrated in figure 2.

An explicit application of the formulation given here to the analysis of nucleate boiling heat transfer will be presented elsewhere.

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REFERENCES

BANKOFF, S. G. 1962 *A.I.Ch.E. J.* **8**, 63.

BANKOFF, S. G. & MIKESELL, R. D. 1953 *Chem. Engng Prog. Symp. Ser.* **29**, 95.

COOPER, M. G. & LLOYD, A. J. P. 1966 *Proc. 3rd Int. Heat Transfer Conf.*, vol. 3, pp. 193–203. New York: A.I.Ch.E.

COOPER, M. G. & LLOYD, A. J. P. 1969 *Int. J. Heat Mass Transfer*, **12**, 895.

GUNTHER, F. C. 1951 *Trans. A.S.M.E.* **73**, 115.

JOHNSON, M. A., DE LA PENA, J. & MESLER, R. B. 1966 *Chem. Eng. Progr. Symp. Ser.* **62**, 1.

JUDD, R. L. & SHOUKRY, S. M. 1975 *J. Heat Transfer*, **97**, 494.

PLESSET, M. S. 1952 *J. Chem. Phys.* **20**, 790.

SNYDER, N. W. 1956 *Summary of Conf. on Bubble Dynamics & Boiling Heat Transfer* (ed. S. G. Bankoff, W. J. Colahan & D. R. Bartz), p. 13. *Jet Propulsion Lab., Caltech. Memo.* no. 20–137.

SNYDER, N. W. & ROBIN, T. T. 1969 *J. Heat Transfer*, **90**, 404.