Two-Dimensional Quarter Space Problems in One-Speed Transport Theory*

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Methods derived from the theory of several complex variables are used as a means of analyzing a class of two-dimensional transport problems in a scattering and absorbing quarter space ($0 \leq x_1, 0 \leq x_2, -\infty \leq x_3 \leq \infty$) described by a linear, one-speed Boltzmann equation. Using Fourier transformation and the Bochner decomposition, the multivariable analog of the Wiener–Hopf factorization, we find the Green's function in transform space, which solves all source problems having a solution bounded at infinity. The transform of the density asymptotically far from the corner ($x_1 = x_2 = 0$) is determined explicitly, while the remainder is given in terms of the solution to a pair of Fredholm equations.

1. INTRODUCTION

For the past forty years, the Wiener–Hopf technique[1] has proven to be a powerful tool in the analysis of integral equations over the half-line with a difference kernel. For that reason, one of its many applications has been to one-speed, linear transport in a half-space.[2] The method is based on Fourier transformation and relies heavily upon the theory of functions of a complex variable.

In this paper, we use a similar approach generalized to two complex variables to study two-dimensional transport in a quarter space. Here the basic integral equation is over a quarter plane, with the kernel depending upon distance in the plane. In Sec. 2, double Fourier transformation of the transport equation yields a two-variable Wiener–Hopf problem for four unknown transforms corresponding to the densities in each quarter space. A similar mathematical problem arises in the theory of electromagnetic wave diffraction from a right angle dielectric wedge. Although an exact solution to the diffraction problem is not yet available,[3] the Bochner decomposition,[4] the multivariable analog of the Wiener–Hopf factorization, was used in one of the analyses[5] and is found to be a useful tool for our analysis as discussed in Sec. 3. In Sec. 4, the asymptotic contribution to the transforms is found explicitly by one-dimensional Wiener–Hopf analyses and is then subtracted, yielding an equation for a new set of four unknown functions representing the transforms of "transient" densities which are nonnegligible only near the corner. The properties of this new equation allow, by subsequent manipulations in Sec. 5, the solution to be expressed in terms of the solution to a pair of Fredholm equations derived in Sec. 6. Analogous to the one-dimensional problem for a finite slab,[6] these Fredholm equations appear to represent the interaction of the "transient" densities in the two quarter spaces adjacent to the scattering and absorbing quarter space. It is shown that this pair of equations may be solved by iteration.

2. FOURIER TRANSFORMATION OF THE TRANSPORT EQUATION

We consider one-speed neutron transport in a quarter space (Q), $0 \leq x_1$, $0 \leq x_2$, $-\infty \leq x_3 \leq \infty$, with isotropic scattering and a given source distribution $S(\mathbf{r}) = S(x_1, x_2)$. The integral transport equation for the neutron density $\rho(\mathbf{r})$ is

$$\rho(\mathbf{r}) = c\int_Q \frac{e^{-|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|^2}\left[\rho(\mathbf{r}') + \frac{S(\mathbf{r}')}{c}\right] d\mathbf{r}', \quad (2.1)$$

where distances are in units of a mean free path, $\mathbf{r}$ and $\mathbf{r}'$ are three-dimensional vectors, and c is the mean number of neutrons emitted per collision. Letting $\rho(\mathbf{r}) = \rho(x_1, x_2)$, $S(\mathbf{r}) = S(x_1, x_2)$, and performing the x_3 integration, the transport equation (2.1) becomes

$$\rho(x_1, x_2) = c\int_0^\infty\int_0^\infty dx_1'\, dx_2' K(|\mathbf{x}-\mathbf{x}'|)\left(\rho(x_1', x_2') + \frac{S(x_1', x_2')}{c}\right), \quad (2.2)$$

where $|\mathbf{x}-\mathbf{x}'|^2 = (x_1 - x_1')^2 + (x_2 - x_2')^2$ and the kernel K has the integral representation

$$K(s) = \frac{1}{2\pi s}\int_1^\infty \frac{e^{-ts}\, dt}{t(t^2-1)^{\frac{1}{2}}}. \quad (2.3)$$

For convenience, we now consider the integral equation for $\varphi(x_1, x_2)$, $-\infty \leq x_1, x_2 < \infty$, with a specific inhomogeneous term as follows:

$$\varphi(x_1, x_2) = c\int_0^\infty\int_0^\infty dx_1'\, dx_2' K(|\mathbf{x}-\mathbf{x}'|)\varphi(x_1', x_2') + \begin{cases}\exp(-a_1x_1 - a_2x_2), & x_1, x_2 \geq 0 \\ 0, & \text{otherwise}\end{cases}. \quad (2.4)$$

If φ is determined for arbitrary a_1 and a_2 in a strip in the complex plane, then φ plays the role of a Green's function in transform space. Specifically, if in (2.2) we let $V(x_1, x_2)$ denote the inhomogeneous term

$$V(x_1, x_2) = \int_0^\infty \int_0^\infty dx_1' \, dx_2' K(|\mathbf{x} - \mathbf{x}'|) S(x_1', x_2')$$

and $\tilde{V}(a_1, a_2)$ be the double Laplace transform of $V(x_1, x_2)$,

$$\tilde{V}(a_1, a_2) = \int_0^\infty \int_0^\infty dx_1 \, dx_2 \exp(-a_1 x_1 - a_2 x_2) V(x_1, x_2),$$

then we assert without proof that the solution to (2.2) may be given in terms of $\tilde{V}$ and φ as follows:

$$\rho(x_1, x_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} \int_{\Gamma_2} \varphi(x_1, x_2; a_1, a_2) \times \tilde{V}(-a_1, -a_2) \, da_1 \, da_2,$$

where Γ_1 and Γ_2 are vertical contours to the left of all the singularities of $\tilde{V}(-a_1, -a_2)$ and to the right of all singularities of $\varphi(x_1, x_2; a_1, a_2) \equiv \varphi(x_1, x_2)$.

We now use a simple device to transform the integral over the quarter plane to an integral over the whole (x_1, x_2) plane so that we may make use of the convolution theorem of Fourier transforms. Let

$$\varphi_i(x_1, x_2) = \varphi(x_1, x_2) \chi_i(x_1, x_2), \qquad (2.5)$$

where χ_i is the characteristic function of the ith quadrant:

$$\chi_1(x_1, x_2) = \begin{cases} 1, & 0 \le x_1 \le \infty, \quad 0 \le x_2 \le \infty \\ 0, & \text{elsewhere} \end{cases};$$

$$\chi_2(x_1, x_2) = \begin{cases} 1, & -\infty \le x_1 < 0, \quad 0 \le x_2 \le \infty \\ 0, & \text{elsewhere} \end{cases};$$

$$\chi_3(x_1, x_2) = \begin{cases} 1, & -\infty \le x_1 < 0, \quad -\infty \le x_2 < 0 \\ 0, & \text{elsewhere} \end{cases};$$

$$\chi_4(x_1, x_2) = \begin{cases} 1, & 0 \le x_1 \le \infty, \quad -\infty \le x_2 < 0 \\ 0, & \text{elsewhere} \end{cases}.$$

In terms of the φ_i, Eq. (2.2) may be rewritten as

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = c \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_1' \, dx_2' K(|\mathbf{x} - \mathbf{x}'|) \varphi_1(x_1', x_2') + \chi_1(x_1, x_2) \exp(-a_1 x_1 - a_2 x_2). \qquad (2.6)$$

Taking the double Fourier transform of the above equation, we find

$$\{1 - c\hat{K}[(k_1^2 + k_2^2)^{\frac{1}{2}}]\} \Phi_1(k_1, k_2) = -\Phi_2(k_1, k_2) - \Phi_3(k_1, k_2) - \Phi_4(k_1, k_2) + 1/(a_1 - ik_1)(a_2 - ik_2), \qquad (2.7)$$

where the Φ_i are the double transforms of the φ_i:

$$\Phi_i(k_1, k_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_1 \, dx_2 \times \exp(ik_1 x_1 + ik_2 x_2) \varphi_i(x_1, x_2), \quad i = 1, \cdots, 4, \qquad (2.8)$$

and

$$\hat{K}(k) = (\tan^{-1} k)/k. \qquad (2.9)$$

With $c < 1$ and $\text{Re}\,(a_1, a_2) \ge 0$, $\varphi_1(x_1, x_2)$ will be a bounded function of x_1 and x_2. From (2.3) and (2.4), the Φ_i will be analytic in the following sets of half-planes:

$$\Phi_1: \{\text{Im}\, k_1 > 0, \text{Im}\, k_2 > 0\},$$
$$\Phi_2: \{\text{Im}\, k_1 < 1, \text{Im}\, k_2 > 0\},$$
$$\Phi_3: \{\text{Im}\, k_1 < 1, \text{Im}\, k_2 < 0\} \cup \{\text{Im}\, k_1 < 0, \text{Im}\, k_2 < 1\} \cup \{(\text{Im}\, k_1)^2 + (\text{Im}\, k_2)^2 < 1\},$$
$$\Phi_4: \{\text{Im}\, k_1 > 0, \text{Im}\, k_2 < 1\}.$$

This is illustrated in Fig. 1. Note that all four of the Φ_i have a common tube of analyticity:

$$T_\Phi = \{\text{Im}\, k_1 > 0, \text{Im}\, k_2 > 0\} \cap \{(\text{Im}\, k_1)^2 + (\text{Im}\, k_2)^2 < 1\}. \qquad (2.10)$$

The goal is to determine $\Phi_1(k_1, k_2)$, which by inverse Fourier transformation gives the neutron density in the quarter space. What we have, then, is

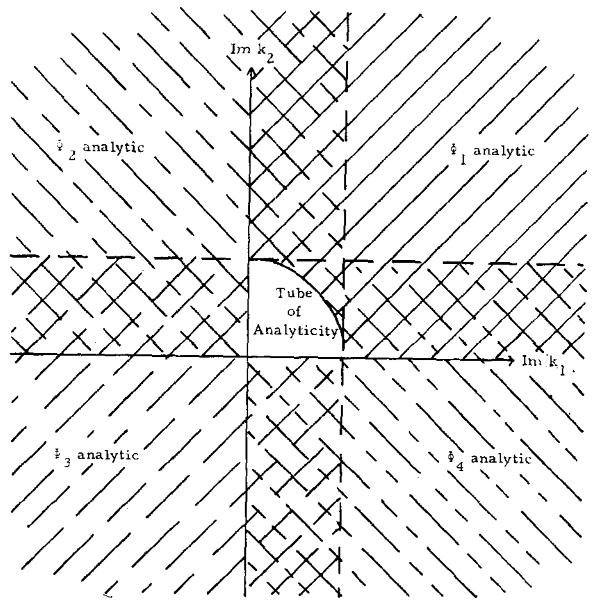

FIG. 1. Domains of analyticity of the $\Phi_i(k_1, k_2)$ with Re a_1, Re $a_2 \ge 0$.

the analogous Wiener–Hopf problem for two complex variables. Instead of two unknown functions which are analytic in a common strip and each analytic in opposite half-planes, we have four functions $\{\Phi_i, i = 1, 2, 3, 4\}$, which are analytic in the common tube T_Φ, each being analytic in a respective set of half-planes. Unfortunately, the factorization of the function $1 - c\hat{K}$, although possible and very useful, as will be shown, does not seem to yield a closed form solution, as it does in the one-dimensional problem.

3. BOCHNER DECOMPOSITION OF $1 - c\hat{K}$

In the analogous one-dimensional problem, one factors the function

$$\Lambda(k) = 1 - c\hat{K}(k)$$

into a product of two functions $H^+(k)$ and $H^-(k)$, respectively analytic in the upper and lower k plane with a common strip of analyticity corresponding to the domains of analyticity of the unknowns Φ^+ and Φ^-. This is the Wiener–Hopf factorization, which, of course, is the key to the Wiener–Hopf technique. Subsequent manipulations and application of the Liouville theorem yield the closed-form solution to the one-dimensional problem.

In the present two-dimensional problem, we intend to make use of the factorization of

$$\Lambda[(k_1^2 + k_2^2)^{\frac{1}{2}}] \equiv 1 - cK((k_1^2 + k_2^2)^{\frac{1}{2}}) \qquad (3.1)$$

into a product of four functions:

$$\Lambda = H_1 H_2 H_3 H_4, \qquad (3.2)$$

where, as we shall see, the H_i have the following regions of analyticity:

$$\begin{aligned} &H_1: \{\mathrm{Im}\, k_1 \geq 0, \mathrm{Im}\, k_2 \geq 0\} \cup T_\Lambda, \\ &H_2: \{\mathrm{Im}\, k_1 \leq 0, \mathrm{Im}\, k_2 \geq 0\} \cup T_\Lambda, \\ &H_3: \{\mathrm{Im}\, k_1 \leq 0, \mathrm{Im}\, k_2 \leq 0\} \cup T_\Lambda, \\ &H_4: \{\mathrm{Im}\, k_1 \geq 0, \mathrm{Im}\, k_2 \leq 0\} \cup T_\Lambda, \end{aligned} \qquad (3.3\text{a})$$

where T_Λ is the tube

$$T_\Lambda = \{(\mathrm{Im}\, k_1)^2 + (\mathrm{Im}\, k_2)^2 < \varkappa_0^2\}, \qquad (3.3\text{b})$$

and $\varkappa_0$ satisfies

$$1 - (c/\varkappa_0) \tanh^{-1} \varkappa_0 = 0. \qquad (3.4)$$

The conditions for the existence of such a factorization and the method of calculation are given in a theorem of Bochner.[4] Let $f(k_1, k_2)$ be analytic and of bounded L_2 norm in a tube T, $\beta_j \leq \mathrm{Im}\, k_j \leq \alpha_j$. The L_2 norm of f is defined by

$$\|f\|_2 = \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(\eta_1 + i\xi_1, \eta_2 + i\xi_2)|^2 \, d\eta_1 \, d\eta_2\right)^{\frac{1}{2}}, \qquad (3.5)$$

where the integration is confined to the tube T. According to Bochner, this function is uniquely decomposable (up to additive constants) into a sum of four functions, $f = f_1 + f_2 + f_3 + f_4$, each of which is analytic and bounded in respective radial tubular domains:

$$\begin{aligned} &f_1: \{\mathrm{Im}\, k_1 > \beta_1, \mathrm{Im}\, k_2 > \beta_2\}, \\ &f_2: \{\mathrm{Im}\, k_1 < \alpha_1, \mathrm{Im}\, k_2 > \beta_2\}, \\ &f_3: \{\mathrm{Im}\, k_1 < \alpha_1, \mathrm{Im}\, k_2 < \alpha_2\}, \\ &f_4: \{\mathrm{Im}\, k_1 > \beta_1, \mathrm{Im}\, k_2 < \alpha_2\}. \end{aligned}$$

The f_i may be given in terms of Cauchy integrals. Letting $[f]_{\sigma_i}\pm$ denote the following integrals of f,

$$[f]_{\sigma_1}\pm = \frac{1}{2\pi i}\int_{\Gamma_1\pm} \frac{f(z_1, k_2)\, dz_1}{z_1 - k_1}, \qquad (3.6)$$

$$[f]_{\sigma_2}\pm = \frac{1}{2\pi i}\int_{\Gamma_2\pm} \frac{f(k_1, z_2)\, dz_2}{z_2 - k_2}, \qquad (3.7)$$

where the contours $\Gamma_j^\pm$ are depicted in Fig. 2, we find that the f_j are given by

$$f_1 = [f]_{\sigma_1^+\sigma_2^+}, \qquad (3.8\text{a})$$

$$f_2 = [f]_{\sigma_1^-\sigma_2^+}, \qquad (3.8\text{b})$$

$$f_3 = [f]_{\sigma_1^-\sigma_2^-}, \qquad (3.8\text{c})$$

$$f_4 = [f]_{\sigma_1^+\sigma_2^-}. \qquad (3.8\text{d})$$

To obtain the product decomposition of $\Lambda = 1 - c\hat{K}$, one must first take the logarithm and determine the additive decomposition of $\ln(\Lambda)$. The desired result is then obtained by exponentiation. For convenience, however, we shall reduce the decomposition problem for Λ to one that has already been considered by Kraut in his analysis of an elastic wave propagation problem.[7]

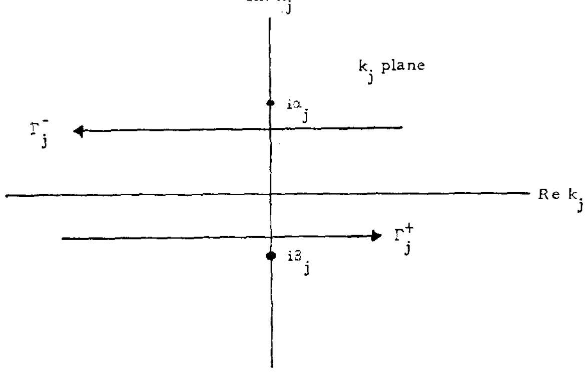

FIG. 2. Integration paths in the tubular domain T.

We have the following integral representation of $\Lambda(k)$:

$$\Lambda(k) = \frac{1-c}{\varkappa_0^2}\frac{k^2+\varkappa_0^2}{k^2+1}\exp\left[\frac{2k^2}{\pi}\int_1^\infty \frac{\theta(1/t)\,dt}{(t^2+k^2)t}\right],$$
$$k^2 = k_1^2 + k_2^2, \quad (3.9)$$

where θ is given by

$$\theta(\sigma) = \tan^{-1}(c\pi\sigma/2\lambda(\sigma)), \quad \theta(0) = 0, \quad (3.10)$$

$$\lambda(\sigma) = 1 - c\sigma\tanh^{-1}\sigma. \quad (3.11)$$

This is a simple variation of a representation well known in one-dimensional transport theory.[8] A more convenient expression is given by

$$\Lambda(k) = \exp\left(-\frac{2}{\pi}\int_1^\infty \frac{\theta(1/t)t\,dt}{(t^2+k^2)} - 2\int_{\varkappa_0}^1 \frac{t\,dt}{t^2+k^2}\right). \quad (3.12)$$

Thus, to achieve the product decomposition of Λ, we see by the above integral representation that we need only additively decompose the function

$$V(k_1, k_2; t) = 1(/t^2 + k_1^2 + k_2^2). \quad (3.13)$$

Since t is real and $\geq \varkappa_0$, V is analytic in the tube T_Λ [Eq. (3.3b)]. It is an easy matter to verify that V also has bounded L_2 norm in T_Λ. By a simple modification of a calculation by Kraut,[7] we obtain the following decomposition:

$$1/(t^2 + k_1^2 + k_2^2) = V_1(k_1, k_2; t) + V_2(k_1, k_2; t) + V_3(k_1, k_2; t) + V_4(k_1, k_2; t), \quad (3.14)$$

with

$$V_1(k_1, k_2; t) = \frac{1}{t^2 + k_1^2 + k_2^2} \times \left[\frac{1}{4} - \frac{k_1}{2\pi(k_2^2+t^2)^{\frac{1}{2}}}\ln\left(\frac{k_2 + (k_2^2+t^2)^{\frac{1}{2}}}{it}\right) - \frac{k_2}{2\pi(k_1^2+t^2)^{\frac{1}{2}}}\ln\left(\frac{k_1 + (k_1^2+t^2)^{\frac{1}{2}}}{it}\right)\right]; \quad (3.15)$$

$$V_2(k_1, k_2; t) = V_1(-k_1, k_2; t), \quad (3.16)$$

$$V_3(k_1, k_2; t) = V_1(-k_1, -k_2; t), \quad (3.17)$$

$$V_4(k_1, k_2; t) = V_1(k_1, -k_2; t). \quad (3.18)$$

In the above, the principle branch of the logarithm is to be taken, and we will arbitrarily choose the branches of the radicals so that $(k_i^2)^{\frac{1}{2}} = +k_i$. Thus, the H_i, which were defined in the factorization of $\Lambda(k)$ [Eq. (3.2)], are given explicitly by

$$H_i(k_1, k_2) = \exp\left(-\frac{2}{\pi}\int_1^\infty \theta\left(\frac{1}{t}\right)V_i(k_1, k_2; t)t\,dt - 2\int_{\varkappa_0}^1 V_i(k_1, k_2; t)t\,dt\right). \quad (3.19)$$

One can now verify without much difficulty that the H_i, as given above, are analytic in their respective domains as per (3.3).

4. SUBTRACTION OF THE ASYMPTOTIC SOLUTION

With the definition (3.1), Eq. (2.7) becomes

$$\Lambda(k_1, k_2)\Phi_1(k_1, k_2) = -\Phi_2(k_1, k_2) - \Phi_3(k_1, k_2) - \Phi_4(k_1, k_2) + 1/(a_1 - ik_1)(a_2 - ik_2), \quad (4.1)$$

where, here and in the following, $\Lambda(k_1, k_2) = \Lambda[(k_1^2 + k_2^2)^{\frac{1}{2}}]$. Using a Bochner decomposition on the term $\Lambda\Phi_1$,[9] we can derive a set of integral equations relating Φ_1, Φ_2, Φ_3, and Φ_4. The decomposition yields

$$\Lambda\Phi_1 = [\Lambda\Phi_1]_1 + [\Lambda\Phi_1]_2 + [\Lambda\Phi_1]_3 + [\Lambda\Phi_1]_4, \quad (4.2)$$

where the operations $[\ ]_i$, $i = 1, \cdots, 4$, are defined by Eqs. (3.6)–(3.8), with the corresponding contours in Fig. 2 confined to the tube T_Φ. The uniqueness of this decomposition allows us to equate terms on the rhs of (4.2) to corresponding terms on the rhs of (4.1) as follows:

$$[\Lambda\Phi_1]_1 = 1/(a_1 - ik_1)(a_2 - ik_2), \quad (4.3)$$

$$[\Lambda\Phi_1]_j = -\Phi_j, \quad j = 2, 3, 4. \quad (4.4)$$

If in (4.3) [with $\mathrm{Re}\,(a_1, a_2) > 0$] the integration contours are taken to be the real axes and k_1 and k_2 approach these contours (from above), we obtain a singular integral equation for $\Phi_1(k_1, k_2)$ on the real axes:

$$\tfrac{1}{4}\Lambda(k_1, k_2)\Phi_1(k_1, k_2) + \frac{1}{4\pi i}\int_{-\infty}^{+\infty}\frac{\Lambda(z_1, k_2)\Phi_1(z_1, k_2)\,dz_1}{z_1 - k_1} + \frac{1}{4\pi i}\int_{-\infty}^{+\infty}\frac{\Lambda(k_1, z_2)\Phi_1(k_1, z_2)\,dz_2}{z_2 - k_2} - \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\Lambda(z_1, z_2)\Phi_1(z_1, z_2)\,dz_1\,dz_2}{(z_1 - k_1)(z_2 - k_2)} = \frac{1}{(a_1 - ik_1)(a_2 - ik_2)}, \quad (4.5)$$

where the integrals are computed as principal values. If one could solve the above equation for Φ_1, then Φ_2, Φ_3, and Φ_4 would follow from (4.4).

In a study of diffraction of electromagnetic waves from a quarter space, Kraut and Lehman[3] encounter similar mathematical problems. They derive an equation analogous to (4.5) and prove that the solution may be obtained by iteration if a certain parameter is less than unity. If this parameter is close to 1, the calculated convergence rate is slow. We could do the same here for $c < 1$. Instead, we will derive an iterative scheme which takes advantage of the analyticity of the various functions and which yields a convergence rate which is relatively fast regardless of the value of c. The transform of the asymptotic flux distribution (away from the corner) is the zeroth-order term in this scheme. Higher-order terms produce a significant correction only very near the corner.

To begin the analysis of (4.1), we redefine the problem in terms of a new set of unknowns $\hat{\Phi}_i$, $i = 1, \cdots, 4$, which represent differences between the Φ_i and the transforms of the asymptotic distributions. The asymptotic distributions are derived in Appendix A by assuming, for example, that if x_2 is large and positive, then, under certain conditions on (a_1, a_2), the purely absorbing quarter space $(x_2 < 0,\ x_1 > 0)$ can be replaced by a medium which has the same properties as quarter space (Q) occupying the positive quadrant. The result is a one-dimensional Wiener–Hopf equation which is easily solved in both cases (large x_1, large x_2).

Referring to (A7) and (A12), we see that Φ_1, given by

$$\Phi_1(k_1, k_2) = 1/[(a_1 - ik_1)(a_2 - ik_2)H_1(k_1, k_2) \times H_2(-ia_1, k_2)H_3(-ia_1, -ia_2) \times H_4(k_1, -ia_2)] + \hat{\Phi}_1(k_1, k_2), \quad (4.6)$$

will produce the desired asymptotic behavior for large x_1 or x_2 with $\hat{\Phi}_1$ analytic in the upper (k_1, k_2) planes and yielding the correction near the corner. Similarly, to get correct asymptotic behavior in the quarter planes adjacent to the position quadrant, we choose to define $\hat{\Phi}_2$ and $\hat{\Phi}_4$ as follows:

$$\Phi_2(k_1, k_2) = \frac{1}{(a_1 - ik_1)}\frac{1}{(a_2 - ik_2)} \times \left(1 - \frac{H_2(k_1, k_2)H_3(k_1, -ia_2)}{H_2(-ia_1, k_2)H_3(-ia_1, -ia_2)}\right) + \hat{\Phi}_2(k_1, k_2), \quad (4.7)$$

$$\Phi_4(k_1, k_2) = \frac{1}{(a_1 - ik_1)}\frac{1}{(a_2 - ik_2)} \times \left(1 - \frac{H_4(k_1, k_2)H_3(-ia_1, k_2)}{H_4(k_1, -ia_2)H_3(-ia_1, -ia_2)}\right) + \hat{\Phi}_4(k_1, k_2). \quad (4.8)$$

Note that in (4.7) we have substituted the ratio $H_2(k_1, k_2)/H_2(-ia_1, k_2)$ in preference to the choice indicated by (A13), namely, $H_2(k_1, -ia_2)/H_2(-ia_1, -ia_2)$, and have made a similar substitution in (4.8). The reason for these changes as well as the definition of Φ_3,

$$\Phi_3(k_1, k_2) = -\frac{1}{(a_1 - ik_1)}\frac{1}{(a_2 - ik_2)} \times \left(1 - \frac{H_3(-ia_1, k_2)H_3(k_1, -ia_2)}{H_3(k_1, k_2)H_3(-ia_1, -ia_2)}\right) + \hat{\Phi}_3(k_1, k_2), \quad (4.9)$$

is given below.

The problem, now defined in terms of hatted variables, is

$$\Lambda(k_1, k_2)\hat{\Phi}_1(k_1, k_2) = -\hat{\Phi}_2(k_1, k_2) - \hat{\Phi}_3(k_1, k_2) - \hat{\Phi}_4(k_1, k_2) + \hat{S}(k_1, k_2), \quad (4.10)$$

where (4.6)–(4.9) were used in (4.1) and the source term $\hat{S}$ is given by

$$\hat{S}(k_1, k_2) = \frac{1}{(a_1 - ik_1)(a_2 - ik_2)H_3(-ia_1, -ia_2)} \times \left(-\frac{H_2(k_1, k_2)H_3(k_1, k_2)H_4(k_1, k_2)}{H_2(-ia_1, k_2)H_4(k_1, -ia_2)} + \frac{H_2(k_1, k_2)H_3(k_1, -ia_2)}{H_2(-ia_1, k_2)} - \frac{H_3(-ia_1, k_2)H_3(k_1, -ia_2)}{H_3(k_1, k_2)} + \frac{H_4(k_1, k_2)H_3(-ia_1, k_2)}{H_4(k_1, -ia_2)}\right). \quad (4.11)$$

Our motivation for making the choice of $\hat{\Phi}_2$, $\hat{\Phi}_3$, and $\hat{\Phi}_4$, as defined by (4.7)–(4.9), becomes clearer by noting that the residues of $\hat{S}$ vanish identically at $k_1 = -ia_1$ and at $k_2 = -ia_2$. Also, because

$$\lim_{k_j \to \infty} H_i(k_1, k_2) = 1 + O(\ln k_j/k_j), \quad j = 1, 2,\ i = 1, \cdots, 4, \quad (4.12)$$

we find that

$$\lim_{k_j \to \infty} \hat{S}(k_1, k_2) = O(\ln k_j/k_j^2), \quad j = 1, 2. \quad (4.13)$$

This latter fact will be useful in our derivation of a convergent iterative solution.

5. SOLUTION FOR $\hat{\Phi}_1$ IN TERMS OF $\hat{\Phi}_2$ AND $\hat{\Phi}_4$

We now demonstrate that assuming that $\hat{\Phi}_2$ and $\hat{\Phi}_4$ are known leads directly to the solution for $\hat{\Phi}_1$. Later we shall determine $\hat{\Phi}_2$ and $\hat{\Phi}_4$. For this purpose and many of the remaining calculations we require the following factorization of Λ in the variable k_2:

$$\Lambda(k_1, k_2) = (\varkappa_0^2 + k_1^2 + k_2^2)E(k_2, k_1)E(-k_2, k_1), \quad (5.1)$$

where

$$E(k_2, k_1) = \exp\left(-\frac{1}{\pi}\int_1^\infty \frac{\theta(1/t)t\,dt}{(t^2+k_1^2)^{\frac{1}{2}}[(t^2+k_1^2)^{\frac{1}{2}} - ik_2]}\right)\Big/(1+k_1^2)^{\frac{1}{2}} - ik_2. \tag{5.2}$$

The above result follows most simply from (3.12). The function $E(k_2, k_1)$ is analytic in the upper k_2 plane while $E(-k_2, k_1)$ is analytic in the lower k_2 plane.

With the substitution (5.1), Eq. (4.10) reads

$$(\varkappa_0^2 + k_1^2 + k_2^2)E(k_2, k_1)E(-k_2, k_1)\hat{\Phi}_1 = -\hat{\Phi}_2 - \hat{\Phi}_3 - \hat{\Phi}_4 + \hat{S}. \tag{5.3}$$

Before proceeding further, we need to derive some useful relations between the $\hat{\Phi}_i$ at infinity. Multiplying (5.3) by k_1 and letting $k_1 \to \infty$, we find that

$$\nu_1(k_2) = -\nu_2(k_2) - \nu_3(k_2) - \nu_4(k_2), \tag{5.4}$$

where

$$\nu_j(k_2) = \lim_{k_1\to\infty} ik_1\hat{\Phi}_j(k_1, k_2), \quad j = 1, 2, 3, 4, \tag{5.5}$$

and where we have used (4.13) and the fact that $\Lambda \to 1$ when either k_1 or $k_2 \to \infty$. Because ν_1 and ν_2 are analytic in the upper half-plane, because ν_3 and ν_4 are analytic in the lower half-plane, and because all the ν_j vanish at infinity, we must have that

$$\nu_1(k_2) + \nu_2(k_2) = 0, \tag{5.6}$$

$$\nu_3(k_2) + \nu_4(k_2) = 0. \tag{5.7}$$

Similarly, if we define the limits

$$\mu_j(k_1) = \lim_{k_2\to\infty} ik_2\hat{\Phi}_j(k_1, k_2), \quad j = 1, 2, 3, 4, \tag{5.8}$$

then we find

$$\mu_1(k_1) + \mu_4(k_1) = 0, \tag{5.9}$$

$$\mu_2(k_1) + \mu_3(k_1) = 0. \tag{5.10}$$

Now, dividing Eq. (5.3) by $E(-k_2, k_1)$ and performing the operation $[\ ]_{\sigma_2+}$ [see Eqs. (3.6)–(3.8)] on the result yields

$$(\varkappa_0^2 + k_1^2 + k_2^2)E(k_2, k_1)\hat{\Phi}_1 + \mu_4(k_1) = \left[\frac{\hat{S} - \hat{\Phi}_2}{E(-k_2, k_1)}\right]_{\tilde{\sigma}_2+}, \tag{5.11}$$

where the operations $[\ ]_{\tilde{\sigma}_i+}$ correspond to the contours $\tilde{\Gamma}_i^{\pm}$, as shown in Fig. 3 and where we have used the facts that (1) the lhs of (5.11) is analytic in the upper k_2 plane and is square-integrable in T_Φ [because of (5.9)] and (2) the same is true of $(\hat{\Phi}_3 + \hat{\Phi}_4)/E(-k_2, k_1)$ except that it is analytic in the lower k_2 plane.

Now dividing (5.11) by $(\varkappa_0^2 + k_1^2 + k_2^2)E(k_2, k_1)$ gives, for $\hat{\Phi}_1$,

$$\hat{\Phi}_1(k_1, k_2) = \frac{1}{(\varkappa_0^2 + k_1^2 + k_2^2)E(k_2, k_1)} \times \left\{\left[\frac{\hat{S} - \hat{\Phi}_2}{E(-k_2, k_1)}\right]_{\sigma_2+} - \mu_4(k_1)\right\}. \tag{5.12}$$

Thus $\hat{\Phi}_1$ is expressed in terms of known functions and the unknown functions $\hat{\Phi}_2$ and μ_4.

A similar expression may be developed in terms of $\hat{\Phi}_4$ and ν_2 by using the factorization of Λ in k_1:

$$\Lambda(k_1, k_2) = (\varkappa_0^2 + k_1^2 + k_2^2)E(k_1, k_2)E(-k_1, k_2). \tag{5.13}$$

We find, analogous to (5.12), that $\hat{\Phi}_1$ may be represented as

$$\hat{\Phi}_1(k_1, k_2) = \frac{1}{(\varkappa_0^2 + k_1^2 + k_2^2)E(-k_1, k_2)} \times \left\{\left[\frac{\hat{S} - \hat{\Phi}_4}{E(-k_1, k_2)}\right]_{\sigma_1} - \nu_2(k_2)\right\}. \tag{5.14}$$

Of course, at this point both $\hat{\Phi}_2$ and $\hat{\Phi}_4$ are unknown. The above expressions for $\hat{\Phi}_1$ were simply derived by a modified one-dimensional Wiener–Hopf analysis, the modification consisting of factoring out the zeros of Λ and using the Wiener–Hopf factorization of the remainder. In the next section, however, a coupled pair of Fredholm equations for $\hat{\Phi}_2$ and $\hat{\Phi}_4$ will be derived which will be shown to be solvable by iteration. Once this is done, we will return to (5.12) or (5.14) to obtain $\hat{\Phi}_1$.

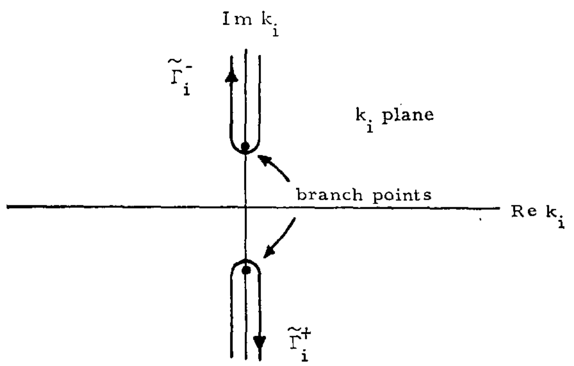

FIG. 3. The contours $\tilde{\Gamma}_i^{\pm}$.

6. COUPLED FREDHOLM EQUATIONS FOR $\hat{\Phi}_2$ AND $\hat{\Phi}_4$

We return to (5.13) and divide by $E(-k_1, k_2)$. This time, however, we perform the operation $[\ ]_{\tilde{\sigma}_1^-}$ and find

$$0 = \left[\frac{\hat{S} - \hat{\Phi}_4}{E(-k_1, k_2)}\right]_{\tilde{\sigma}_1^-} - \frac{\hat{\Phi}_2 + \hat{\Phi}_3}{E(-k_1, k_2)} + \nu_2 - \nu_4. \quad (6.1)$$

Now we multiply the above equation by $E(-k_1, k_2)$ and perform the operation $[\ ]_{\sigma_2^+}$. This filters out the term $\hat{\Phi}_3$, which is analytic in the lower k_2 plane, giving

$$\hat{\Phi}_2(k_1, k_2) = -\left[E(-k_1, k_2)\left(\left[\frac{\hat{\Phi}_4 - \hat{S}}{E(-k_1, k_2)}\right]_{\tilde{\sigma}_1^-} + \nu_2 - \nu_4\right)\right]_{\sigma_2^+}. \quad (6.2)$$

Another equation relating $\hat{\Phi}_2$ and $\hat{\Phi}_4$ may be derived by dividing by $E(-k_2, k_1)$ in (5.3) and performing the operation $[\ ]_{\tilde{\sigma}_2^-}$. We find

$$0 = \left[\frac{\hat{S} - \hat{\Phi}_2}{E(-k_2, k_1)}\right]_{\tilde{\sigma}_2^-} - \frac{\hat{\Phi}_3 + \hat{\Phi}_4}{E(-k_2, k_1)} - \mu_2 + \mu_4. \quad (6.3)$$

Multiplying (6.3) by $E(-k_2, k_1)$ and performing $[\ ]_{\sigma_1^+}$ gives the second desired equation:

$$\hat{\Phi}_4(k_1, k_2) = -\left[E(-k_2, k_1)\left(\left[\frac{\hat{\Phi}_2 - \hat{S}}{E(-k_2, k_1)}\right]_{\tilde{\sigma}_2^-} - \mu_2 + \mu_4\right)\right]_{\sigma_1^+}. \quad (6.4)$$

The contours Γ_i^+ (corresponding to the operations $[\ ]_{\sigma_i^+}$) are as shown in Fig. 2 and, for the moment, are confined to the tube T_Λ. This is sufficient to ensure analyticity of $S(z_1, z_2)$ if $(ia_1, ia_2) \in T_\Lambda \cap \{\text{Im}\,(ia_1), \text{Im}\,(ia_2) \geq 0\}$.

If (6.2) is evaluated for $k_1 \in \Gamma_1^+$ and $k_2 \in \tilde{\Gamma}_2^-$ to obtain $\hat{\Phi}_2(k_1, k_2)$ and (6.4) is evaluated for $k_1 \in \tilde{\Gamma}_1^-$ and $k_2 \in \Gamma_2^+$ to obtain $\hat{\Phi}_4(k_1, k_2)$, then the two equations represent a pair of coupled Fredholm equations for $\hat{\Phi}_2$ and $\hat{\Phi}_4$. The quantities $1/(z_i - k_i)$ are clearly bounded. In contemplating a Neumann series solution to (6.2) and (6.4), it would be advantageous to make $1/(z_i - k_i)$ as small as possible by lowering the Γ_i^+ contours as far as possible into the lower half-planes. Guided by our experience in one-dimensional problems, we might hope to achieve deformation of the contours Γ_i^+ to $\tilde{\Gamma}_i^+$ shown in Fig. 3, which possibly would have the additional advantage

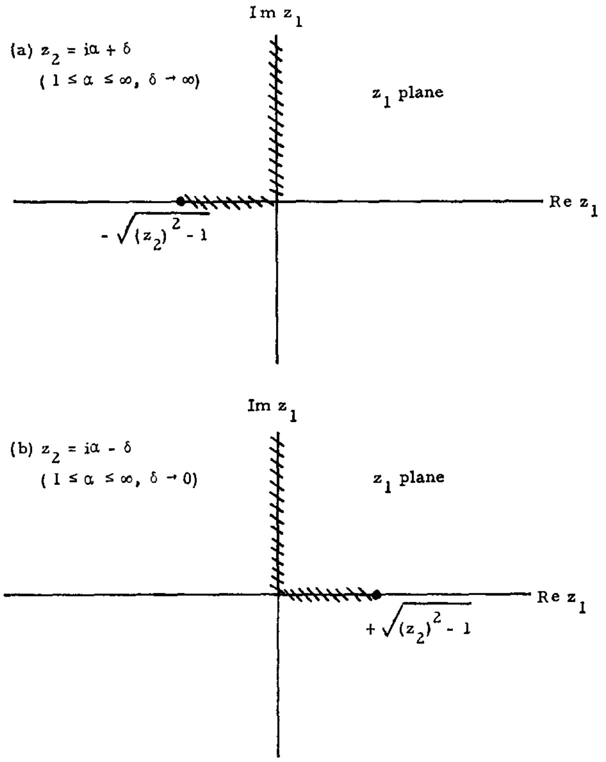

FIG. 4. Possible choice of branch cuts of $E(-z_1, z_2)$.

of producing a real-valued kernel. However, this deformation is not possible for the following reason. In (6.2), for example, if the integration variable z_2 approaches the line $(-i\infty, -i)$, then the branch point of $E(-z_1, z_2)$ in the z_1 plane approaches the real z_1 axis at a point depending on the value of z_2 [refer to the representation of $E(-z_1, z_2)$ given by (5.2)]. This is shown in Fig. 4. Thus it would not be possible to deform Γ_1^+ and Γ_2^+, as shown in Fig. 3, because the contours $\tilde{\Gamma}_1^-$ and $\tilde{\Gamma}_2^-$ could not be preserved.

An iterative scheme which avoids this difficulty and leads to real-valued kernels will now be derived. First, we note that because

$$\left[\frac{\hat{\Phi}_4}{E(-k_1, k_2)}\right]_{\tilde{\sigma}_1^-} = \frac{\hat{\Phi}_4}{E(-k_1, k_2)} - \mu_4 - \left[\frac{\hat{\Phi}_4}{E(-k_1, k_2)}\right]_{\tilde{\sigma}_1^+}, \quad (6.5)$$

it follows that (6.2) may be rewritten as

$$\hat{\Phi}_2(k_1, k_2) = \left[E(-k_1, k_2)\left[\frac{\hat{\Phi}_4}{E(-k_1, k_2)}\right]_{\tilde{\sigma}_1^+}\right]_{\sigma_2^+} + R_2(k_1, k_2), \quad (6.6)$$

and, similarly, (6.4) may be rewritten as

$$\hat{\Phi}_4(k_1, k_2) = \left[E(-k_2, k_1)\left[\frac{\hat{\Phi}_2}{E(-k_2, k_1)}\right]_{\tilde{\sigma}_2^+}\right]_{\sigma_1^+} + R_4(k_1, k_2), \tag{6.7}$$

where

$$R_j(k_1, k_2) = T_j(k_1, k_2) + U_j(k_1, k_2), \quad j = 2, 4, \tag{6.8}$$

and

$$T_2(k_1, k_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_2^+} \frac{dz_2 E(-k_1, z_2)}{(z_2 - k_2)} \int_{\tilde{\Gamma}_1^-} \frac{dz_1 \hat{S}(z_1, z_2)}{E(-z_1, z_2)(z_1 - k_1)}, \tag{6.9}$$

$$T_4(k_1, k_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1^+} \frac{dz_1 E(-k_2, z_1)}{(z_1 - k_1)} \int_{\tilde{\Gamma}_2^-} \frac{dz_2 \hat{S}(z_1, z_2)}{E(-z_2, z_1)(z_2 - k_2)}, \tag{6.10}$$

$$U_2(k_1, k_2) = \frac{1}{2\pi i} \int_{\Gamma_2^+} \frac{dz_2 E(-k_1, z_2)\nu_2(z_2)}{z_2 - k_2}, \tag{6.11}$$

$$U_4(k_1, k_2) = \frac{1}{2\pi i} \int_{\Gamma_1^+} \frac{dz_1 E(-k_2, z_1)\mu_4(z_1)}{z_1 - k_1}. \tag{6.12}$$

The contours in (6.6) and (6.7) may now be deformed into the lower half-planes. From a careful inspection of the functions R_2, R_4, $E(-k_2, k_1)$, and $E(-k_1, k_2)$, we find that $\hat{\Phi}_2$ and $\hat{\Phi}_4$ have branch cuts as shown in Fig. 5. Accordingly, it is useful to define discontinuities of various functions as follows:

$k_1, z_1 \in (-i\infty, -i0)$:

$$\hat{\Phi}_{2+}(z_1, z_2) - \hat{\Phi}_{2-}(z_1, z_2) \equiv \begin{cases} \psi_2(z_1, z_2), & z_2 \in (-i, -i\varkappa_0) \\ \chi_2(z_1, z_2), & z_2 \in (-i\infty, -i) \end{cases}, \tag{6.13}$$

$$\frac{E_+(-k_1, z_2)}{E_+(-z_1, z_2)} - \frac{E_-(-k_1, z_2)}{E_-(-z_1, z_2)} \equiv G(k_1, z_1, z_2), \quad z_2 \in (-i\infty, -i), \tag{6.14}$$

$k_2, z_2 \in (-i\infty, -i0)$:

$$\hat{\Phi}_4^+(z_1, z_2) - \hat{\Phi}_4^-(z_1, z_2) \equiv \begin{cases} \psi_4(z_1, z_2), & z_1 \in (-i, -i\varkappa_0) \\ \chi_4(z_1, z_2), & z_1 \in (-i\infty, -i) \end{cases}, \tag{6.15}$$

$$\frac{E^+(-k_2, z_1)}{E^+(-z_2, z_1)} - \frac{E^-(-k_2, z_1)}{E^-(-z_2, z_1)} \equiv G(k_2, z_2, z_1), \quad z_1 \in (-i\infty, -i), \tag{6.16}$$

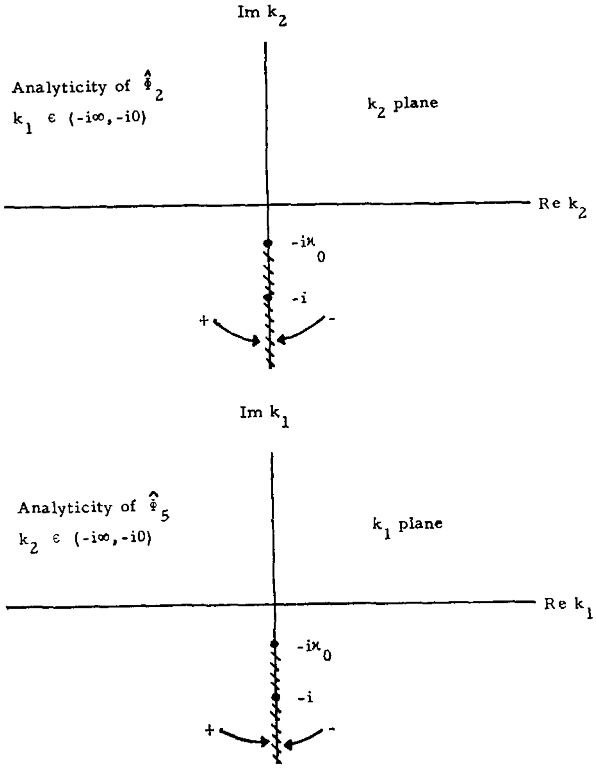

FIG. 5. Branch cuts of $\hat{\Phi}_2$ and $\hat{\Phi}_4$.

where the superscript ($\pm$) denotes a limit in the k_1 plane and the subscript ($\pm$) denotes a limit in the k_2 plane (see Fig. 5).

Collapsing the Γ_i^+ about the negative imaginary axes,[10] we can rewrite (6.6) and (6.7) as

$$\hat{\Phi}_2(k_1, k_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{-i} \frac{dz_2}{z_2 - k_2} \times \left(\int_{-i\infty}^{-i} \frac{dz_1 G(k_1, z_1, z_2)\chi_4(z_1, z_2)}{z_1 - k_1} + \int_{-i}^{-i\varkappa_0} \frac{dz_1 G(k_1, z_1, z_2)\psi_4(z_1, z_2)}{z_1 - k_1} \right) + R_2(k_1, k_2), \tag{6.17}$$

$$\hat{\Phi}_4(k_1, k_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{-i} \frac{dz_1}{z_1 - k_1} \times \left(\int_{-i\infty}^{-i} \frac{dz_2 G(k_2, z_2, z_1)\chi_2(z_1, z_2)}{z_2 - k_2} + \int_{-i}^{-i\varkappa_0} \frac{dz_2 G(k_2, z_2, z_1)\psi_2(z_1, z_2)}{z_2 - k_2} \right) + R_4(k_1, k_2). \tag{6.18}$$

In Appendix B, integral representations for T_2 and T_4, the known parts of R_2 and R_4, are derived by using formulas analogous to (6.5) in (6.9) and (6.10) with contour integration.

Thus we need only determine ψ_2, χ_2, ν_2, ψ_4, χ_4, and μ_4 to obtain $\hat{\Phi}_2$ and $\hat{\Phi}_4$ from (6.17) and (6.18), respectively. The discontinuities of $\hat{\Phi}_2$ and $\hat{\Phi}_4$ as calculated by just these equations provide four of the equations for the above functions. Computing the discontinuity of (6.17) across the cut $(-i, -i\varkappa_0)$ in the k_2 plane, we find that $\psi_2(k_1, k_2)$ is simply given as the discontinuity of R_2:

$$\psi_2(k_1, k_2) = R_{2+}(k_1, k_2) - R_{2-}(k_1, k_2),$$
$$k_1 \in (-i\infty, -i0), \quad k_2 \in (-i, -i\varkappa_0). \quad (6.19)$$

Similarly, from (6.18) we compute $\psi_4(k_1, k_2)$ to be given by

$$\psi_4(k_1, k_2) = R_4^+(k_1, k_2) - R_4^-(k_1, k_2),$$
$$k_1 \in (-i, -i\varkappa_0), \quad k_2 \in (-i\infty, -i0). \quad (6.20)$$

Next, computing the discontinuities of (6.17) and (6.18) across $(-i\infty, -i)$ in the k_1 and k_2 planes, respectively, yields two coupled Fredholm equations for χ_2 and χ_4:

$$\chi_2(k_1, k_2) = \frac{1}{2\pi i}\int_{-i\infty}^{-i} \frac{dz_1 G(k_1, z_1, k_2)\chi_4(z_1, k_2)}{z_1 - k_1} + [R_{2+}(k_1, k_2) - R_{2-}(k_1, k_2)], \quad (6.21)$$

$$\chi_4(k_1, k_2) = \frac{1}{2\pi i}\int_{-i\infty}^{-i} \frac{dz_2 G(k_2, z_2, k_1)\chi_2(k_1, z_2)}{z_2 - k_2} + [R_4^+(k_1, k_2) - R_4^-(k_1, k_2)]. \quad (6.22)$$

The kernels are real and are continuous because $G(k_i, z_i, k_j)$ is analytic in k_i and vanishes at $k_i = z_i$.

We define the linear operators L_2 and L_4,

$$L_2(\gamma)(k_1, k_2) = \frac{1}{2\pi i}\int_{-i\infty}^{-i} \frac{dz_1 G(k_1, z_1, k_2)\gamma(z_1, k_2)}{z_1 - k_1}, \quad (6.23)$$

$$L_4(\gamma)(k_1, k_2) = \frac{1}{2\pi i}\int_{-i\infty}^{-i} \frac{dz_2 G(k_2, z_2, k_1)\gamma(k_1, z_2)}{z_2 - k_2}, \quad (6.24)$$

and write an iterative scheme for solving (6.21) and (6.22) as follows:

$$\chi_2^{(n)} = L_2(\chi_4^{(n)}) + [R_{2+} - R_{2-}], \quad (6.25)$$

$$\chi_4^{(n)} = L_4(\chi_4^{(n-1)}) + [R_4^+ - R_4^-],$$
$$\chi_4^{(0)} \equiv 0, \quad n = 1, 2, \cdots. \quad (6.26)$$

To justify this procedure, we must demonstrate that the norm of L_4L_2 satisfies $\|L_4L_2\| < 1$ in some Banach space. In Appendix C, we show indeed that this is true for all $0 < c \gtrsim 2$ and that the convergence is uniform.

As a final step, we relate the unknown functions ν_2 and μ_4 to the functions ψ_2, χ_2 and ψ_4, χ_4 as follows. As in the development of (6.2), but using the factorization $\Lambda = (H_1H_4)(H_2H_3)$, we find that

$$\hat{\Phi}_2(k_1, k_2) = \left[H_2H_3\left[\frac{\hat{\Phi}_4}{H_2H_3}\right]_{\tilde{\sigma}_1^+}\right]_{\tilde{\sigma}_2^+} + \left[H_2H_3\left[\frac{\hat{S}}{H_2H_3}\right]_{\tilde{\sigma}_1^-}\right]_{\sigma_2^+}. \quad (6.27)$$

Multiplying (6.27) by ik_2 and letting $k_2 \to \infty$, we have

$$\nu_2(k_2) = \frac{i}{4\pi^2}\int_{-i\infty}^{-i\varkappa_0} \frac{dz_2}{z_2 - k_2} \times \int_{-i\infty}^{-i\varkappa_0} \frac{dz_1[\hat{\Phi}_4^+(z_1, z_2) - \hat{\Phi}_4^-(z_1, z_2)]}{H_3(z_1, z_2)} \times \left(\frac{1}{H_{2+}(z_1, z_2)} - \frac{1}{H_{2-}(z_1, z_2)}\right), \quad (6.28)$$

where $\hat{\Phi}_4^+ - \hat{\Phi}_4^-$ is related to ψ_4 and χ_4 by (6.15). Similarly, we find that $\mu_4(k_1)$ is given by

$$\mu_4(k_1) = \frac{i}{4\pi^2}\int_{-i\infty}^{-i\varkappa_0} \frac{dz_1}{z_1 - k_1} \times \int_{-i\infty}^{-i\varkappa_0} \frac{dz_2[\hat{\Phi}_{2+}(z_1, z_2) - \hat{\Phi}_{2-}(z_1, z_2)]}{H_3(z_1, z_2)} \times \left(\frac{1}{H_4^+(z_1, z_2)} - \frac{1}{H_4^-(z_1, z_2)}\right), \quad (6.29)$$

where $\hat{\Phi}_{2+} - \hat{\Phi}_{2-}$ is related to ψ_2 and χ_2 by (6.13).

Equations (6.19)–(6.22) and (6.28) and (6.29) comprise the required set of six equations for the six unknowns ψ_2, χ_2, ν_2, ψ_4, χ_4, and μ_4.

7. CONCLUSION

Assuming an inhomogeneous term $\exp(-a_1x_1 - a_2x_2)$ in the transport equation (2.2), we find that the double Fourier transform of the flux in the quarter space, $\Phi_1(k_1, k_2)$, is given by (4.6) in terms of the transform of the flux away from the corner whose properties are discussed in Appendix A, and a correction $\hat{\Phi}_1(k_1, k_2)$. The function $\hat{\Phi}_1(k_1, k_2)$ is given in terms of $\hat{\Phi}_2(k_1, k_2)$ or $\hat{\Phi}_4(k_1, k_2)$ by (5.12) or (5.14), respectively. The functions $\hat{\Phi}_2$ and $\hat{\Phi}_4$ have singularities only on the imaginary axes and are given by (6.17) and (6.18) in terms of functions $\psi_i(z_1, z_2)$, $\chi_i(z_1, z_2)$, $i = 2, 4$, $\nu_2(k_2)$, and $\mu_4(k_1)$. The ψ_i are given by (6.19) and (6.21), while the χ_i satisfy Fredholm equations (6.21) and (6.22). Uniform convergence to the solution of these equations may be obtained by iteration for values of c: $0 \le c \lesssim 2$. The functions ν_2 and μ_4 are related to the ψ_i and χ_i by (6.28) and (6.29).

A more direct approach to the solution appears to be possible by obtaining Fredholm equations for Φ_2 and Φ_4 (instead of $\hat{\Phi}_2$ and $\hat{\Phi}_4$) starting from (4.1) and bypassing the subtraction of the asymptotic terms. There are two objections to this approach. First, the asymptotic behavior of the flux does not appear in a natural, relatively simple way. Second, the properties of the resultant Fredholm equations for Φ_2 and Φ_4 appear to be very sensitive to the value of $\varkappa_0$ (and hence the value of c) and to the location of the points ia_1 and ia_2. Convergence is an open question. In particular, residues at the poles $k_1 = -ia_1$ and $k_2 = -ia_2$ are unknown and must be carried along during the iteration.

APPENDIX A: ASYMPTOTIC SOLUTIONS

Far away from the boundary $x_1 = 0$, the spatial distribution of the density $\varphi(x_1, x_2)$ in x_1 will tend to the source distribution $\exp(-a_1 x)$ if $\mathrm{Re}(a_1)$ and $\mathrm{Re}(a_2)$ are small enough. Later in this appendix we give specific upper bounds which must be satisfied. Thus,

$$\varphi(x_1, x_2) \to \exp(-a_1 x_1)\alpha_2(x_2), \quad x_1 \to +\infty. \quad \text{(A1)}$$

To determine the distribution in x_2 for large x_1 given by α_2, we substitute the rhs above into the transport equation (2.4) and extend the integration on x_1 to $(-\infty, +\infty)$:

$$\begin{aligned}\exp(-a_1x_1)\alpha_2(x_2) &= c\int_{-\infty}^{\infty} dx_1' \int_0^\infty dx_2' K(|\mathbf{x}-\mathbf{x}'|)\exp(-a_1x_1')\alpha_2(x_2') \\ &\quad + \begin{cases}\exp(-a_1x_1 - a_2x_2), & x_2 \ge 0\\ 0, & x_2 < 0\end{cases}. \quad \text{(A2)}\end{aligned}$$

Equation (A2) represents a one-dimensional Wiener-Hopf problem in x_2. Fourier transformation in x_2 of the above equation yields

$$\Lambda[(k_2^2 - a_1^2)^{\frac{1}{2}}]A_2^+(k_2) = -A_2^-(k_2) + 1/(a_2 - ik_2), \quad \text{(A3)}$$

where Λ is defined in (3.1) and $A_2^\pm$ is the transform of α_2 for $x_2 \gtrless 0$:

$$A_2^+(k_2) = \int_0^\infty \exp(ik_2x_2)\alpha_2(x_2)\,dx_2, \quad \text{(A4)}$$

$$A_2^-(k_2) = \int_{-\infty}^0 \exp(ik_2x_2)\alpha_2(x_2)\,dx_2. \quad \text{(A5)}$$

The appropriate factorization of $\Lambda[(k_2^2 - a_1^2)^{\frac{1}{2}}]$ is given by

$$\Lambda[(k_2^2 - a_1^2)^{\frac{1}{2}}] = [H_1(-ia_1, k_2)H_2(-ia_1, k_2)] \times [H_3(-ia_1, k_2)H_4(-ia_1, k_2)], \quad \text{(A6)}$$

where the first factor in square brackets above is analytic and nonzero in the upper k_2 plane while the same is true for the second factor in the lower k_2 plane.

Using the factorization (A6), we easily solve (A3) to give

$$A_2^+(k_2) = [(a_2 - ik_2)H_1(-ia_1, k_2)H_2(-ia_1, k_2) \times H_3(-ia_1, -ia_2)H_4(-ia_1, -ia_2)]^{-1} \quad \text{(A7)}$$

and

$$A_2^-(k_2) = \frac{1}{a_2 - ik_2} \times \left(1 - \frac{H_3(-ia_1, k_2)H_4(-ia_1, k_2)}{H_3(-ia_1, -ia_2)H_4(-ia_1, -ia_2)}\right). \quad \text{(A8)}$$

To interpret the above results without going into any great detail, we see from (5.13) and (5.2) that

$$H_1(-ia_1, k_2)H_2(-ia_1, k_2) = \left(\frac{(\varkappa_0^2 - a_1^2)^{\frac{1}{2}} - ik_2}{(1 - a_1^2)^{\frac{1}{2}} - ik_2}\right) \times \exp\left(-\frac{1}{\pi}\int_1^\infty \frac{\theta(1/t)t\,dt}{(t^2 - a_1^2)^{\frac{1}{2}}[(t^2 - a_1^2)^{\frac{1}{2}} - ik_2]}\right). \quad \text{(A9)}$$

Thus H_1H_2 for $k_1 = -ia_1$ has a zero at $k_2 = -i(\varkappa_0^2 - a_1^2)^{\frac{1}{2}}$ and a branch point at $k_2 = -i(1 - a_1^2)^{\frac{1}{2}}$ with a corresponding branch cut in the lower half-plane. (The apparent pole at the branch point is canceled by the exponential term which goes to zero at that point.) Fourier inversion of A_2^+ will give by contour integration two discrete exponential terms for $\alpha_2(x_2)$, $x_2 > 0$, plus an integral over the branch cut as follows:

$$\begin{aligned}\lim_{x_1\to\infty} \varphi(x_1, x_2) &= \exp(-a_1x_1)\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(-ik_2x_2)A_2^+(k_2)\,dk_2 \\ &= \exp(-a_1x_1)\Big(R_1\exp(-a_2x_2) + R_2\exp[-(\varkappa_0^2 - a_1^2)^{\frac{1}{2}}x_2] \\ &\quad + \int_{(1-a_1^2)^{\frac{1}{2}}}^\infty \exp(-kx_2)D(k)\,dk\Big), \quad \text{(A10)}\end{aligned}$$

where residues R_1 and R_2 and the function $D(k)$ associated with the discontinuity across the branch cut are easily obtained by using (A7) in (A9). For $x_2 < 0$, $\alpha_2(x_2)$ will consist only of an integral term over the branch cut of $H_3(-ia_1, k_2)H_4(-ia_1, k_2)$ in the upper half k_2 plane. We have thus determined the behavior of $\varphi(x_1, x_2)$ for large positive x_1 for all x_2.

Similarly, we expect for large positive x_2 that

$$\varphi(x_1, x_2) \to \exp(-a_2 x_2)\alpha_1(x_1), \quad x_2 \to +\infty. \quad (A11)$$

If Fourier transforms of α_1 are defined as follows:

$$A_1^+(k_1) = \int_0^\infty \exp(ik_1x_1)\alpha_1(x_1)\,dx_1,$$

$$A_1^-(k_1) = \int_{-\infty}^0 \exp(ik_1x_1)\alpha_1(x_1)\,dx_1,$$

then an analysis similar to that above for α_2 gives

$$A_1^+(k_1) = [(a_1 - ik_1)H_1(k_1, -ia_2)H_4(k_1, -ia_2) \times H_2(-ia_1, -ia_2)H_3(-ia_1, -ia_2)]^{-1}, \quad (A12)$$

and

$$A_1^-(k_1) - \frac{1}{a_1 - ik_1} \times \left(1 - \frac{H_2(k_1, -ia_2)H_3(k_1, -ia_2)}{H_2(-ia_1, -ia_2)H_3(-ia_1, -ia_2)}\right). \quad (A13)$$

The large x_2 behavior of $\varphi(x_1, x_2)$ for $x_1 > 0$ will have the same form as given in (A10), with a_1, a_2, x_1, and x_2 replaced by a_2, a_1, x_2, and x_1, respectively.

An earlier statement was made that Re (a_1) and Re (a_2) must be small enough if (A1) and (A11) are to be valid expansions. From (A10) and the equivalent expression for large x_2 we see that Re (a_1) and Re (a_2) must satisfy

$$\mathrm{Re}\,(a_1) < \mathrm{Re}\,(\varkappa_0^2 - a_2^2)^{\frac{1}{2}}$$

and

$$\mathrm{Re}\,(a_2) < \mathrm{Re}\,(\varkappa_0^2 - a_1^2)^{\frac{1}{2}},$$

or, if a_1 and a_2 are real,

$$a_1^2 + a_2^2 < \varkappa_0^2.$$

APPENDIX B: THE FUNCTIONS $T_2(k_1, k_2)$ AND $T_4(k_1, k_2)$

The function T_2 is given by (6.9) or, equivalently, by

$$T_2(k_1, k_2) = \left[E(-k_1, k_2)\left[\frac{\hat{S}(k_1, k_2)}{E(-k_1, k_2)}\right]_{\tilde{\sigma}_1 -}\right]_{\sigma_2 +}, \quad (B1)$$

where $\hat{S}$ is given by (4.11). For the first two terms within the brackets in (4.11), we apply the operations shown on the rhs of (B1), and for the second pair of terms we use, for any suitably behaved $B(k_1, k_2)$,

$$\left[\frac{B(k_1, k_2)}{E(-k_1, k_2)}\right]_{\sigma_1 -} = \frac{B(k_1, k_2)}{E(-k_1, k_2)} - \left[\frac{B(k_1, k_2)}{E(-k_1, k_2)}\right]_{\sigma_1 +}. \quad (B2)$$

After subsequent manipulations we find

$$\begin{aligned} H_3(-ia_1, -ia_2)T_2(k_1, k_2) &= \frac{H_3(k_1, -ia_2)}{(a_1 - ik_1)(a_2 - ik_2)} \times \left(\frac{H_2(k_1, k_2)}{H_2(-ia_1, k_2)} - \frac{H_2(k_1, -ia_2)}{H_2(-ia_1, -ia_2)}\right) \\ &- \frac{1}{2\pi i(a_1 - ik_1)H_4(-ia_1, -ia_2)} \times \int_{-i\infty}^{-i} \frac{dz_2 G(k_1, -ia_1, z_2)H_4(-ia_1, z_2)}{(z_2 - k_2)(a_2 - iz_2)} \\ &+ \frac{1}{2\pi i}\int_{-i\infty}^{-i\varkappa_0} \frac{dz_2}{(z_2 - k_2)(a_2 - iz_2)} \times \left(\frac{E_+(-k_1, z_2)}{H_{2+}(-ia_1, z_2)} - \frac{E_-(-k_1, z_2)}{H_{2-}(-ia_1, z_2)}\right) \\ &- \frac{1}{4\pi^2}\int_{-i\infty}^{-i} \frac{dz_2 H_3(-ia_1, z_2)}{(z_2 - k_2)(a_2 - iz_2)} \times \int_{-i\infty}^{-i\varkappa_0} \frac{dz_1 G(k_1, z_1, z_2)}{(z_1 - k_1)(a_1 - iz_1)} \\ &\times \left(\frac{H_4^+(z_1, z_2)}{H_4^+(z_1, -ia_2)} - \frac{H_4^-(z_1, z_2)}{H_4^-(z_1, -ia_2)}\right). \end{aligned} \quad (B3)$$

Similarly, we find that $T_4(k_1, k_2)$ is given by

$$\begin{aligned} H_3(-ia_1, -ia_2)T_4(k_1, k_2) &= \frac{H_3(-ia_1, k_2)}{(a_1 - ik_1)(a_2 - ik_2)} \times \left(\frac{H_4(k_1, k_2)}{H_4(k_1, -ia_2)} - \frac{H_4(-ia_1, k_2)}{H_4(-ia_1, -ia_2)}\right) \\ &- \frac{1}{2\pi i}\frac{1}{(a_2 - ik_2)H_2(-ia_1, -ia_2)} \times \int_{-i\infty}^{-i} \frac{dz_1 G(k_2, -ia_2, z_1)H_2(z_1, -ia_2)}{(z_1 - k_1)(a_1 - iz_1)} \\ &+ \frac{1}{2\pi i}\int_{-i\infty}^{-i\varkappa_0} \frac{dz_1}{(z_1 - k_1)(a_1 - iz_1)} \times \left(\frac{E^+(-k_2, z_1)}{H_4^+(z_1, -ia_2)} - \frac{E^-(-k_2, z_1)}{H_4^-(z_1, -ia_2)}\right) \\ &- \frac{1}{4\pi^2}\int_{-i\infty}^{-i} \frac{dz_1 H_3(z_1, -ia_2)}{(z_1 - k_1)(a_1 - iz_1)} \times \int_{-i\infty}^{-i\varkappa_0} \frac{dz_2 G(k_2, z_2, z_1)}{(z_2 - k_2)(a_2 - iz_2)} \\ &\times \left(\frac{H_{2+}(z_1, z_2)}{H_{2+}(-ia_1, z_2)} - \frac{H_{2-}(z_1, z_2)}{H_{2-}(-ia_1, z_2)}\right). \end{aligned} \quad (B4)$$

APPENDIX C: CONVERGENCE OF THE ITERATION SCHEME

In this appendix we will show that the Fredholm equations derived in Sec. 6 may be solved by iteration and that the convergence is uniform. The integral operator in question is L_2L_4 given by (6.23) and (6.24). Since the Fredholm kernels belonging to L_2 and L_4 will be seen to be positive, we can define the norm of the operator L_2L_4 as

$$\|L_2L_4\| = \max_{k_1,k_2\in(-i\infty,-i)} \left| \frac{B(k_1,k_2)}{4\pi^2}\int_{-i\infty}^{-i} dz_1 \times \int_{-i\infty}^{-i} \frac{dz_2 G(k_1,z_1,k_2)G(k_2,z_2,z_1)}{(z_1-k_1)(z_2-k_2)B(z_1,z_2)}\right|, \quad \text{(C1)}$$

where $B(k_1, k_2)$ is a positive, bounded function to be chosen later. The corresponding norm on the function space is

$$\|f\| = \max_{k_1,k_2\in(-i\infty,-i)} |B(k_1,k_2)f(k_1,k_2)|. \quad \text{(C2)}$$

To prove uniform convergence, we must show that $\|L_2L_4\| < 1$. For convenience we switch to positive real variables

$$ik_j \to \eta_j, \quad iz_j \to \zeta_j, \quad j = 1, 2$$

and consider the quantity $-(1/2\pi)G(k_2, z_2, z_1)/(z_2 - k_2)$. Using (6.16) and (5.2), we find that

$$-\frac{1}{2\pi}\frac{G(k_2,z_2,z_1)}{z_2-k_2} = \frac{1}{2\pi i}\frac{G(-i\eta_2,-i\zeta_2,-i\zeta_1)}{\zeta_2-\eta_2} = \frac{D_0D_1D_2}{\pi}\frac{\sin(\psi_0-\psi_1)}{\eta_2-\zeta_2}, \quad \text{(C3)}$$

where

$$D_0 = \left(\frac{\zeta_1^2+\zeta_2^2-1}{\zeta_1^2+\eta_2^2-1}\right)^{\frac{1}{2}}, \quad \text{(C4)}$$

$$D_1 = \exp\left[-\frac{1}{\pi}\int_1^{\zeta_1}\theta(1/t)t\,dt \times \left(\frac{1}{\zeta_1^2+\eta_2^2-t^2} - \frac{1}{\zeta_1^2+\zeta_2^2-t^2}\right)\right], \quad \text{(C5)}$$

$$D_2 = \exp\left[-\frac{1}{\pi}\int_{\zeta_1}^{\infty}\frac{\theta(1/t)t\,dt}{(t^2-\zeta_1^2)^{\frac{1}{2}}} \times \left(\frac{1}{\eta_2+(t^2-\zeta_1^2)^{\frac{1}{2}}} - \frac{1}{\zeta_2+(t^2-\zeta_1^2)^{\frac{1}{2}}}\right)\right], \quad \text{(C6)}$$

$$\psi_0 = \sin^{-1}\left(\frac{(\eta_2-\zeta_2)(\zeta_1^2-1)^{\frac{1}{2}}}{(\zeta_1^2+\zeta_2^2-1)^{\frac{1}{2}}(\zeta_1^2+\eta_2^2-1)^{\frac{1}{2}}}\right), \quad \text{(C7)}$$

$$\psi_1 = \frac{1}{\pi}\int_1^{\zeta_1}\frac{\theta(1/t)t\,dt}{(\zeta_1^2-t^2)^{\frac{1}{2}}}\left(\frac{\zeta_2}{\zeta_1^2+\zeta_2^2-t^2} - \frac{\eta_2}{\zeta_1^2+\eta_2^2-t^2}\right). \quad \text{(C8)}$$

The quantities ψ_0, ψ_1, and $\eta_2 - \zeta_2$ always have the same sign. Furthermore,

$$|\psi_0| \le \pi/2,$$

and, using $\theta(1/t) \le \pi$, we obtain

$$|\psi_1| \le \left|\int_1^{\zeta_1}\frac{t\,dt}{(\zeta_1^2-t^2)^{\frac{1}{2}}}\left(\frac{\zeta_2}{\zeta_1^2+\zeta_2^2-t^2} - \frac{\eta_2}{\zeta_1^2+\eta_2^2-t^2}\right)\right| = \left|\tan^{-1}\left(\frac{(\zeta_1^2-1)^{\frac{1}{2}}}{\zeta_2}\right) - \tan^{-1}\left(\frac{(\zeta_1^2-1)^{\frac{1}{2}}}{\eta_2}\right)\right| = |\psi_0|. \quad \text{(C9)}$$

Thus the left-hand side of (C3) is positive, real, and bounded by

$$-\frac{1}{2\pi i}\frac{G(-i\eta_2,-i\zeta_2,-i\zeta_1)}{\zeta_2-\eta_2} \le \frac{D_0D_1D_2}{\pi}\frac{\sin\theta_0}{\eta_2-\zeta_2} = \frac{D_1D_2D_3}{\pi}, \quad \text{(C10)}$$

where

$$D_3 = \frac{(\zeta_1^2-1)^{\frac{1}{2}}}{\zeta_1^2+\eta_2^2-1}. \quad \text{(C11)}$$

We now choose

$$B(z_1,z_2) = B(-i\zeta_1,-i\zeta_2) = (\zeta_1^2+\zeta_2^2-1)\exp\left[\frac{1}{\pi}\int_1^{\zeta_1}\frac{\theta(1/t)t\,dt}{\zeta_1^2+\zeta_2^2-t^2} + \frac{1}{\pi}\int_{\zeta_1}^{\infty}\frac{\theta(1/t)t\,dt}{(t^2-\zeta_1^2)^{\frac{1}{2}}}\left(\frac{1}{\zeta_2+(t^2-\zeta_1^2)^{\frac{1}{2}}}\right)\right], \quad \text{(C12)}$$

consistent with the behavior of χ_2 and χ_4 at infinity. Substituting (C12) and (C10) into (C1) and using carefully selected inequalities which are too numerous to repeat here, we find that the norm of L_2L_4 is bounded by

$$\|L_2L_4\| = \max_{\eta_1,\eta_2\in(1,\infty)}\left|\frac{\exp(K)}{\pi^2}\int_1^{\infty}\frac{(\eta_2^2-1)^{\frac{1}{2}}\,d\zeta_1}{\zeta_1^2+\eta_2^2-1} \times \int_1^{\infty}\frac{(\zeta_1^2-1)^{\frac{1}{2}}\,d\zeta_2}{\zeta_1^2+\zeta_2^2-1}\right| \le \tfrac{1}{4}\exp(K), \quad \text{(C13)}$$

where

$$K = \max_{\lambda\in(1,\infty)}\left|\frac{1}{\pi}\int_1^{\infty}\frac{\theta(1/t)t\,dt}{(t^2-1)^{\frac{1}{2}}[1+(t^2-1)^{\frac{1}{2}}]} - \left(\frac{1}{\pi}\int_1^{\infty}\frac{\theta(1/t)t\,dt}{(t^2-1)^{\frac{1}{2}}[\lambda+(t^2-1)^{\frac{1}{2}}]} - \frac{1}{\pi}\int_1^{\lambda}\frac{\theta(1/t)t\,dt}{\lambda^2+1-t^2}\right)\right|. \quad \text{(C14)}$$

Finally, we obtain the following interesting result for $0 \le c \not< 2$. In this case one can show that the term in braces above is positive, so that

$$K \le \frac{1}{\pi}\int_1^{\infty}\frac{\theta(1/t)t\,dt}{(t^2-1)^{\frac{1}{2}}[1+(t^2-1)^{\frac{1}{2}}]} = \ln\left(\frac{1}{E(i,i)}\right), \quad \text{(C15)}$$

where $E(k_2, k_1)$ is defined in (5.2). Thus,

$$\|L_2L_4\| \leq 1/4E(i, i). \qquad (C16)$$

The result (C16) should be compared with analogous results in one-dimensional slab geometry.[11]

Using the inequality $\theta(1/t)t \leq \pi$, valid also for $0 < c \lessapprox 2$, we find further that

$$\|L_2L_4\| \leq \frac{1}{4}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^{1/\sqrt{2}} \approx 0.87. \qquad (C17)$$

Due to the numerous inequalities employed to achieve this result, we suspect that the actual norm of L_2L_4 is significantly lower than the above upper bound.

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[9] The density $\varphi(x_1, x_2)$ is bounded along the edge of the quarter space so that $\Phi_1 \sim 1/k_i$ for large k_i, and hence $\Lambda\Phi_1$ has a bounded L_2 norm in the tube T_Φ.
[10] The contours at infinity give no contribution. See Ref. 9.
[11] See Ref. 6, p. 331.

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Exactly Solvable Cell Model with a Melting Transition

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A new cell model for classical particle systems is presented and analyzed. In this model the particles are confined to congruent, interconnected, cubic cells of volume ω centered on the points of a cubic lattice with lattice spacing $1/\gamma$. The particles interact via a 2-body potential of the form $q(\mathbf{r}) + \omega^{-1}K(\gamma\mathbf{r})$. The paper deals with the limiting form of this model in which the cells are very large but their separation is much larger. The free energy density is defined by

$$a(\rho, T) \equiv \lim_{\omega\to\infty} \lim_{\gamma\to 0} \tilde{a}(\rho, T, \gamma, \omega),$$

where $\tilde{a}(\rho, T, \gamma, \omega)$ is the free energy density at density ρ, temperature T, and arbitrary γ and ω. For a very general class of functions q and K, it is proved that $a(\rho, T)$ is given by a variational principle. For a certain class of functions K (including $K \leq 0$), $a(\rho, T)$ is given by the Lebowitz–Penrose generalization of the van der Waals–Maxwell theory. For a different class of functions K the system has crystalline states. When K is chosen so that only particles in nearest-neighbor cells interact and K is isotropic, it is proved that the most general crystalline state of the system has a density distribution with two values ρ_+ and ρ_- arranged in a checkerboard (sodium chloride) pattern. For the special case with K repulsive, $K(0) = 0$ and $q = 0$, the system has a second-order melting transition from a crystalline to a fluid state, with no critical temperature. Various correlation functions are defined and evaluated. In the 1-dimensional nearest-neighbor case, the results include exact versions of the Ornstein–Zernike theory for both fluid and crystalline states. Magnetic systems are also considered. Different special cases of the model yield precisely the Weiss theory of ferromagnetism and the Néel–van Vleck theory of antiferromagnetism.

1. INTRODUCTION

This paper deals with a new cell model for many-body systems. The model is of a general type in that it applies to particle systems and magnetic spin systems, and allows a wide choice of interaction potentials. It is not a realistic model for these systems, but, nevertheless, it exhibits many of their properties and has the advantage of being very amenable to exact treatment. In particular, it has crystalline (or anti-ferromagnetically ordered) states and a melting transition which can be studied in detail.

The explanation of the crystalline state and the phenomenon of melting from the principles of statistical mechanics is an outstanding unsolved problem of theoretical physics. Several simplified models have been studied, but even these are not very well understood. The early theories are of the mean-field type, due mainly to Kirkwood and Monroe.[1] The Lennard-Jones and Devonshire theory,[2] and the model of this paper, are related to these. Recent work[3] has shown that these theories are derivable from the statistical mechanics of a model system. It has been shown[4] that