SIAM J. APPL. MATH.
Vol. 28, No. 3, May 1975

# FLUID DYNAMICS OF LIQUID HELIUM*

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**Abstract.** Liquid helium at low temperatures owes its existence to $\hbar$ through the zero point energy; classically it should be solid. $^4$He the common isotope, owes its peculiar behavior as a fluid to its spin and hence again to $\hbar$; classically the difference between $^3$He and $^4$He should be trivial.

In liquid helium flow we deal with a system which still shows all the usual behavior of a liquid plus some additional strange properties which reflect directly macroscopic quantum effects. The governing equations of motion due largely to Landau and London are, except in their linearized form, not as well founded and most certainly less well confirmed than one would like. Consequently, the experimental fluid dynamicist working with helium should have a field day exploring flow problems in an atmosphere more adventureous than with any ordinary fluid. This indeed is often the case. One does, however, ruefully discover that some of the more interesting and significant flow configurations which one likes to study in this strange field are by no means sufficiently well explored in the corresponding classical cases. One therefore likes to design simple fluid flow experiments which bring out the essentially new properties of He II and permit an experimental contribution to, or decision among, the theories of He II flow. In this spirit, experiments associated with the propagation of shock waves in liquid helium have been initiated at GALCIT. The design and construction of a cryogenic shock tube and its application to liquid helium are discussed in this paper.

**1. Introduction.** The development of a cryogenic shock tube and its application to the study of superfluid dynamics resulted simply from cross-fertilization between parallel work at GALCIT on shock waves, and on He II flow problems. Anyone familiar with elementary shock tube theory realizes that the shock Mach number is limited by the ratio of the velocity of sound in the driver and driven gases and hence for perfect gases by the temperature ratio $T_4/T_1$. This ratio is conventionally increased by increasing $T_4$, but it is obviously even more effective to decrease $T_1$. Indeed, with room temperature helium as driver and cold helium as the driven gas, $T_4/T_1$ can be made as large as 200. This simple idea was the origin of the shock tube development by Rupert and Cummings [1], [2], [3]. The possibility of producing very closely controlled heat and pressure pulses at cryogenic temperatures opens the way to the study of a host of interesting problems in both fluids and solids. The most obvious and spectacular application seemed to us the production and propagation of shock waves in liquid helium. This paper should be considered a progress report on this work.

**2. Shock tube.** The cryogenic shock tube in its present version is shown in Figs. 1 and 2. It differs from the usual shock tube by a unique diaphragm setup which, using a long strip of mylar, permits a change of diaphragms without opening the shock tube. This feature was necessary since it is impossible to open a cryogenically cooled tube without condensation of air, $CO_2$, etc. from the room. Measurements of the $x,t$-diagrams were based on sensitive temperature gauges,

* Received by the editors March 15, 1974. Presented by invitation at an International Symposium on Modern Developments in Fluid Dynamics in honor of the 70th birthday of Sydney Goldstein, held at Haifa, Israel, December 16–23, 1973.

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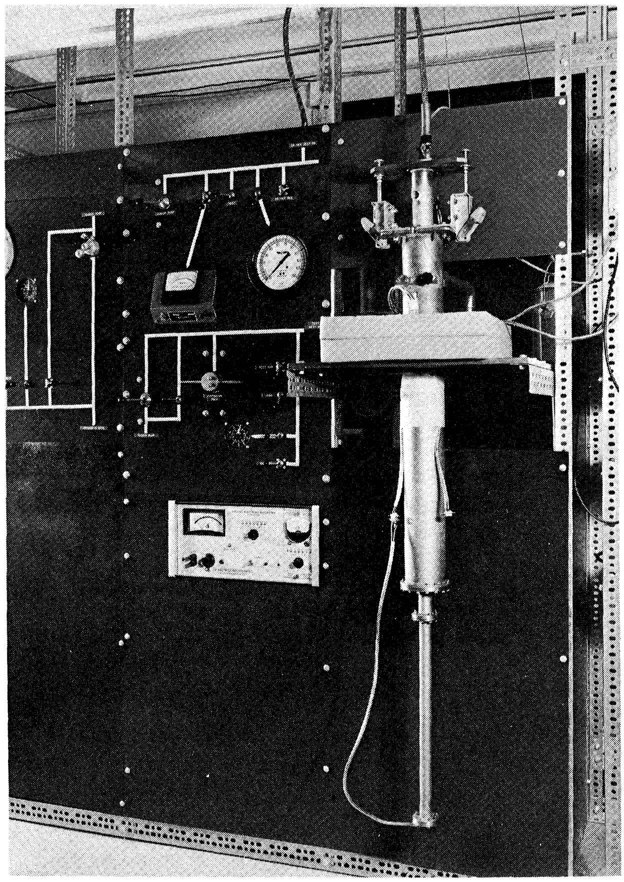


FIG. 1

standard thin film gauges at the higher temperatures and semiconductor elements at cryogenic temperatures. Details of the design and performance can be found in [3].

Shock tube theory applied to the case of the same monatomic gas as driver and driven gas gives, for the limiting shock Mach number $M = c/a_1$,

$$M^2 \leqq 16\frac{T_4}{T_1}.$$

Similar simple limit expressions can be found for the density and temperature ratios across the shock $\rho_2/\rho_1$ and $T_2/T_1$ as well as for the temperature and pressure $T_5$ and $p_5$ reached behind a shock wave reflected from an ideal endwall. Thus

$$\rho_2/\rho_1 \leqq 20T_4/T_1, \qquad T_2 \leqq 5T_4,$$

$$p_5/p_1 \leqq 120T_4/T_1, \qquad T_5 \leqq 12T_4.$$

The pertinent numbers in the present sets of experiments are $T_4 = 300°\text{K}$, $T_1$

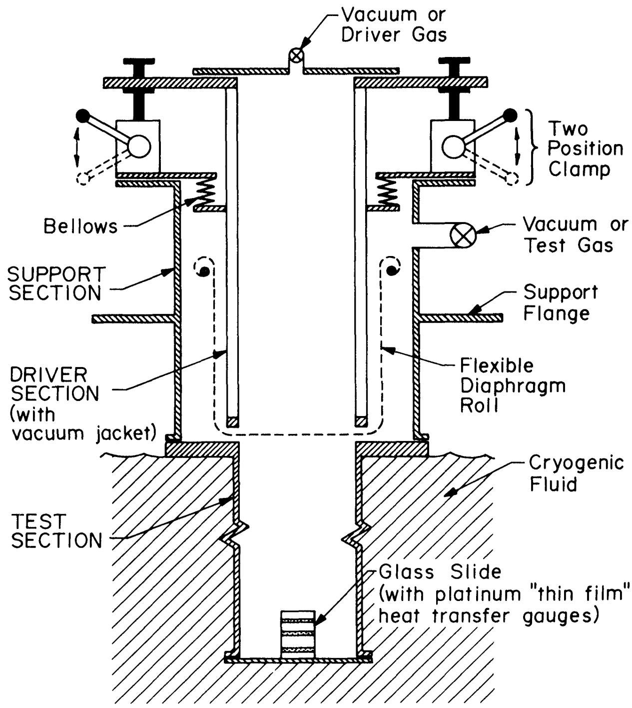



FIG. 2

$\geqq$ 1.3°K and hence for the gas temperature range in the shock tube

$$1.3°\text{K} < T < 3600°\text{K}.$$

Helium for all practical purposes is a perfect gas in this temperature and appropriate pressure range. Indeed the cryogenic shock tube is the only facility for which the simple perfect gas relations apply all the way up to Mach numbers of the order of 40 or more.

The possible range of variables of state in a shock tube gas are limited by the condition that the tube diameter $D$ should be large compared to the shock thickness and hence to the mean free path $\Lambda$. This is essentially a condition on the density $\rho_1$ with some small temperature effects due to the dependence of the collision cross section and mean free path on the temperature. Thus $\rho_1$ cannot be too small because of viscous effects; it cannot be too large in order to avoid condensation. For a perfect gas, consequently,

$$\rho_l R T_1 < p < p_s,$$

where $\rho_l$ denotes the limiting density $\rho_l = \rho_l(D/\Lambda)$, and $p_s$ the saturation pressure.

Since $T_1$ is very low, the inequality is not very stringent, and the viscous effects at a given pressure level are much reduced by cooling. Figure 3 shows Cummings' measurements of shock Mach number as a function of the pressure ratio $p_4/p_1$ with $T_1$ as parameter compared to ideal shock tube theory. The approach to ideal behavior as $T_1$ is lowered is evident.

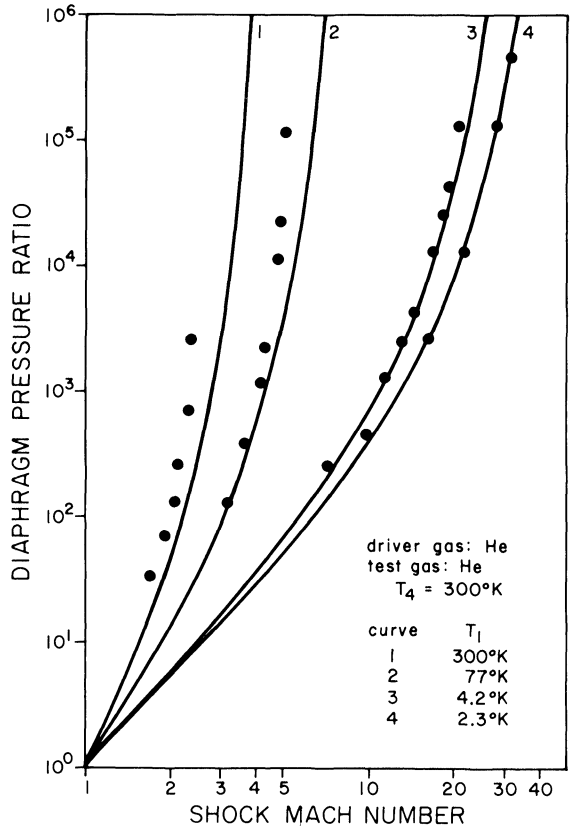



FIG. 3

At first glance it may appear that even stronger shock waves can be obtained using a heavy monatomic gas in the driven section, because then

$$M^2 \leqq 16 \frac{T_4}{T_1} \frac{m_1}{m_4},$$

where $m_1$ and $m_4$ are the molecular masses of the driven and driver gas respectively. Unfortunately, the vapor pressure at a given temperature is correspondingly lower for the heavier gases and hence, the range of shock Mach numbers cannot be substantially increased in this fashion. Of course, the use of a hydrogen driver does increase the performance of a cryogenic helium shock tube.

**3. Macroscopic quantum effects.** A fundamental consequence of quantum mechanics is the existence of a zero point energy, i.e., a finite energy content of matter as $T \to 0$. Hence, at sufficiently low temperatures, any substance will exhibit quantum mechanical effects on a macroscopic scale. For a perfect gas or an ideal solid the necessary dimensionless variables are easily written down using the uncertainty principle. For a gas, the momentum per particle is of order $\sqrt{mkT}$, the characteristic length, the average distance between particles $l$, is related to the number of particles per unit volume $n$ by $l \sim n^{-1/3}$, and hence

$$\frac{\sqrt{mkT}}{n^{1/3}} \sim \hbar$$

relates $n$ and $T$. The corresponding temperature is the degeneracy temperature of gases, very high for electrons in metals, of order $2°K$ for helium. For ideal solids the momentum of the phonons is related to the velocity of sound, $a$, and the uncertainty principle gives

$$\frac{kT}{n^{1/3}a} \sim \hbar,$$

i.e., $T$ is here the Debye-temperature, $\theta$, which for most solids is of order $10^{2}$°K.

This quantum mechanical regime of both fluids and solids is well within the temperature and density range accessible with the cryogenic shock tube. The most obvious of these quantum effects are:

(a) The characteristic difference in the allowed collisions for particles with integer and half-integer spin. This results in a different mean free path for $^3$He and $^4$He respectively, and hence to a different shock wave thickness. The effect has been computed and verified in viscosity measurements. It should be observable without too much trouble.

(b) The specific heat due to the lattice vibrations in a solid vanishes as $(T/\theta)^3$ and consequently, the heat capacity of solid boundaries in the temperature regime of interest here is much less than a liquid or gas at the same temperature, in spite of the large density ratio.

This fact is crucial for the heat transfer from a fluid to a solid and hence for the reflection and transmission of shock waves at fluid-solid interfaces.

(c) The most spectacular of the macroscopic quantum effects occurs in the liquid phase of $^4$He below the $\lambda$-line. The study of shock waves in He II promises to be of real interest for the understanding of the appropriate quantum hydrodynamical equations of motion. The first steps in this direction are reported below.

**4. Shock waves in liquid He II.** The phase diagram of $^4$He is well known (Fig. 4). Helium does not solidify under its own vapor pressure even as $T \to 0$, but exists in two liquid modifications divided by a phase transition line, the so-called $\lambda$-line.

He II is a "superfluid", i.e., it behaves in many respects quite differently than a classical liquid. In particular there exists a real wave velocity for temperature waves, i.e., small temperature variations satisfy a hyperbolic wave equation unlike the parabolic, diffusion-like, equation for ordinary fluids. This wave motion predicted first by Tisza and experimentally realized by Peshkov is, unfortunately, known as "second sound". Thus, a finite reversible heat flux is possible in He II and indeed the transmission of heat from, say, one heated plane boundary to another is quite like the corresponding heat pipe problem in classical heat transfer, and interpreted as a counterflow of a "normal", entropy carrying fluid and a return flow of a "superfluid" with zero entropy.

To illustrate the difference between He II and a classical gas consider, with a view to later shock tube application, the "piston problem" (Fig. 5). The fluid in a semi-infinite tube is set into motion by a piston advancing impulsively with

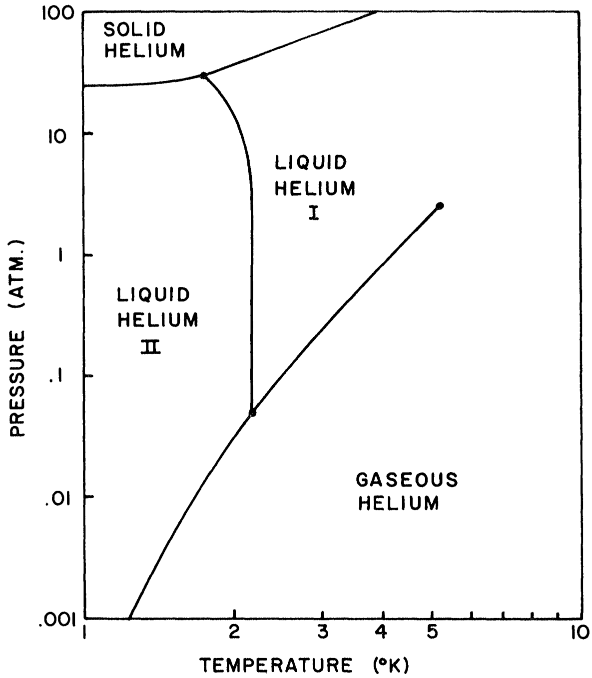

SOLID HELIUM
LIQUID HELIUM I
LIQUID HELIUM II
GASEOUS HELIUM
PRESSURE (ATM.)
100
10
1
.1
.01
.001
1
2
3
4
5
10
TEMPERATURE (°K)

FIG. 4

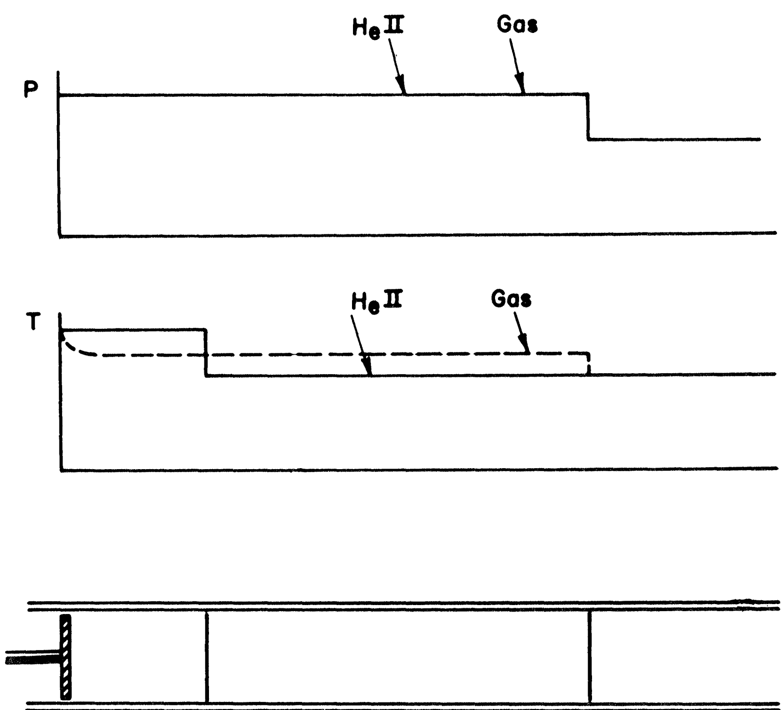

P
He II
Gas
T
He II
Gas

FIG. 5

velocity $U$. The resulting wave propagates with a velocity $c$ related to the piston velocity and piston pressure by

$$p - p_0 = \rho_0 U c$$

for the classical fluid and by a similar expression for He II (identical in fact for weak waves). Hence, the propagation of a pressure wave is similar in the two cases. The corresponding temperature increase, due to the pressure wave, is much smaller in He II than in a gas.

Now consider a heated piston, i.e., a case where not only the velocity but the temperature as well is prescribed at the piston surface. In the classical case, the temperature distribution is as shown in Fig. 5. If the temperature rise through the pressure wave does not happen to coincide with the temperature prescribed on the piston surface, the adjustment must occur via a local temperature boundary layer in a continuous fashion. In He II, however, the adjustment is made discontinuously by means of a temperature wave propagating with a definite wave velocity quite like a shock wave. On the basis of these facts, which stem from the Bose character of helium, London and Landau have developed the two fluid equations for He II, which can be written down conveniently in terms of two velocities $\vec{v}$ and $\vec{w}$ and the ratio $\xi = \rho_n/\rho$,

$$\rho\vec{v} = \rho_n\vec{u}_n + \rho_s\vec{u}_s,$$
$$\vec{w} = \vec{u}_n - \vec{u}_s,$$

in terms of the velocities and densities of the normal and super fluid, respectively. The resulting equations are, of course, much more complicated than the Navier–Stokes equation but the characteristics as well as the shock jump conditions are easily extracted.

The existence of an additional undamped wave motion, second sound, implies an additional term in the thermodynamic identities: $w$ or $\rho_n/\rho \equiv \xi$, say, can be used as new variables of state. For example, the chemical potential $\mu$ in He II is a function of $p$, $T$ and $w$:

$$d\mu = -s\,dT + (1/\rho)\,dp - \xi w\,dw,$$

to second order

$$\mu(p, T, w) = \mu_0(p, T) - \tfrac{1}{2}\xi w^2,$$

as given by Landau. (London uses $\xi$ as an independent variable. The resulting expressions can be transformed into Landau's by a Legendre transformation.)

At first, neglecting quadratic and higher order terms in $w$ we can write

$$dp = a^2 d\rho + b^2\frac{\rho}{s}\,ds,$$

$$dT = \alpha^2\frac{1}{\rho s}\,d\rho + \frac{\beta^2}{s^2}\,ds.$$

$a, b, \alpha, \beta$ have the dimensions of velocities. The vanishing of the characteristic determinant of the equations of motion gives for the angle of the characteristic $\varphi$,

$$\tan^4 \varphi - \left(a^2 + \frac{\rho_s}{\rho_n}\beta^2\right) \tan^2 \varphi - \frac{\rho_s}{\rho_n}(\alpha^2 b^2 - a^2\beta^2) = 0.$$

In terms of the velocity of sound $a$, the ratio of specific heats $\gamma$ and

$$a_2 = \left(\frac{\rho_s}{\rho_n} \frac{s^2 T}{c_p}\right)^{1/2},$$

the corresponding equation for the wave velocity $c$ becomes

$$c^4 - (a^2 + \gamma a_2^2)c^2 + a^2 a_2^2 = 0.$$

Since $\gamma \approx 1$ and $a_2 \ll a$, two nearly uncoupled wave motions with

$$c_1 = a,$$

$$c_2 = a_2$$

result.

The full, nonlinear set of equations of motion relates pressure $p$, density $\rho$, enthalpy $h$ and chemical potential $\mu$ to the wave velocity $c$, the fluid velocities $v$ and $w$ and $\xi$. For a normal shock propagating into undisturbed fluid ($v_0 = w_0 = 0$), these jump conditions are

$$\rho(c - v) = \rho_0 c,$$

$$p - p_0 = \rho_0 c v - \rho\xi(1 - \xi)w^2,$$

$$h - h_0 = v\left(c - \frac{v}{2}\right) - \xi w^2\left[2(1 - \xi) + \frac{1}{2}\xi\right] + \frac{1 - \xi}{c} w \frac{\rho}{\rho_0}[sT + \xi(1 - \xi)w^2],$$

$$\mu - \mu_0 = v\left(c - \frac{v}{2}\right) - \xi w(c - v) - \frac{1}{2}\xi w^2,$$

where $s$ is the entropy/mass, i.e., $sT = h - \mu$.

The set is very complex and has so far been studied very little. A few results are, however, easily seen.

In the linear approximation, i.e., for $v$ and $w$ small,

$$\Delta h = T\Delta S + (1/\rho)\,\Delta p.$$

The system of equations can be split into two parts, a pressure wave for which $\Delta h \approx \Delta p/\rho$ and a "temperature wave" for which $\Delta h \approx T\Delta S$. Thus

$$\Delta\rho = \rho_0 \frac{v}{c}, \qquad \text{i.e.,}\ \frac{\Delta p}{\Delta\rho} = c^2,$$
$$\Delta p = \rho_0 c v,$$

and the pressure wave satisfies the usual sound wave relation

$$c^2 = \left(\frac{dp}{d\rho}\right)_s.$$

On the other hand, with $\Delta h = T\Delta S$ and correspondingly, $\Delta\mu = T\Delta S - \Delta(ST)$

$= -S\Delta T$, we have, since $v \sim \Delta\rho \to 0$,

$$T\Delta S = \frac{1-\xi}{c} wST,$$

$$-S\Delta T = -\xi wc,$$

$$c^2 = \frac{1-\xi}{\xi} S^2 \frac{\Delta T}{\Delta S}.$$

Since

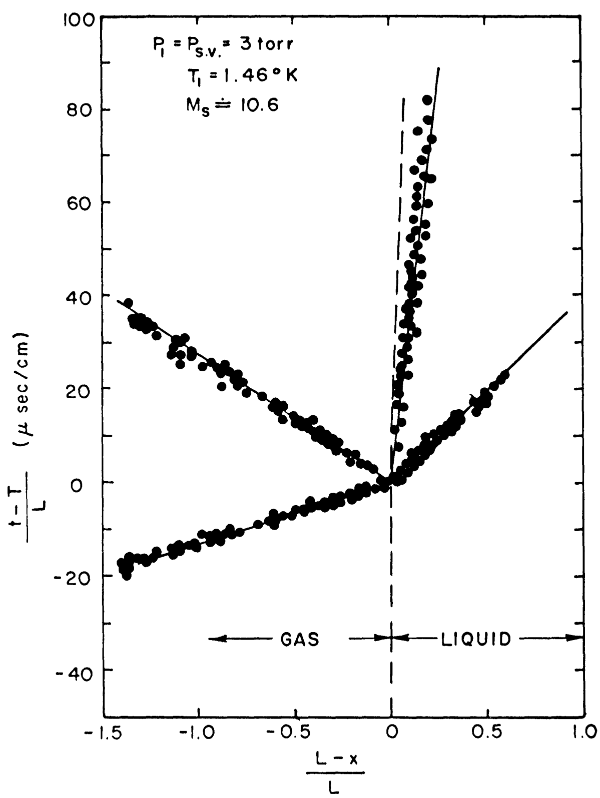
$$\frac{\Delta T}{\Delta S} \to \left(\frac{dT}{dS}\right)_p = \frac{T}{c_p},$$

we have again

$$c^2 = \frac{1-\xi}{\xi} \frac{TS^2}{c_p}$$

for the velocity of "second sound".

The second order theory, i.e., up to terms in $w^2$, is still within reach, and the




L = liquid depth
T = arrival time of incident shock at gas-liquid interface

FIG. 6

results have been given by Kalatnikov. For stronger waves, analytical results become difficult because the functional dependence of the thermodynamic state variables, such as $\mu$ on $w$, are known only up to quadratic terms in $w$.

In any case, one always has to expect two shock waves, one a pressure wave with a relatively small jump in temperature, the other a temperature wave with a relatively small change in pressure.

This behavior is demonstrated by Cummings' experiment in which a shock wave in the cryogenic tube was transmitted through the surface into liquid He II (Fig. 6). The two transmitted shock waves are clearly seen in the $x, t$-diagram. A similar experiment about the $\lambda$-line (Fig. 7) results, as expected, in only one transmitted shock.

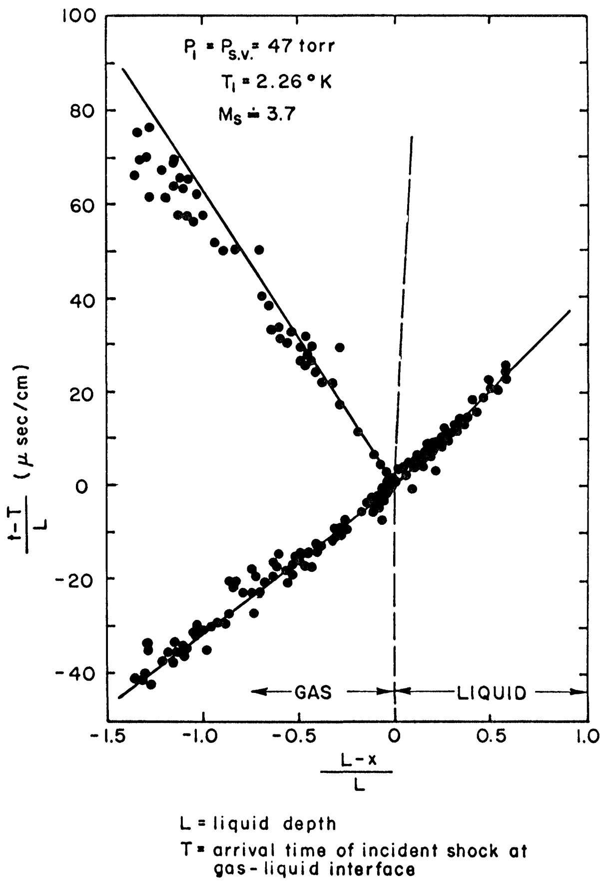



FIG. 7

**5. Conclusion and outlook.** The work done so far has demonstrated that strong shock waves can be produced at cryogenic temperatures, and that these waves can be transmitted into liquid helium. The obvious first step is the study of shock wave propagation in He II. It is interesting enough to study the possible discontinuities and their interaction contained in the complex set of jump conditions of the two-fluid model. It is expected, but by no means certain, that the model is correct beyond the linear and second order terms. Thus a more complete experimental and theoretical exploration of the $x, t$-diagrams is certainly called for. Beyond this rather obvious problem, a host of exciting possibilities exist, e.g., triple intersections and curved shock waves with their vortex sheets are particularly interesting in a medium in which continuous vorticity is restricted to the normal fluid only. Besides this, the study of interfaces and of phase boundaries appears to be quite promising, especially in helium, e.g., it is possible to transmit a shock wave into helium such that the shock transition straddles the $\lambda$-line. Corresponding experiments at the melting line are equally possible.

The general direction of the research was outlined by the author some time ago; it could not have been carried out without the very active participation of Drs. J. E. Broadwell, J. C. Cummings, P. E. Dimotakis and V. C. Rupert who, at various stages of the research, contributed work, ideas and criticism.

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