# Wave propagation in a random medium: A complete set of the moment equations with different wavenumbers

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Propagation of waves in a random medium is studied under the "quasioptics" and the "Markov random process" approximations. Under these assumptions, a Fokker–Planck equation satisfied by the characteristic functional of the random wave field is derived. A complete set of the moment equations with different transverse coordinates and *different* wavenumbers is then obtained from the Fokker–Planck equation of the characteristic functional. The applications of our results to the pulse smearing of the pulsar signal and the frequency correlation function of the wave intensity in interstellar scintillation are briefly discussed.

## I. INTRODUCTION

Phenomena such as the twinkling of starlight and the ionospheric, interplanetary, and interstellar radio wave scintillations involve the propagation of an electromagnetic wave in a random medium. A complete statistical description of the wave field requires the solution of all moments of the wave field with different positions and different wavenumbers.

A complete set of the moment equations of the wave field with different transverse coordinates but the *same* wavenumbers has been derived under the "quasioptics" and the "Markov random process" approximations,[1,2] which can be applied to both weak and strong scatterings. However, such a set of the moment equations with the *same* wavenumbers is not sufficient to describe all the statistical properties of the random wave field. Some observed quantities in interstellar scintillations, such as the pulse smearing and the correlation function of the intensity fluctuation with different wavenumbers,[3-5] need the solution of the moment equations with different wavenumbers. It is the purpose of this paper to derive a complete set of the moment equations with different transverse positions and *different* wavenumbers under the quasioptics and the Markov random process approximations. The results reduce to those of Tatarskii[1,2] in the case of the *same* wavenumbers. It is noted that the method of the derivation used here is new, and simpler than that by Tatarskii.[1,2]

It is the idea of Hopf[6] to introduce the "characteristic functional" as an alternative way to describe the complete statistical properties of a random field. In Sec. II, we will derive a Fokker–Planck equation for the characteristic functional of the random electromagnetic field. In Sec. III, a complete set of the moment equations will be derived from the Fokker–Planck equation satisfied by the characteristic functional. Some applications of the results will be briefly discussed in Sec. IV.

## II. FOKKER–PLANCK EQUATION FOR THE CHARACTERISTIC FUNCTIONAL OF THE WAVE FIELD

We consider the propagation of a monochromatic wave $E_\omega(\mathbf{r}, t)$ obeying the scalar wave equation

$$\nabla^2\Phi_\omega(\mathbf{r}) + (\omega^2/c^2)\epsilon_\omega(\mathbf{r})\Phi_\omega(\mathbf{r}) = 0, \tag{1}$$

where

$$E_\omega(\mathbf{r}, t) = \Phi_\omega(\mathbf{r})e^{-i\omega t}. \tag{2}$$

$\Phi_\omega(\mathbf{r})$ may be regarded as a Fourier component in time of a general wavefunction. Here $(\omega/2\pi)$ is the frequency of the monochromatic wave, $c$ is the speed of light, and $\epsilon_\omega(\mathbf{r})$ is the refractive index of the medium in which the wave propagates.

The refractive index $\epsilon_\omega(\mathbf{r})$ is a random function and depends on both the position $\mathbf{r}$ and the wave frequency $\omega$. As an example, we will consider in this paper the propagation of the high frequency waves with $\omega \gg \omega_p$, the plasma frequency of the medium, in the plasma medium. This applies to the propagation of the radio waves in the ionosphere, the interplanetary space, or the interstellar medium. If $N_e$ is the electron density, then we have

$$\epsilon_\omega(\mathbf{r}) = 1 - \omega_p^2/\omega^2 \tag{3}$$

and

$$\omega_p^2 = 4\pi N_e e^2/m, \tag{4}$$

where $m$ is the mass and $e$ is the charge of an electron.

Now $N_e$ and $\epsilon_\omega(\mathbf{r})$ fluctuate irregularly. Let $\langle\ \rangle$ denote an average over an ensemble of propagation volumes. Then define

$$\langle\epsilon_\omega(\mathbf{r})\rangle = \epsilon_{\omega 0}(\mathbf{r}),$$

$$N_e(\mathbf{r}) = \langle N_e(\mathbf{r})\rangle + \delta N_e(\mathbf{r}),$$

$$\beta(\mathbf{r}) = -4\pi e^2 \delta N_e(\mathbf{r})/mc^2.$$

We have

$$\nabla^2\Phi_\omega(\mathbf{r}) + k^2[1 + \beta(\mathbf{r})/k^2]\Phi_\omega(\mathbf{r}) = 0, \tag{5}$$

where now $\beta(\mathbf{r})$ is a wave-frequency independent random variable with zero mean and where the wavenumber $k = (\omega/c)\sqrt{\epsilon_{\omega 0}}$.

It is useful to define

$$\Phi_\omega(\mathbf{r}) = u(k, \mathbf{r})e^{ikz}, \tag{6}$$

from which we obtain

$$2ik\frac{\partial u(k,\mathbf{r})}{\partial z} + \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(k,\mathbf{r}) + \beta(\mathbf{r})u(k,\mathbf{r}) = 0. \tag{7}$$

Let

$$\mathbf{r} = (z, \rho), \quad \rho = (x, y), \quad \text{and} \quad s = (\rho, k).$$

In order to proceed further, we will make two assumptions about the wave equation and the properties of the medium.

  

First, we assume that the term $\partial^2 u/\partial z^2$ in Eq. (7) can be neglected. This is called the "quasioptics" approximation or "parabolic" approximation. Physically this assumption is equivalent to neglect the reflected wave since the equation has been reduced to one with a first-order derivative in $z$ from the one with a second-order derivative. Thus we have

$$\frac{\partial}{\partial z}u(z,\rho,k)+\frac{1}{2ik}\nabla_\rho^2 u(z,\rho,k)+\frac{1}{2ik}\beta(z,\rho)u(z,\rho,k)=0, \quad (8)$$

where

$$\nabla_\rho^2=\partial^2/\partial x^2+\partial^2/\partial y^2.$$

Second, we assume that $\beta(z,\rho)$ is delta-correlated in $z$ direction. This is called the Markov random process approximation. As we can see later, this is equivalent to assume that the correlation scale of $\beta(z,\rho)$ in $z$ direction is much less than the correlation scale of the wave field $u$ in $z$ direction. We then have

$$\langle\beta(z,\rho)\beta(z',\rho')\rangle=2\delta(z-z')A(\rho-\rho') \quad (9a)$$

and

$$A(\rho-\rho')=\int_{-\infty}^{z}\langle\beta(z,\rho)\beta(z',\rho')\rangle\,dz'. \quad (9b)$$

Note that the $z$ dependence of $A(\rho)$ is not explicitly expressed for convenience.

The validity of the above two assumptions has been discussed.[2,7] We will only note that the "quasioptics" approximation and the "Markov" approximation can be applied in the strong scattering cases.

It is known that the probability distribution function at time $t$ of a random variable $x(t)$ that satisfies a differential equation of the first order in time with a delta-correlated external random force satisfies the Fokker–Planck equation. In our case, $z$ plays the role of time. However, for a fixed value of $z$, the random field $u(z,\rho,k)$ does not have just a discrete value but has an infinite number of values and is a function of $\rho$ and $k$. It is the idea of Hopf[6] to introduce a characteristic functional $\Psi$ to describe the statistical properties of a random field. One defines the characteristic functional as

$$\begin{aligned}\Psi(z,\nu,\nu^*)&=\langle\exp(iR_z)\rangle\\&=\langle\exp\{i\int\int[u(z,\rho,k)\nu(\rho,k)\\&\quad+u^*(z,\rho,k)\nu^*(\rho,k)]\,d^2\rho\,dk\}\rangle,\end{aligned} \quad (10)$$

where * denotes complex conjugate and the range of integration is over all the allowed values of $\rho$ and $k$. Here $\nu$ and $\nu^*$ are treated as independent functions of $\rho$ and $k$.

It is the purpose of this section to derive a Fokker–Planck equation for the characteristic functional $\Psi$ defined above. Tatarskii[1] derived an equation for the characteristic functional with constant wavenumber $k$. It is noted that we treat in Eq. (10) the wavenumber $k$ as a variable.

Using $s=(\rho,k)$, we write Eq. (10) as

$$\Psi(z,\nu,\nu^*)=\langle\exp\{i\int[u(z,s)\nu(s)+u^*(z,s)\nu^*(s)]\,ds\}\rangle. \quad (10')$$

We differentiate Eq. (10) with respect to $z$ and obtain by Eq. (8)

$$\begin{aligned}\frac{\partial}{\partial z}\Psi(z,\nu,\nu^*)&=\Big\langle\exp(iR_z)i\int\Big[\Big(\frac{-1}{2ik}\Big)[\nabla_\rho^2u(z,s)+\beta(z,\rho)u(z,s)]\nu(s)\\&\quad+\Big(\frac{1}{2ik}\Big)[\nabla_\rho^2u^*(z,s)+\beta(z,\rho)u^*(z,s)]\nu^*(s)\Big]\,ds\Big\rangle. \end{aligned}\quad (11)$$

First we calculate the terms $\langle\exp(iR_z)\nabla_\rho^2u(z,s)\rangle$ and $\langle\exp(iR_z)\nabla_\rho^2u^*(z,s)\rangle$ in Eq. (11). From Eq. (10), we have

$$\frac{\delta\Psi(z,\nu,\nu^*)}{d\nu(s)}=i\langle u(z,s)\exp(iR_z)\rangle \quad (12a)$$

and

$$\frac{\delta\Psi(z,\nu,\nu^*)}{\delta\nu^*(s)}=i\langle u^*(z,s)\exp(iR_z)\rangle. \quad (12b)$$

The operators $\delta/\delta\nu(s)$ and $\delta/\delta\nu^*(s)$ denote functional derivatives.[6,8] Operating $\nabla_\rho^2$ on Eqs. (12a) and (12b), we have respectively

$$\langle\nabla_\rho^2u(z,s)\exp(iR_z)\rangle=\frac{1}{i}\nabla_\rho^2\frac{\delta\Psi(z,\nu,\nu^*)}{\delta\nu(s)} \quad (13a)$$

and

$$\langle\nabla_\rho^2u^*(z,s)\exp(iR_z)\rangle=\frac{1}{i}\nabla_\rho^2\frac{\delta\Psi(z,\nu,\nu^*)}{\delta\nu^*(s)}. \quad (13b)$$

Next we consider the other terms in Eq. (11), namely, $\langle\exp(iR_z)\beta(z,\rho)u(z,s)\rangle$ and $\langle\exp(iR_z)\beta(z,\rho)u^*(z,s)\rangle$. We define

$$g(\nu,\nu^*,z,s)=\langle\exp(iR_z)\beta(z,\rho)\rangle. \quad (14)$$

Expand $\exp(iR_z)$ in power series as follows:

$$\exp(iR_z)=\sum_{m=0}^{\infty}\frac{1}{m!}\{i\int[u(z,s)\nu(s)+u^*(z,s)\nu^*(s)]\,ds\}^m. \quad (15)$$

Then we have

$$\begin{aligned}g(\nu,\nu^*,z,s)&=\sum_{m=0}^{\infty}\frac{i^m}{m!}\langle[\int(u_1\nu_1+u_1^*\nu_1^*)\,ds_1]\\&\quad\times[\int(u_2\nu_2+u_2^*\nu_2^*)\,ds_2]\cdots[\int(u_m\nu_m+u_m^*\nu_m^*)\,ds_m]\beta(z,\rho)\rangle,\end{aligned} \quad (16)$$

where we define $s_i=(\rho_i,k_i)$, $\nu_i=\nu(s_i)$, $u_i=u(z,s_i)$, and etc. for $i=1,2,3,\cdots$. In the expansion of Eq. (16), the existence of moments of all orders is assumed.

Consider now the term in Eq. (16) like $\langle u_1^{\alpha_1}u_2^{\alpha_2}\cdots u_m^{\alpha_m}\beta\rangle$, where $u_i^{\alpha_i}$ denotes either $u_i$ or $u_i^*$. From Eq. (8), we may write $u(z,s)$ as

$$u(z,s)=u(0,s)+\frac{i}{2}\int_0^z\frac{1}{k}\times[\nabla_\rho^2u(z',s)+\beta(z',\rho)u(z',s)]\,dz'. \quad (17)$$

Note that $u(z,s)$ does not depend on $\beta(z',s)$ for $z'>z$. Let $\Delta z$ be an increment in $z$, which is larger than the correlation scale of $\beta(z,\rho)$ in $z$ direction, and write

$$u(z,s)=u(z-\Delta z,s)+\frac{i}{2}\int_{z-\Delta z}^{z}\frac{i}{k}\times[\nabla_\rho^2u(z',s)+\beta(z',\rho)u(z',s)]\,dz', \quad (18)$$

where $u(z-\Delta z, s)$ has no correlation with $\beta(z,\rho)$. Suppose $\Delta z$ is small, and expand $u(z,s)$ as

$$u(z,s)=u(z-\Delta z,s)+\frac{i}{2}\left(\frac{\Delta z}{k}\right)\nabla_\rho^2 u(z-\Delta z,s)+\frac{i}{2}\,\frac{u(z-\Delta z,s)}{k}\int_{z-\Delta z}^{z}\beta(z',\rho)\,dz'+O(\Delta^2 z). \tag{19}$$

Under the Markov approximation, the correlation scale of $\beta(z,\rho)$ in $z$ direction is zero. Therefore, we let $\Delta z \to 0$. We note that

$$\lim_{\Delta z\to 0} u(z-\Delta z,s)=u(z,s) \tag{20}$$

and

$$\langle\beta(z,\rho')\int_{z-\Delta z}^{z}\beta(z',\rho)\,dz'\rangle=A(\rho-\rho'). \tag{21a}$$

For higher moments such as

$$T_i=\langle\beta(z,\rho)\int_{z-\Delta z}^{z}\beta(z_1,\rho_1)\,dz_1\cdots\times\int_{z-\Delta z}^{z}\beta(z_i,\rho_i)\,dz_i\rangle,\quad i\geq 2,$$

we will assume as in the derivation of ordinary Fokker—Planck equation[9]

$$\lim_{\Delta z\to 0}T_i=0,\quad i\geq 2. \tag{21b}$$

This assumption can be satisfied if the random function $\beta(z,\rho)$ has a Gaussian, or normal statistics. However, the assumption made in (21b) is more general and does not require the Gaussian statistics of $\beta(z,\rho)$ in general.

It follows directly from Eqs. (16), (19), (20), (21a), and (21b) that, as $\Delta z\to 0$,

$$\langle u(z,s_1)\beta(z,\rho)\rangle=(i/2k_1)\langle u(z,s_1)\rangle A(\rho-\rho_1)$$

and, in general,

$$\begin{aligned}&\langle(u_1\nu_1+u_1^*\nu_1^*)\cdots(u_m\nu_m+u_m^*\nu_m^*)\beta(z,\rho)\rangle\\&=\sum_{j=1}^{m}A(\rho-\rho_j)\left(\frac{i}{2k_j}\right)\langle(u_1\nu_1+u_1^*\nu_1^*)\cdots\\&\quad\times(u_{j-1}\nu_{j-1}+u_{j-1}^*\nu_{j-1}^*)(u_j\nu_j-u_j^*\nu_j^*)\\&\quad\times(u_{j+1}\nu_{j+1}+u_{j+1}^*\nu_{j+1}^*)\cdots(u_m\nu_m+u_m^*\nu_m^*)\rangle\end{aligned} \tag{22}$$

by noting that $\langle u(z-\Delta z,s)\beta(z,\rho')\rangle=0$. Other than the assumption made in (21b), Eq. (22) is exact under the delta-correlation assumption. But we see that really we only require the existence of an intermediate scale $\Delta z$ which is larger than the coherence scale of $\beta(z,\rho)$ but smaller than the scale of variation of $u$ such that $u(z-\Delta z,s)\approx u(z,s)$. The existence of the intermediate scale and Eq. (21) are the essence of the Fokker-Planck equation.

Substituting Eq. (22) into (16) and noting that all the $s_i$'s are dummy variables, we then have

$$\begin{aligned}&g(\nu,\nu^*,z,s)\\&=\sum_{m=1}^{\infty}\left(\frac{i^m}{m!}\right)\left(\frac{i}{2k_m}\right)(m)\int\cdots\int A(\rho-\rho_m)\langle(u_1\nu_1+u_1^*\nu_1^*)\cdots\\&\quad\times(u_{m-1}\nu_{m-1}+u_{m-1}^*\nu_{m-1}^*)(u_m\nu_m-u_m^*\nu_m^*)\rangle\,ds_1\cdots ds_m.\end{aligned} \tag{23}$$

We can also write Eq. (23) as

$$\begin{aligned}&g(\nu,\nu^*,z,s)\\&=\sum_{m=1}^{\infty}\frac{i^{m-1}}{(m-1)!}\left(-\frac{1}{2}\right)\int\cdots\int\frac{1}{k'}A(\rho-\rho')\langle(u'\nu'-u^{*\prime}\nu^{*\prime})\\&\quad\times(u_1\nu_1+u_1^*\nu_1^*)\cdots(u_{m-1}\nu_{m-1}+u_{m-1}^*\nu_{m-1}^*)\rangle\,ds'ds_1\cdots ds_{m-1}.\end{aligned} \tag{24}$$

where $s'=(\rho',k')$, $u'=u(z,s')$, and $\nu'=\nu(s')$.

Setting $m-1=n$, we have

$$\begin{aligned}&g(\nu,\nu^*,z,s)\\&=\sum_{n=0}^{\infty}\frac{(i)^n}{n!}\left(\frac{-1}{2}\right)\int\cdots\int\frac{i}{k'}A(\rho-\rho')\langle(u'\nu'-u^{*\prime}\nu^{*\prime})\\&\quad\times(u_1\nu_1+u_1^*\nu_1^*)\cdots(u_n\nu_n+u_n^*\nu_n^*)\rangle\,ds'ds_1\cdots ds_n.\end{aligned} \tag{25}$$

From Eq. (25) it is easy to show

$$\begin{aligned}g(\nu,\nu^*,z,s)=\left(\frac{-1}{2}\right)\int\frac{ds'}{k'}A(\rho-\rho')[&\nu(s')\langle u(z,s')\exp(iR_z)\rangle\\&-\nu^*(s')\langle u^*(z,s')\exp(iR_z)\rangle].\end{aligned} \tag{25'}$$

By Eqs. (12a) and (12b), we write $g$ as

$$\begin{aligned}&g(\nu,\nu^*,z,s)\\&=\left(\frac{i}{2}\right)\int\frac{ds'}{k'}A(\rho-\rho')\left(\nu(s')\frac{\delta\Psi}{\delta\nu(s')}-\nu^*(s')\frac{\delta\Psi}{\delta\nu^*(s')}\right).\end{aligned} \tag{26}$$

Define the operator $\hat{M}(s)$ as

$$\hat{M}(s)=\nu(s)\frac{\delta}{\delta\nu(s)}-\nu^*(s)\frac{\delta}{\delta\nu^*(s)}. \tag{27}$$

We then have

$$g(\nu,\nu^*,z,s)=\left(\frac{i}{2}\right)\int\frac{ds'}{k'}A(\rho-\rho')\hat{M}(s')\Psi(z,\nu,\nu^*). \tag{28}$$

We also note that

$$\langle\beta(z,\rho)u(z,s)\exp(iR_z)\rangle=\frac{1}{i}\,\frac{\delta g(\nu,\nu^*,z,s)}{\delta\nu(s)} \tag{29a}$$

and

$$\langle\beta(z,\rho)u^*(z,s)\exp(iR_z)\rangle=\frac{1}{i}\,\frac{\delta g(\nu,\nu^*,z,s)}{\delta\nu^*(s)}. \tag{29b}$$

By Eqs. (11), (13a), (13b), (28), (29a), and (29b), we obtain

$$\begin{aligned}\frac{\delta\Psi(z,\nu,\nu^*)}{\delta z}=&\left(\frac{i}{2}\right)\int\frac{ds}{k}\left(\nu(s)\nabla_\rho^2\frac{\delta\Psi}{\delta\nu(s)}-\nu^*(s)\nabla_\rho^2\frac{\delta\Psi}{\delta\nu^*(s)}\right)\\&-\frac{1}{4}\iint\frac{ds\,ds'}{kk'}A(\rho-\rho')\hat{M}(s)\hat{M}(s')\Psi.\end{aligned} \tag{30}$$

This is the Fokker—Planck equation for the characteristic functional $\Psi$ of the random electromagnetic field $u(z,\rho,k)$. Since the characteristic functional is the Fourier transform of the probability functional, Eq. (30) is in fact the Fourier transform of the Fokker-Planck equation. Our technique used here can also be applied to the derivation of the Fokker-Planck equation for the ordinary characteristic function of a random function $x(t)$.

## III. MOMENT EQUATIONS

We want to derive a complete set of moment equations in this section. First, we expand $\Psi(z,\nu,\nu^*)$ as a power series

$$\Psi(z,\nu,\nu^*)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^{m+n}}{m!n!}\left(\int u(z,s)\nu(z,s)\,ds\right)^m \times\left(\int u^*(z,s')\nu^*(z,s')\,ds'\right)^n$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^{m+n}}{m!n!}K_{m,n}(z,\nu,\nu^*), \tag{31}$$

where

$$K_{m,n}(z,\nu,\nu^*)=\int\cdots\int\Gamma_{m,n}(z,s_1,\cdots,s_m;s_1',\cdots,s_n') \times\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime}\,ds_1\cdots ds_m ds_1'\cdots ds_n'$$

and

$$\Gamma_{m,n}(z,s_1,\cdots,s_m;s_1',\cdots,s_n')=\langle u_1u_2\cdots u_m u_1^{*\prime}\cdots u_n^{*\prime}\rangle. \tag{32}$$

$\Gamma_{m,n}$ is the $m$-$n$th moment of the random field $u(z,s)$. The object of this section is to derive a differential equation satisfied by $\Gamma_{m,n}$.

We note that, for any function $f(s)$ of $s$, we have

$$\int f(s)\nu(s)\frac{\delta}{\delta\nu(s)}K_{m,n}(z,\nu,\nu^*)\,ds =\int\cdots\int\sum_{i=1}^{m}\Gamma_{m,n}(z,s,\cdots,s_m;s_1',\cdots,s_n')f(s_i) \times\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime}\,ds_1\cdots ds_m ds_1'\cdots ds_n' \tag{33a}$$

and

$$\int f(s)\nu^*(s)\frac{\delta}{\delta\nu^*(s)}K_{m,n}(z,\nu,\nu^*)\,ds =\int\cdots\int\sum_{i=1}^{n}\Gamma_{m,n}(z,s_1\cdots s_m;s_1'\cdots s_n')f(s_i') \times\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime}\,ds_1\cdots ds_m ds_1'\cdots ds_n'. \tag{33b}$$

From (33a) and (33b) we obtain

$$-\frac{1}{4}\int\int\frac{ds\,ds'}{kk'}A(\rho-\rho')\hat{M}(s)\hat{M}(s')K_{m,n}(z,\nu,\nu^*) =-\frac{1}{4}\int\cdots\int\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{A(\rho_i-\rho_j)}{k_ik_j} -\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{[A(\rho_i-\rho_j')+A(\rho_j'-\rho_i)]}{k_ik_j'} +\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{A(\rho_i'-\rho_j')}{k_i'k_j'}\Bigg)\Gamma_{m,n}\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime} \times ds_1\cdots ds_m ds_1'\cdots ds_n'. \tag{34}$$

We also note that

$$\int\nu(s)\nabla_\rho^2\frac{\delta K_{m,n}}{\delta\nu(s)}\,ds =\int\cdots\int(\nabla_1^2+\nabla_2^2+\cdots+\nabla_m^2)\Gamma_{m,n}\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime} \times ds_1\cdots ds_m ds_1'\cdots ds_n' \tag{35a}$$

and

$$\int\nu^*(s)\nabla_\rho^2\frac{\delta K_{m,n}}{\delta\nu^*(s)}\,ds =\int\cdots\int(\nabla_1'^2+\nabla_2'^2+\cdots+\nabla_n'^2)\Gamma_{m,n}\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime} \times ds_1\cdots ds_m ds_1'\cdots ds_n', \tag{35b}$$

where $\nabla_j^2=\nabla_{\rho j}^2$ and $\nabla_j'^2=\nabla_{\rho j'}^2$.

By Eqs. (31), (34), (35a), and (35b), we can write Eq. (30) as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^{m+n}}{m!n!}\int\cdots\int\Bigg[\frac{\partial\Gamma_{m,n}}{\partial z}-\frac{i}{2}\left(\frac{\nabla_1^2}{k_1}+\cdots+\frac{\nabla_m^2}{k_m}-\frac{\nabla_1'^2}{k_1'}-\cdots-\frac{\nabla_n'^2}{k_n'}\right)\Gamma_{m,n}+\frac{1}{4}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{A(\rho_i-\rho_j)}{k_ik_j} -\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(A(\rho_i-\rho_j')+A(\rho_j'-\rho_i))}{k_ik_j'}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{A(\rho_i'-\rho_j')}{k_i'k_j'}\Bigg)\Gamma_{m,n}\Bigg]\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime}ds_1\cdots ds_m ds_1'\cdots ds_n'=0. \tag{36}$$

Since $\nu(s)$ and $\nu^*(s)$ are arbitrarily defined, the quantity inside the bracket in Eq. (36) must be zero. We have then the following differential equation for the moment function $\Gamma_{m,n}$:

$$\frac{\partial\Gamma_{m,n}}{\partial z}(z,s_1\cdots s_m,s_1'\cdots s_n') =\frac{i}{2}\left[\frac{\nabla_1^2}{k_1}+\cdots+\frac{\nabla_m^2}{k_m}-\frac{\nabla_1'^2}{k_1'}-\cdots-\frac{\nabla_n'^2}{k_n'}\right]\Gamma_{m,n} -\frac{1}{4}\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{A(\rho_i-\rho_j)}{k_ik_j}-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{[A(\rho_i-\rho_j')+A(\rho_j'-\rho_i)]}{k_ik_j'} +\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{A(\rho_i'-\rho_j')}{k_i'k_j'}\right)\Gamma_{m,n}. \tag{37}$$

It is noted that we can also derive the moment equation (37) directly from the wave equation (8), using the same technique in obtaining Eq. (22). Equation (37) thus gives us a complete set of the moment equations of the random wave field with different transverse coordinates and *different* wavenumbers.

## IV. APPLICATIONS

First we note that we have derived a complete set of the moment equations with different transverse coordinates and *different* wavenumbers for the high-frequency waves propagating in a plasma medium. However, we can easily extend the argument to the other cases when the index of refraction $\epsilon_\omega(\mathbf{r})$ has a different frequency dependence.

Next we consider some applications.

### A. Identical wavenumbers

When all the wavenumbers are identical, Eq. (37) becomes

$$\frac{\partial \Gamma_{m,n}}{\partial z}(z, \rho_1 \cdots \rho_m, \rho_1' \cdots \rho_n')$$

$$= \frac{i}{2k}(\nabla_1^2 + \cdots + \nabla_m^2 - \nabla_1'^2 - \cdots - \nabla_n'^2)\Gamma_{m,n} - \frac{1}{4k^2}$$

$$\times\Big(\sum_{i=1}^{m}\sum_{j=1}^{n} A(\rho_i - \rho_j)$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}[A(\rho_i - \rho_j') + A(\rho_j' - \rho_i)]$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} A(\rho_i' - \rho_j')\Big)\Gamma_{m,n}, \tag{38}$$

which is identical to that obtained by Tatarskii.[2] However, the derivation by Tatarskii requires that the refraction index fluctuations possess Gaussian statistics while we do not require the assumption of Gaussian statistics in our derivation in general.

### B. $\Gamma_{1,1}\ (z, s_1, s_2)$

When $m=1$, and $n=1$, Eq. (37) gives

$$\frac{\partial \Gamma_{1,1}}{\partial z}(z, \rho_1, k_1, \rho_2, k_2)$$

$$= \frac{i}{2}\left(\frac{\nabla_1^2}{k_1} - \frac{\nabla_2^2}{k_2}\right)\Gamma_{1,1} - \frac{1}{4}\left[\left(\frac{1}{k_1^2} + \frac{1}{k_2^2}\right)A(0)\right.$$

$$\left. - \frac{2A(\rho_1 - \rho_2)}{k_1 k_2}\right]\Gamma_{1,1}, \tag{39}$$

where $\Gamma_{1,1}(z, \rho_1, k_1, \rho_2, k_2) = \langle u(z, \rho_1, k_1)u^*(z, \rho_2, k_2)\rangle$.

Equation (39) can be used to calculate the mean intensity profile $\langle I(\mathbf{r}, t)\rangle$ at position $\mathbf{r}$. Consider the random wave observed by a detector with a bandwidth function $f_B(k)$. Then we have the total observed wave amplitude $h(z, \rho, t)$ at position $z, \rho$ and time $t$

$$h(z, \rho, t) = \int_{-\infty}^{\infty} u(z, \rho, k) f_B(k) \exp\{i[kz - \omega(k)t]\}\, dk. \tag{40}$$

The average intensity profile is then

$$\langle I(\mathbf{r}, t)\rangle = \langle h(z, \rho, t) h^*(z, \rho, t)\rangle$$

$$= \int\int_{-\infty}^{\infty} \langle u(z, \rho, k_1)u^*(z, \rho, k_2)\rangle f_B(k_1) f_B(k_2)$$

$$\times \exp\{i[k_1 z - \omega(k_1)t]\}\exp\{-i[k_2 z - \omega(k_2)t]\}\, dk_1 dk_2. \tag{41}$$

Thus $\Gamma_{1,1}$ is related to the average intensity profile $\langle I(\mathbf{r}, t)\rangle$ by Eq. (41). Equations (39) and (41) have been applied to calculate the pulse profile of pulsar in interstellar scintillation. The details will be given in a later paper.

### C. $\Gamma_{2,2}$

When $m=2$, and $n=2$, Eq. (39) becomes

$$\frac{\partial}{\partial z}\Gamma_{2,2}(z, s_1, s_2, s_3, s_4)$$

$$= \frac{i}{2}\left(\frac{\nabla_1^2}{k_1} + \frac{\nabla_2^2}{k_2} - \frac{\nabla_3^2}{k_3} - \frac{\nabla_4^2}{k_4}\right)\Gamma_{2,2}$$

$$- \frac{1}{4}\left[\left(\frac{1}{k_1^2} + \frac{1}{k_2^2} + \frac{1}{k_3^2} + \frac{1}{k_4^2}\right)A(0) + 2\frac{A(\rho_1 - \rho_2)}{k_1 k_2}\right.$$

$$+ 2\frac{A(\rho_3 - \rho_4)}{k_3 k_4} - 2\frac{A(\rho_1 - \rho_3)}{k_1 k_3} - 2\frac{A(\rho_1 - \rho_4)}{k_1 k_4}$$

$$\left. - 2\frac{A(\rho_2 - \rho_3)}{k_2 k_3} - 2\frac{A(\rho_2 - \rho_4)}{k_2 k_4}\right]\Gamma_{2,2}, \tag{42}$$

where

$$\Gamma_{2,2}(z, s_1, s_2, s_3, s_4) = \langle u(z, s_1)u(z, s_2)u^*(z, s_3)u^*(z, s_4)\rangle. \tag{43}$$

If one sets $s_3 = s_1$, $s_4 = s_2$, and $\rho_1 = \rho_2$, then

$$\Gamma_{2,2}(z, s_1, s_2, s_3, s_4)$$

$$= \langle |u(z, \rho_1, k_1)|^2 |u(z, \rho_1, k_2)|^2\rangle = \langle I(z, \rho_1, k_1),$$

$$I(z, \rho_1, k_2)\rangle = P_I(k_1 - k_2). \tag{44}$$

Here $I$ is the intensity and $P_I$ is the correlation function of intensity at different frequencies. Thus $\Gamma_{2,2}$ gives in this special case the intensity correlation function $P_I(k_1 - k_2)$ at a given observation point with different wavenumbers. The intensity correlation function has been measured in interstellar scintillations,[3–5,10] and Eq. (42) provides a theoretical base of interpretation.

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