

Delensing CMB Polarization with External Datasets

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Abstract

One of the primary scientific targets of current and future CMB polarization experiments is the search for a stochastic background of gravity waves in the early universe. As instrumental sensitivity improves, the limiting factor will eventually be B-mode power generated by gravitational lensing, which can be removed through use of so-called “delensing” algorithms. We forecast prospects for delensing using lensing maps which are obtained externally to CMB polarization: either from large-scale structure observations, or from high-resolution maps of CMB temperature. We conclude that the forecasts in either case are not encouraging, and that significantly delensing large-scale CMB polarization requires high-resolution polarization maps with sufficient sensitivity to measure the lensing B-mode. We also present a simple formalism for including delensing in CMB forecasts which is computationally fast and agrees well with Monte Carlos.

1 Introduction

The last two decades of cosmological observations have resulted in a model which is consistent with a wide variety of datasets (e.g. [1, 2, 3, 4, 5]) to an impressive level of precision. The observations are consistent with a flat Λ CDM expansion history, and Gaussian adiabatic initial conditions with a power spectrum which is slightly redder than scale-invariant ($n_s = 0.963 \pm 0.012$).

A major observational frontier in the near future will be the search for a stochastic background of gravity waves on cosmological scales, parameterized by the tensor-to-scalar ratio r . In single-field slow-roll inflation, r is related to the total field variation of the inflaton in Planck units ($\Delta\phi = (M_{Pl}/8\sqrt{\pi}) \int r^{1/2} dN$); “large-field” models with Planck-scale field excursions correspond to $r \gtrsim 10^{-2}$, whereas “small-field” models can have arbitrarily small values of r [6]. The simplest models of inflation, such as a power-law slow-roll potential, have $r \sim 10^{-2}$, and it has been argued that obtaining r smaller than this requires fine-tuning [7]. However, there are also models with a microphysical reason for requiring small r , such as the string-inspired KKLMMT model [8], so this is not a firm prediction. Increasingly precise constraints on r in the next few years will sharpen this picture considerably.

On the observational front, current upper limits are $r \leq 0.36$ (95% C.L.) from large-scale CMB temperature [1], or $r \leq 0.72$ (95% C.L.) from large-scale CMB B-mode polarization [9]. Because the large-scale CMB temperature contains large primary contributions (i.e. Sachs-Wolfe and Integrated Sachs-Wolfe anisotropy), the temperature constraint on r is limited by sample variance (and uncertainty in other cosmological parameters) and cannot be qualitatively improved. In contrast, large-scale B-modes are only generated by gravity waves (or other non-scalar sources, in models which contain them) or nonlinear effects [10, 11], and the polarization constraint on r is currently noise-limited. As instrumental sensitivity improves, the statistical error will vary roughly as $\sigma(r) \propto 1/T$, where T is the sensitivity of the experiment in detector-hours. Since detector sensitivities and array sizes of CMB polarization experiments are currently in a state of rapid development, polarization experiments will give the best constraints on r in the near future, and should soon probe the $r \sim 10^{-3}$ to 10^{-2} range where qualitative classes of early universe models can be ruled out.

In the limit of small instrumental noise, the largest “guaranteed” source of B-mode power on large angular scales is gravitational lensing by large-scale structure, which converts the E-modes generated during recombination to a mixture of E and B [12, 13]. The lensed B-mode power spectrum $C_\ell^{B_{\text{len}}}$ is roughly constant on large angular scales, and can thus be interpreted as an extra source of white noise, whose amplitude is found to be $4.4 \mu\text{K-arcmin}$ for the fiducial cosmology used in this paper. When instrumental noise levels fall below this level, experiments will be *lensing-limited*: the limiting factor in constraining r will be the lensing B-mode, which acts as the dominant source of noise.

“Delensing” algorithms have been proposed [14, 15, 16] which statistically separate the gravity wave and lensing B-mode signals, offering the prospect for reducing the effective noise level below the $4.4 \mu\text{K-arcmin}$ floor due to gravitational lensing. These algorithms are based on the idea that, given a noisy observation of the CMB E-mode polarization and an estimate for the CMB lensing deflection field, one can form an estimate for the large-scale lensing B-mode and subtract it from the observed B-mode to reduce the level of lensing contamination. The required estimate for the deflection field is usually obtained “internally”, using a lens reconstruction estimator [17, 18, 19, 20]

which estimates the lensing deflection field directly from observations of the lensed CMB polarization. High angular resolution is required for delensing (even if one is only interested in delensing B-modes on large angular scales), since the large-scale lensing B-mode receives contributions from E-mode and deflection power on small scales. In this way, after B-mode experiments become lensing-limited, the small and large scales will be intimately linked: the effective noise level for the large-scale B-mode will be determined by the fidelity with which the small-scale polarization is measured, and used to estimate the deflection field.

The purpose of this paper is twofold: first, we present an approximate scheme for doing forecasts which include delensing. This scheme is easy to implement, computationally fast (a few CPU-seconds are needed per forecast) and we find that it agrees well with Monte Carlo based results in the literature; second, we investigate whether delensing can be made more effective by making use of lensing maps which are obtained externally from CMB polarization. Small-scale CMB *temperature* maps will soon be available (e.g. from the Planck satellite [21], or from ground-based experiments such as ACT [22] or SPT [23]) which can be used to reconstruct the CMB lensing map with reasonable signal-to-noise. Approximate CMB lensing maps can also be obtained using observations of large-scale structure (e.g. cosmic shear), but such maps will not have precisely the same redshift weighting as the CMB, where the contribution from large-scale structure at high redshifts is non-negligible. Do either of these sources of lensing information suffice (or help significantly) to delens large-scale CMB polarization?

Throughout this paper, we use the WMAP7+BAO+ H_0 cosmology from [24], with parameters $\Omega_b h^2 = 0.227$, $\Omega_c h^2 = 0.111$, $h = 0.714$, $n_s = 0.969$, $\Delta_{\mathcal{R}}^2(k_0) = 2.38 \times 10^{-9}$ at wavenumber $k_0 = 0.002$ Mpc^{-1} , and $\tau = 0.086$. All power spectra were calculated using CAMB [25], using HALOFIT [26] to model nonlinear evolution.

2 Preliminaries

Gravitational lensing by large-scale structure preserves surface brightness, and therefore remaps the temperature anisotropy and polarization of the CMB (for recent reviews see [27, 28]). The remapping is described mathematically by introducing a vector field $\mathbf{d}(\hat{\mathbf{n}})$ (the deflection field) such that lensed and unlensed CMB temperature and polarization fields are related by:

$$T_{\text{len}}(\hat{\mathbf{n}}) = T_{\text{unl}}(\hat{\mathbf{n}} + \mathbf{d}(\hat{\mathbf{n}})) \quad (1)$$

$$(Q \pm iU)_{\text{len}}(\hat{\mathbf{n}}) = (Q \pm iU)_{\text{unl}}(\hat{\mathbf{n}} + \mathbf{d}(\hat{\mathbf{n}})). \quad (2)$$

This equation is exact in the flat-sky limit, but schematic on the curved sky: here the remapping $\hat{\mathbf{n}} + \mathbf{d}$ represents an angular displacement $|\mathbf{d}|$ from $\hat{\mathbf{n}}$ along a geodesic in the direction given by \mathbf{d} , and there is an implicit parallel transport of the $Q \pm iU$ pseudo-vector [29].

To lowest order in perturbation theory, the deflection field $\mathbf{d}(\hat{\mathbf{n}})$ is the gradient of a scalar lensing potential (i.e. $\mathbf{d}(\hat{\mathbf{n}}) = \nabla\phi(\hat{\mathbf{n}})$) which can be written as a line-of-sight integral:

$$\phi(\hat{\mathbf{n}}) = -2 \int_0^{z_{\text{rec}}} \frac{dz}{H(z)} \Psi(z, D(z)\hat{\mathbf{n}}) \left(\frac{1}{D(z)} - \frac{1}{D(z_{\text{rec}})} \right), \quad (3)$$

where $H(z)$ is the Hubble factor, $\Psi(z, \mathbf{x})$ is the Newtonian potential, $D(z)$ denotes the comoving distance to redshift z , and $\Omega_K = 0$ has been assumed. In the Limber approximation, which we use throughout this paper, the angular power spectrum of the lensing potential is given by

$$C_\ell^{\phi\phi} = 4 \int_0^{z_{\text{rec}}} \frac{dz}{H(z)D(z)^2} P_\Psi(z, k = \ell/D(z)) \left(\frac{1}{D(z)} - \frac{1}{D(z_{\text{rec}})} \right)^2, \quad (4)$$

where $P_\Psi(z, k)$ is the power spectrum of the potential at redshift z . The lensed B-mode $B_{\ell_1 m_1}^{\text{len}}$ is given in terms of the unlensed E-mode $E_{\ell_2 m_2}$ at first order in the lensing potential $\phi_{\ell m}$ by:

$$B_{\ell_1 m_1}^{\text{len}} = \sum_{\ell_2 m_2 \ell m} f_{\ell_1 \ell_2 \ell}^{EB} \begin{pmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{pmatrix} E_{\ell_2 m_2}^* \phi_{\ell m}^*. \quad (5)$$

Here, the coupling coefficient $f_{\ell_1 \ell_2 \ell}^{EB}$ is given by

$$f_{\ell_1 \ell_2 \ell}^{EB} = \frac{F_{\ell_1 \ell_2 \ell}^{-2} - F_{\ell_1 \ell_2 \ell}^2}{2i}, \quad (6)$$

where

$$F_{\ell_1 \ell_2 \ell_3}^s \stackrel{\text{def}}{=} [-\ell_1(\ell_1 + 1) + \ell_2(\ell_2 + 1) + \ell_3(\ell_3 + 1)] \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{16\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -s & s & 0 \end{pmatrix}. \quad (7)$$

A short calculation now shows that the power spectrum of the lensed B-mode is given by:

$$C_{\ell_1}^{B_{\text{len}}} = \frac{1}{2\ell_1 + 1} \sum_{\ell_2 \ell} |f_{\ell_1 \ell_2 \ell}^{EB}|^2 C_{\ell_2}^{EE} C_\ell^{\phi\phi}. \quad (8)$$

In writing down the form (5) of the lensed B-mode, we have made the so-called gradient approximation, i.e. keeping only the lowest order term in the lensing potential, but this has been shown to be an excellent approximation for B-modes with $\ell \lesssim 2000$ [30].

To delens the observed (lensed) B-modes, two pieces of data are required: (1) a measurement $E_{\ell m}^{\text{obs}}$ of the E-mode with noise power spectrum N_ℓ^{EE} , and (2) a measurement $\phi_{\ell m}^{\text{obs}}$ of the lensing potential ϕ with noise power spectrum $N_\ell^{\phi\phi}$. Most commonly, the measurement of the lensing potential is obtained ‘‘internally’’ from the CMB, by applying a lens reconstruction estimator, but the discussion in this section is general and would also apply to an ‘‘external’’ measurement (e.g. estimating ϕ from observations of large-scale structure). One feature of delensing that we wish to emphasize is that both measurements must go from intermediate to small angular scales (roughly $30 \lesssim \ell \lesssim 1000$), even if B-mode delensing is only required on large scales, since small-scale lenses contribute large-scale B-mode power. This is shown directly in Fig. 1.

Intuitively, the idea of delensing is that a lensing-limited measurement $B_{\ell_1 m_1}^{\text{obs}}$ of the large-scale B-mode can be improved by subtracting a quadratic combination of $E_{\ell_2 m_2}^{\text{obs}}$ and $\phi_{\ell m}^{\text{obs}}$ which approximates the lensed B-mode (Eq. (5)) as closely as possible. More formally, the delensing estimator is given by

$$B_{\ell_1 m_1}^{\text{del}} = B_{\ell_1 m_1}^{\text{obs}} - \sum_{\ell_2 m_2 \ell m} f_{\ell_1 \ell_2 \ell}^{EB} \begin{pmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{pmatrix} \left(\frac{C_{\ell_2}^{EE} E_{\ell_2 m_2}^{\text{obs}*}}{C_{\ell_2}^{EE} + N_{\ell_2}^{EE}} \right) \left(\frac{C_\ell^{\phi\phi} \phi_{\ell m}^{\text{obs}*}}{C_\ell^{\phi\phi} + N_\ell^{\phi\phi}} \right). \quad (9)$$

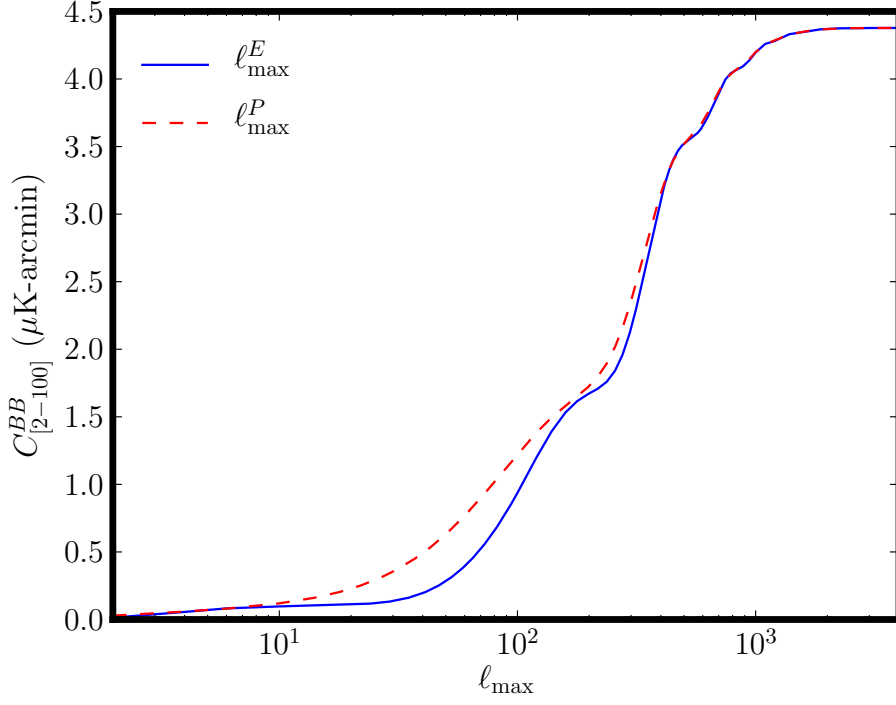


Figure 1: Large scale lensing B -mode power spectrum, interpreted as a pixel noise level in $\mu\text{K}\text{-arcmin}$, for varying choices of maximum E-mode multipole ℓ_{max}^E or deflection multipole ℓ_{max}^P in Eq. (8). It is seen that the large-scale B-mode is generated by E-mode and deflection power on a wide range of angular scales (roughly $30 \lesssim \ell \lesssim 1000$).

If the power spectrum of the original B-mode measurement is the sum of contributions from tensor modes, lensing, and instrumental noise:

$$C_{\ell}^{B_{\text{obs}}} = C_{\ell}^{B_{\text{tens}}} + C_{\ell}^{B_{\text{len}}} + N_{\ell}^{BB}, \quad (10)$$

then the power spectrum of the delensed B-mode will be the sum of contributions from tensors, noise, and residual lensing:

$$C_{\ell}^{B_{\text{del}}} = C_{\ell}^{B_{\text{tens}}} + C_{\ell}^{B_{\text{res}}} + N_{\ell}^{BB}, \quad (11)$$

where the residual lensed B-mode power spectrum after delensing is given by:

$$C_{\ell_1}^{B_{\text{res}}} = \frac{1}{2\ell_1 + 1} \sum_{\ell_2 \ell} |f_{\ell_1 \ell_2 \ell}^{EB}|^2 \left[C_{\ell_2}^{EE} C_{\ell}^{\phi\phi} - \left(\frac{(C_{\ell_2}^{EE})^2}{C_{\ell_2}^{EE} + N_{\ell_2}^{EE}} \right) \left(\frac{(C_{\ell}^{\phi\phi})^2}{C_{\ell}^{\phi\phi} + N_{\ell}^{\phi\phi}} \right) \right] \quad (12)$$

and satisfies $C_{\ell}^{B_{\text{res}}} \leq C_{\ell}^{B_{\text{len}}}$.

The form (9) of the delensing estimator can be understood intuitively as Wiener filtering the observed E and ϕ fields, and using the filtered fields to estimate a lensed B-mode, which is then subtracted from the observed B-mode. A formal derivation is obtained by solving for the weights

on the right-hand side of (9) which minimize the residual power spectrum $C_\ell^{B\text{res}}$. Details of this calculation are given in Appendix A.

On large angular scales ($\ell \lesssim 100$), we find empirically that the residual lensing power spectrum $C_\ell^{B\text{res}}$ is independent of ℓ to an excellent approximation¹ across the wide range of instrumental specifications considered in this paper. This simplifies the interpretation and some implementational details of the forecasts which will follow: the residual lensed B-mode can be interpreted as a source of white noise, parameterized by a single number (the pixel noise in units $\mu\text{K-arcmin}$) which will depend on the instrumental specifications. If no delensing is performed (i.e. $C_\ell^{B\text{res}} = C_\ell^{B\text{len}}$), the pixel noise level is $4.4 \mu\text{K-arcmin}$ in the fiducial cosmology from §1.

3 Forecasts

In this paper, our basic figure of merit will be the statistical error $\sigma(r)$ on the tensor-to-scalar ratio r , in the limit of no tensor modes after delensing the large-scale B-mode, given by:

$$\sigma(r) = \left[\frac{f_{\text{sky}}}{2} \sum_{\ell \geq \ell_{\text{min}}} (2\ell + 1) \left(\frac{\partial C_\ell^{BB} / \partial r}{C_\ell^{B\text{res}} + N_\ell^{BB}} \right)^2 \right]^{-1/2}. \quad (13)$$

Let $\sigma_0(r)$ denote the statistical error without performing delensing; it is given by replacing $C_\ell^{B\text{res}} \rightarrow C_\ell^{B\text{len}}$ in the above expression:

$$\sigma_0(r) = \left[\frac{f_{\text{sky}}}{2} \sum_{\ell \geq \ell_{\text{min}}} (2\ell + 1) \left(\frac{\partial C_\ell^{BB} / \partial r}{C_\ell^{B\text{len}} + N_\ell^{BB}} \right)^2 \right]^{-1/2}. \quad (14)$$

We have included a parameter ℓ_{min} which represents the largest angular scale which can be measured in the presence of a sky cut and large-scale foregrounds. If this scale is large enough to include the signal at $\ell \lesssim 10$ from tensor modes which enter the horizon during reionization, then the error on r will be very sensitive to the value of ℓ_{min} which is assumed. This is illustrated in Fig. 2, which shows $\sigma_0(r)$ as a function of noise level for a few choices of ℓ_{min} . For an ideal all-sky survey with $\ell_{\text{min}} = 2$, the gravity wave measurement gets $\approx 90\%$ of its signal-to-noise from reionization scales ($\ell \lesssim 10$), and $\approx 50\%$ of its signal-to-noise from the B-mode quadrupole ($\ell = 2$) alone. However, these large angular scales are very difficult to measure in suborbital experiments (although not impossible with long-duration ballooning) and it is not clear whether the measuring the reionization signal will be practical in the presence of foregrounds. Although the reionization signal at $\ell \sim 8$ has a higher signal-to-noise ratio than the recombination signal at $\ell \sim 60$, the signal-to-foreground ratio is expected to be higher for the recombination signal [31], assuming that a patch of sky is observed which is chosen to minimize foreground contamination. For these reasons, it is currently unclear whether the reionization or recombination signal will offer better observational prospects for constraining r .

¹To be more quantitative about this, $C_\ell^{B\text{res}}$ is not observationally distinguishable (in the Fisher matrix sense) from a best-fit constant power spectrum, even with cosmic variance limited all-sky measurements for $\ell \leq 100$.

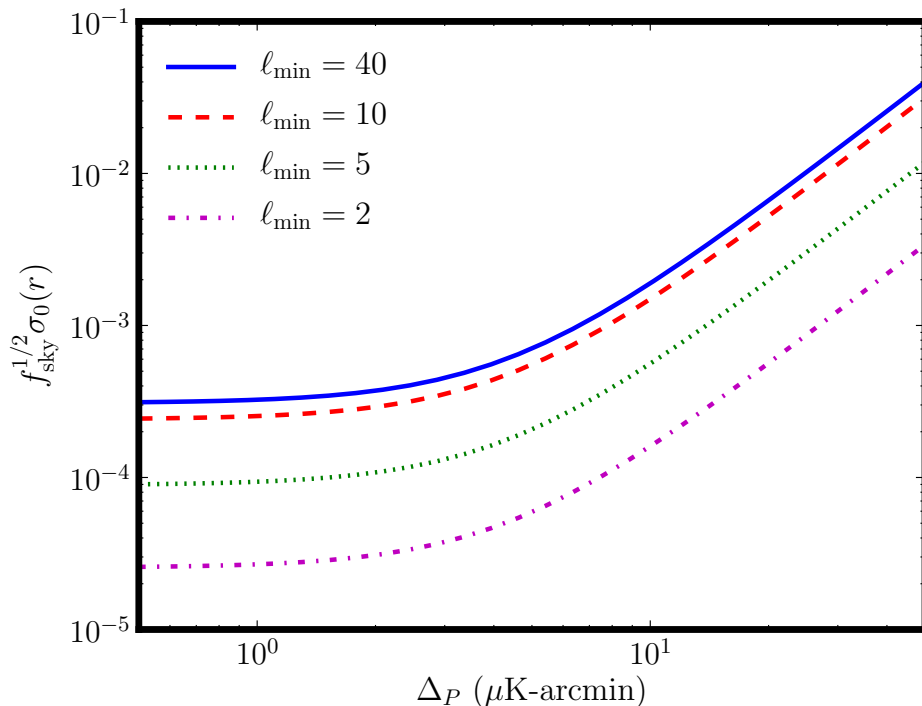


Figure 2: Statistical error $\sigma_0(r)$ on the tensor-to-scalar ratio for varying noise level, assuming no delensing has been done. The forecast is strongly dependent on whether the reionization B-mode signal at $\ell \lesssim 10$ is assumed measurable in the presence of sky cuts and Galactic foregrounds. The value of $\sigma_0(r)$ levels off for noise levels $\lesssim 4.4 \mu\text{K-arcmin}$ since the observations have become lensing-limited.

As remarked in the previous section, the lensed B-mode power spectrum $C_\ell^{B_{\text{len}}}$ and delensed power spectrum $C_\ell^{B_{\text{res}}}$ are independent of ℓ on large scales. We will also assume that the instrumental beam is sufficiently small that N_ℓ^{BB} is also approximately independent of ℓ on large scales. In this case, if we define an improvement factor $\alpha = \sigma_0(r)/\sigma(r)$, then α has the simple form:

$$\alpha = \frac{\sigma_0(r)}{\sigma(r)} = \frac{C_\ell^{B_{\text{len}}} + N_\ell^{BB}}{C_\ell^{B_{\text{res}}} + N_\ell^{BB}}, \quad (15)$$

where the RHS is independent of ℓ (and the value of ℓ_{min} which is assumed).

Eq. (15) states that $\sigma(r)$ factors into two pieces: a factor $\sigma_0(r)$ which depends mainly on survey geometry (i.e. f_{sky} and ℓ_{min}), and a factor $(1/\alpha)$ which represents the improvement due to delensing.² This is a very convenient simplification because it separates the issue of whether the reioinzation B-mode signal is measurable from the issue of how well delensing improves the r constraint. In the following subsections, we will show forecasts for α given various observational scenarios, and

²We have shown this only for a “sharp” cutoff in ℓ parameterized by a cutoff multipole ℓ_{min} , but it would also apply in the case where the number of modes per multipole continuously drops to zero near the fundamental mode, e.g. because ambiguous modes [32, 33] have been projected out.

remain agnostic on the question of whether the reionization signal can be used to obtain a clean measurement of r . The reader should bear in mind that the statistical error $\sigma(r)$ is given by $\sigma_0(r)/\alpha$, where the value of $\sigma_0(r)$ can be read off from Fig. 2 for a given survey geometry.

3.1 Polarization delensing

A B-mode measurement which extends to small angular scales ($\ell \gtrsim 200$) can be used to estimate the lensing potential, by means of lens reconstruction estimators which cross-correlate E and B-modes [17, 18, 19, 20]. By taking this noisy estimate of ϕ as the input to the delensing estimator (9), delensing can be performed in a way which is purely internal to a CMB polarization dataset.

We assume that the E and B-modes have been measured with instrumental noise power spectrum given by

$$N_\ell^{EE} = N_\ell^{BB} = \Delta_P^2 \exp\left(\frac{\theta_{\text{FWHM}}^2 \ell^2}{8 \log 2}\right), \quad (16)$$

where Δ_P is the pixel noise level of the experiment (used with units μK -radian in the above equation, but typically quoted in μK -arcmin).

The noise power spectrum of the lens reconstruction obtained using the EB estimator is given by [34]:

$$N_\ell^{\phi\phi} = \left[\frac{1}{2\ell + 1} \sum_{\ell_1 \ell_2} |f_{\ell_1 \ell_2 \ell}^{EB}|^2 \left(\frac{1}{C_{\ell_1}^{BB} + N_{\ell_1}^{BB}} \right) \left(\frac{(C_{\ell_2}^{EE})^2}{C_{\ell_2}^{EE} + N_{\ell_2}^{EE}} \right) \right]^{-1}. \quad (17)$$

We can plug this into Eq. (12) to obtain $C_\ell^{B^{\text{res}}}$, and then use Eq. (15) to obtain a forecast for α . However, the resulting forecast is incomplete since it does not include the improvements that could be obtained using iterative delensing.

In polarization, the quadratic estimators for the lensing potential and the delensed B-mode that have been discussed so far can be significantly improved for low noise levels using an iterative, likelihood-based approach [20]. The intuition behind the iterative estimators is that lensed B-mode power acts as a source of noise for the lens reconstruction estimator (as can be seen from Eq. (17)), so that the delensed B-mode (which has less power than the lensed B-mode) can be used as the input for a second round of delensing with improved statistical errors, and so on iteratively. One qualitative difference between the two estimators is that in the limit of zero instrumental noise ($N_\ell^{BB} = 0$), the iterative estimator can achieve perfect reconstruction of the lens potential ϕ and perfect delensing of the B-mode. As the instrumental noise improves, there is no fundamental limit to the value of r which can be detected [16], unlike the case of the quadratic estimator [14, 15], although of course there will be some practical limit due to foregrounds and instrumental systematics.

Based on this intuitive picture, we propose the following procedure for forecasting B-mode measurements which use iterative delensing. After computing the residual lensing B-mode power spectrum $C_\ell^{B^{\text{res}}}$ using Eqs. (17), (12) as described above, we recompute $C_\ell^{B^{\text{res}}}$ iteratively until convergence, using the value of $C_\ell^{B^{\text{res}}}$ from the previous iteration in place of $C_\ell^{B^{\text{len}}}$. To state this in a completely

formal way, our forecasting procedure is to iterate the pair of equations

$$N_\ell^{\phi\phi} = \left[\frac{1}{2\ell + 1} \sum_{\ell_1 \ell_2} |f_{\ell_1 \ell_2 \ell}^{EB}|^2 \left(\frac{1}{C_{\ell_1}^{B_{\text{res}}} + N_{\ell_1}^{BB}} \right) \left(\frac{(C_{\ell_2}^{EE})^2}{C_{\ell_2}^{EE} + N_{\ell_2}^{EE}} \right) \right]^{-1} \quad (18)$$

$$C_{\ell_1}^{B_{\text{res}}} = \frac{1}{2\ell_1 + 1} \sum_{\ell_2 \ell} |f_{\ell_1 \ell_2 \ell}^{EB}|^2 \left[C_{\ell_2}^{EE} C_\ell^{\phi\phi} - \left(\frac{(C_{\ell_2}^{EE})^2}{C_{\ell_2}^{EE} + N_{\ell_2}^{EE}} \right) \left(\frac{(C_\ell^{\phi\phi})^2}{C_\ell^{\phi\phi} + N_\ell^{\phi\phi}} \right) \right] \quad (19)$$

to convergence, starting by taking $C_\ell^{B_{\text{res}}} = C_\ell^{B_{\text{len}}}$ in the first iteration.

We have arrived at this forecasting procedure via a heuristic argument, but we can test its validity by comparing with the results in Table I of [16], which show values of $C_\ell^{B_{\text{res}}}$ obtained from Monte Carlo simulations of an iterative delensing estimator, for a wide range of instrumental parameters. We find that all entries in the table agree at the $\approx 10\%$ level, showing that this simple heuristic procedure actually provides rather accurate forecasts. Given the implementational complexity and computational cost of the iterative delensing estimator, this forecasting procedure is one of the main results of this paper. Using the optimizations from Appendix B, iterative delensing forecasts can be generated in a few CPU-seconds.

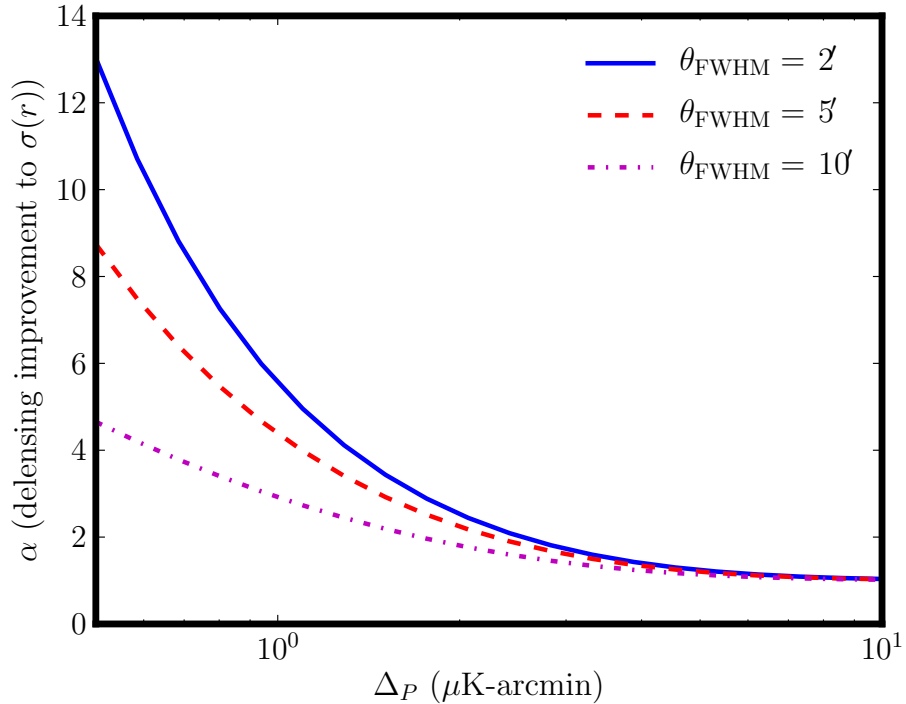


Figure 3: Forecasted improvement $\alpha = \sigma_0(r)/\sigma(r)$ in the statistical error on r due to polarization delensing, for varying noise level and beam. In the limit of low noise and high resolution, we find no limit (from delensing residuals alone) to how well r can be measured.

In Fig. 3, we show forecasts for “internal” lens reconstruction using small-scale CMB polarization, for varying noise level and beam and taking $\ell_{\text{max}} = 4000$ throughout. Delensing can significantly

improve $\sigma(r)$ if the instrumental noise is $\lesssim 4.4 \mu\text{K-arcmin}$ (so that the large-scale B-mode is lensing-limited rather than noise limited) and the beam is small (so that the small-scale lensing B-modes needed for lens reconstruction are measured). As the instrumental noise and resolution improve, we do not find any fundamental limit (due to delensing residuals alone) to how well r can be measured, in agreement with [16].

3.2 Temperature delensing

Lens reconstruction estimators can also be used to estimate the lensing potential from CMB *temperature* measurements which extend to small angular scales [35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]. Lens reconstruction from temperature has been applied to detect CMB lensing in WMAP, in cross-correlation with large-scale structure [46, 47] and is expected to be a powerful source of cosmological information in the near future [48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59]. It is therefore natural to ask whether the lens reconstruction from *temperature* datasets is sufficient to delens large-scale CMB *polarization*. Such a question would be relevant in a scenario where a lensing-limited B-mode measurement has been made on large angular scales, using an instrumental beam which is too large to observe the small-scale lensing B-mode needed for “internal” delensing. Note that, even if an “external” estimate of ϕ is available from CMB temperature, polarization delensing still requires a measurement of E-mode polarization on intermediate to small scales (since the delensing estimator (9) combines E and ϕ on a wide range of scales, in order to delens B on large scales). Therefore, it is not possible to avoid the use of small-scale polarization completely, but the small-scale E-mode measurement is much easier than the small-scale B-mode measurement needed for polarization delensing. (One possible source of the small-scale E-mode measurement would be the Planck dataset, which will release small-scale E-mode measurements on the full sky in the near future.)

Since CMB lens reconstruction estimators are based on the presence of small deviations from Gaussian statistics induced by lensing, any non-Gaussian contribution to the CMB temperature can potentially bias lens reconstruction. Although a detailed characterization of such biases is largely unknown territory, some studies have already found significant biases from non-Gaussian signals such as the kinetic SZ effect [60]. Therefore, the limiting factor in temperature lens reconstruction is likely to be the presence of non-Gaussian secondary anisotropies (which become increasingly important as ℓ increases), rather than instrumental sensitivity or resolution. However, at the time of this writing it is unclear what range of scales will be “sufficiently Gaussian” to use for lens reconstruction in practice. We will model this unclear situation in a rough way by introducing a cutoff multipole ℓ_{max}^T , and assuming that temperature multipoles $\ell \leq \ell_{\text{max}}^T$ can be used for lens reconstruction with the full statistical power of a Gaussian field (i.e. without introducing extra systematic error from secondary anisotropies), but multipoles $\ell > \ell_{\text{max}}^T$ are not used. We will furthermore assume that the measurements for $\ell \leq \ell_{\text{max}}^T$ have been measured with negligible instrumental noise.

Under these assumptions, the noise power spectrum of the lens reconstruction obtained using the TT estimator is given by [34]:

$$N_{\ell}^{TT} = \left[\frac{1}{2(2\ell + 1)} \sum_{\ell_1, \ell_2 \leq \ell_{\text{max}}^T} \frac{(F_{\ell_1 \ell_2 \ell}^0 C_{\ell_2}^{TT} + F_{\ell_2 \ell_1 \ell}^0 C_{\ell_1}^{TT})^2}{(C_{\ell_1}^{TT} + N_{\ell_1}^{TT})(C_{\ell_2}^{TT} + N_{\ell_2}^{TT})} \right]^{-1}. \quad (20)$$

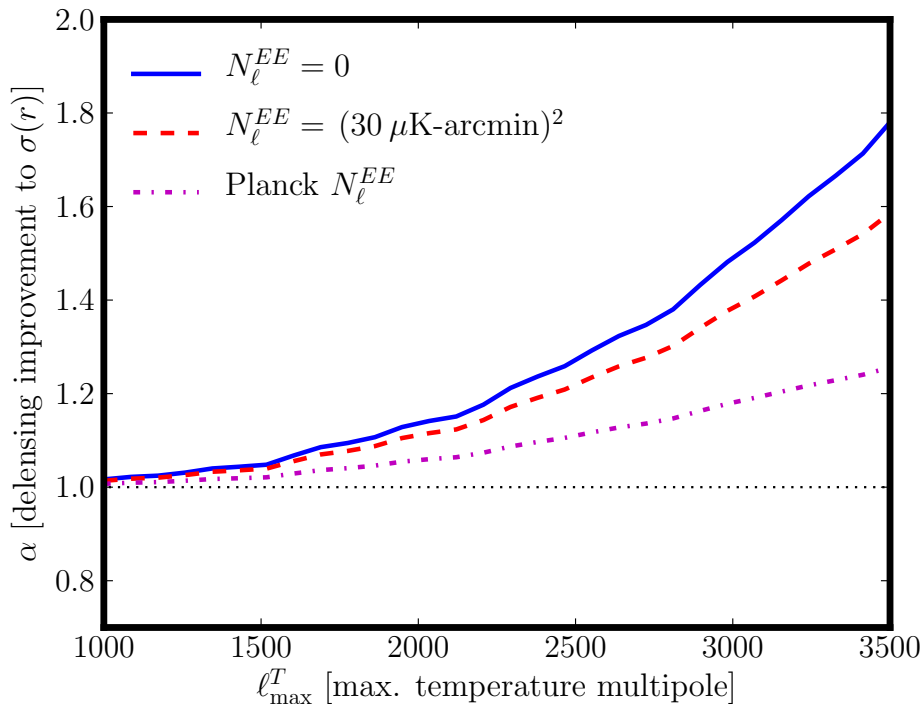


Figure 4: Forecasted improvement $\alpha = \sigma_0(r)/\sigma(r)$ in the statistical error on r , assuming a lensing-limited large-scale B-mode which is delensed using a lens reconstruction from small-scale *temperature*. We assume that the temperature measurements are cosmic-variance limited up to maximum multipole ℓ_{\max}^T (beyond which non-Gaussian secondary anisotropy inhibits lens reconstruction). Large-scale B-mode delensing also requires a measurement of the small-scale E-mode; we consider either the E-mode measurement expected from Planck, a future E-mode measurement with $\Delta_P = 30 \mu\text{K-arcmin}$ and zero beam, or a perfect E-mode measurement.

To obtain a delensing forecast, we must also specify two pieces of data:

1. The noise power spectrum N_ℓ^{BB} on large scales (needed to forecast α in Eq. (15)),
2. The noise power spectrum N_ℓ^{EE} on intermediate to small scales (needed to compute $C_\ell^{B_{\text{res}}}$ in Eq. (12)).

For the first of these, we will take $N_\ell^{BB} = 0$ in order to assume that the large-scale B-mode is purely lensing-limited. For the second, we will consider three possibilities: either (a) Planck noise levels³, (b) E-mode measurements with $\Delta_P = 30 \mu\text{K-arcmin}$ and zero beam, and (c) E-mode measurements with zero noise. Option (c) is artificial, since a zero-noise measurement of the small-scale E-modes would be accompanied by a zero-noise measurement of the lensing B-modes and one could just do

³When making forecasts which include Planck E-mode measurements, we use the parameters from [61]: we assume that the 143 GHz channel ($\Delta_P = 78 \mu\text{K-arcmin}$, $\theta_{\text{FWHM}} = 7.1 \text{ arcmin}$) and the 217 GHz channel ($\Delta_P = 135 \mu\text{K-arcmin}$, $\theta_{\text{FWHM}} = 5 \text{ arcmin}$) can be used with full sensitivity, while channels at lower or higher frequency are “consumed” by foreground cleaning.

polarization delensing, but we include it for the sake of having an upper bound on the possible improvement from temperature delensing, by neglecting uncertainty in the small-scale E-mode.

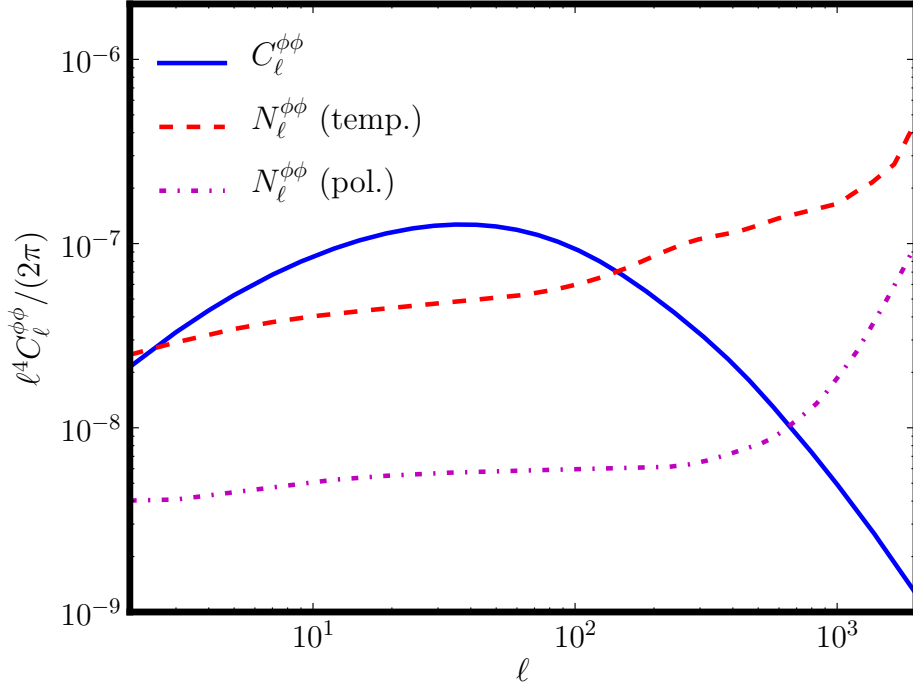


Figure 5: Reconstruction noise power spectra $N_\ell^{\phi\phi}$ resulting from (a) temperature lens reconstruction assuming noise-free measurements to maximum multipole $\ell_{\max}^T = 2500$, (b) polarization lens reconstruction with $\Delta_P = 2 \mu\text{K-arcmin}$ and $\theta_{\text{FWHM}} = 5 \text{ arcmin}$. The signal power spectrum $C_\ell^{\phi\phi}$ is shown for comparison.

In Fig. 4, we show the improvement in $\sigma(r)$ that can be obtained from temperature delensing, for varying ℓ_{\max}^T and small-scale E-mode noise. It is seen that the improvement in $\sigma(r)$ due to delensing is always modest, if the small-scale CMB temperature is the source of the lens reconstruction. Obtaining an improvement larger than $\approx 10\%$ would require both measurements of the small-scale E-mode which are less noisy than Planck, and making the optimistic assumption that temperature multipoles out to $\ell_{\max}^T \approx 3000$ can be cleaned of secondary anisotropy at a level where they can be used for lens reconstruction. This negative result can be understood in a more qualitative way by comparing the noise power spectra $N_\ell^{\phi\phi}$ obtained from temperature and polarization lens reconstruction (Fig. 5). The lensing potential on large scales ($\ell \lesssim 100$) is reconstructed with high signal-to-noise in both cases, but reconstructing lenses on small scales ($\ell \sim 1000$) requires polarization. Since small-scale lenses do generate B-mode power on large scales (by coupling to small-scale E-modes), the temperature reconstruction is of limited effectiveness in delensing the large-scale B-mode.

3.3 Delensing using large-scale structure

Finally, one could ask whether an “external” estimate of ϕ obtained from large-scale structure observations can be used to delens large-scale B-mode polarization. One could imagine using different flavors of large-scale structure data (for example, cosmic shear [62] or 21-cm temperature [63, 64, 65]), with redshift and ℓ weighting optimized to approximate the CMB lensing potential ϕ as closely as possible. The problem encountered in the previous subsection, namely that the lens reconstruction from CMB temperature does not extend to small scales, is not generally an issue for large-scale structure, which can go to even smaller scales than the CMB polarization reconstruction. However, a new problem arises: the CMB lensing potential receives contributions from matter fluctuations at high redshift, and it is not possible to include these contributions if ϕ is being estimated from large-scale structure data at low z .

Let us make the crude assumption that the matter fluctuations have been observed perfectly for redshifts $z \leq z_{\max}$, and have not been observed at all for $z > z_{\max}$. (In a real survey, there will be a smooth transition between redshifts at which the matter density field is signal-dominated versus noise-dominated, but we will use a sharp cutoff to get a rough idea of what to expect.)

A small modification to Eq. (12) for forecasting $C_\ell^{B\text{res}}$ is needed in the large-scale structure case. In deriving Eq. (12), we have assumed that the measurement $\phi_{\ell m}^{\text{obs}}$ of the lensing potential is the sum of the true potential and an uncorrelated noise term (i.e. that $(\phi_{\ell m}^{\text{obs}} - \phi_{\ell m})$ and $\phi_{\ell m}$ are uncorrelated fields). In the large-scale structure case, we write the true potential as the sum of uncorrelated contributions from low and high redshifts: $\phi_{\ell m} = \phi_{\ell m}^{\text{lo}} + \phi_{\ell m}^{\text{hi}}$. We then assume that $\phi_{\ell m}^{\text{lo}}$ has been observed perfectly, that $\phi_{\ell m}^{\text{hi}}$ is unobserved, and that we have an observation $E_{\ell m}^{\text{obs}}$ of the small-scale E-mode with noise power spectrum N_ℓ^{EE} . Given this setup, the minimum-variance delensing estimator and residual lensing B-mode power spectrum are given by:

$$B_{\ell_1 m_1}^{\text{del}} = B_{\ell_1 m_1}^{\text{obs}} - \sum_{\ell_2 m_2 \ell m} f_{\ell_1 \ell_2 \ell}^{EB} \left(\frac{C_{\ell_2}^{EE} E_{\ell_2 m_2}^{\text{obs}*}}{C_{\ell_2}^{EE} + N_{\ell_2}^{EE}} \right) \phi_{\ell m}^{\text{lo}*} \quad (21)$$

$$C_{\ell_1}^{B\text{res}} = \frac{1}{2\ell_1 + 1} \sum_{\ell_2 \ell} |f_{\ell_1 \ell_2 \ell}^{EB}|^2 \left[C_{\ell_2}^{EE} C_\ell^{\phi\phi} - \left(\frac{(C_{\ell_2}^{EE})^2}{C_{\ell_2}^{EE} + N_{\ell_2}^{EE}} \right) C_\ell^{\phi_{\text{lo}}} \right]. \quad (22)$$

This is shown in Appendix A, although the estimator (21) is easy to guess: we simply subtract an estimate for the lensed B-mode which is obtained by combining the Wiener-filtered E-mode and the observed part of the lensing potential. Note that the power spectrum $C_\ell^{\phi_{\text{lo}}}$ can be calculated in the Limber approximation by simply restricting the line-of-sight integral (4) to the redshift range $0 \leq z \leq z_{\max}$.

In Fig. 6, we show the improvement in $\sigma(r)$ that can be obtained from large-scale structure delensing, for varying z_{\max} . As in the temperature case from the previous subsection, this type of delensing also requires a measurement of the small-scale E-mode polarization, and we consider the same three cases: a Planck E-mode measurement, a measurement with $\Delta_P = 30 \mu\text{K-arcmin}$, and a zero-noise measurement. It is seen that the improvement in $\sigma(r)$ due to large-scale structure delensing is modest, at least for survey parameters which are likely to be attainable for the next few generations of CMB polarization experiments. An ambitious galaxy/quasar survey which is deep

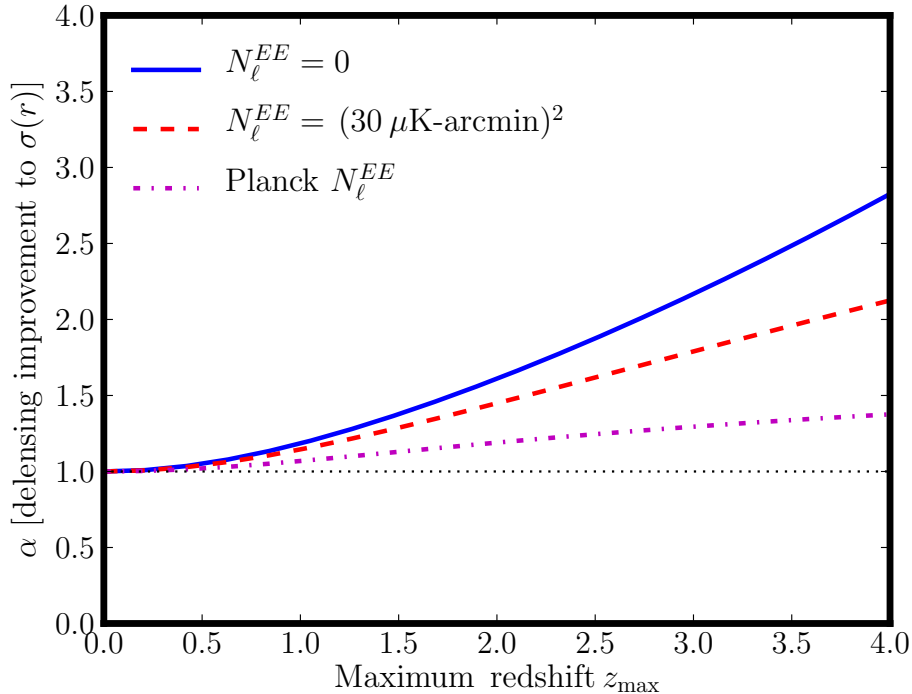


Figure 6: Forecasted improvement $\alpha = \sigma_0(r)/\sigma(r)$ in the statistical error on r , assuming a lensing-limited large-scale B-mode which is delensed using perfect large-scale structure measurements out to redshift z_{\max} . Large-scale B-mode delensing also requires a measurement of the small-scale E-mode; we consider either the E-mode measurement expected from Planck, a future E-mode measurement with $\Delta_P = 30 \mu\text{K-arcmin}$ and zero beam, or a perfect E-mode measurement.

enough to probe structure to $z \sim 3$ with high signal-to-noise can delens by a factor ~ 2 , in combination with small-scale E-mode measurements which are a few times more sensitive than Planck.

Qualitatively similar results were obtained in [62], in a detailed analysis which concentrated on cosmic shear as the large-scale structure observable. Here, we have shown that these conclusions are generic to any observable which does not probe the matter distributions at high redshift. Even with perfect knowledge of the matter fluctuations out to redshift ~ 3 , only relatively modest improvements in $\sigma(r)$ can be obtained.

4 Discussion

In this paper, we have presented forecasts for delensing the gravity wave B-mode, aimed at assessing whether high-sensitivity measurements of CMB polarization on small scales are a requirement, or whether the lensing map can come from a different source. We have also presented a simple framework for including delensing in forecasts, which is approximate but agrees very well with Monte Carlo simulations.

One technical but important point is that calculations throughout this paper have treated all

fields as Gaussian, even “composite” fields which are formed from products of others (such as the lensed B-mode, which is a composite of E and ϕ). In more detail, we have made three approximations as follows. First, the Fisher forecasts for $\sigma(r)$ and $\sigma_0(r)$ (Eqs. (13), (14)) have assumed that the bandpower covariance of the lensed B-mode $B_{\ell m}^{\text{len}}$ and delensed B-mode $B_{\ell m}^{\text{res}}$ are related to the power spectrum in the same way as for a Gaussian field. For the lensed B-mode, it has been shown [66] that this is a good approximation for narrow bandpowers, but breaks down for bandpowers wider than the coherence scale of the lenses ($\Delta\ell \gtrsim 200$). Since the effective bandpower for constraining r is narrow ($\Delta\ell \sim 40$), the approximation should be accurate in this case. It seems plausible that the same qualitative statements are true for the delensed B-mode, but we defer investigation of this issue to further work. The second approximation appears in Eq. (12): when calculating the residual B-mode power spectrum $C_\ell^{B^{\text{res}}}$, we have used Wick’s theorem to compute expectation values of products of the fields B^{unl} , E^{obs} , ϕ^{obs} . More precisely, we have neglected the $(B^{\text{unl}}E^{\text{obs}}\phi^{\text{obs}})$ three-point function and the $(E^{\text{obs}}E^{\text{obs}}\phi^{\text{obs}}\phi^{\text{obs}})$ connected four-point function. The validity of this approximation is unclear in the case where ϕ^{obs} is obtained by applying a quadratic lens reconstruction estimator to the CMB; in this case the 3-point and 4-point terms which have been neglected will “expand” to a nontrivial CMB 4-point or 6-point function. The situation becomes even murkier if the delensing is iterative, since iteration will lead to N -point functions of even higher order. The third approximation we have made is treating the lensing potential ϕ as a Gaussian field. We have included nonlinear corrections to the power spectrum $C_\ell^{\phi\phi}$, but nonlinear evolution also generates higher N -point functions which have been neglected. A complete study of these approximations and potential biases is outside the scope of this paper, and will be pursued separately [67]. As remarked in §3.1, the close agreement between the forecasts in this paper and the Monte Carlo results from [16] suggests that the impact of these approximations is small in forecasts, but may be important in actual data analysis.

We considered the possibility of using the CMB lens reconstruction from small scale *temperature* to help delens large-scale polarization. We found that this is not a promising strategy and that it will be difficult to achieve a reduction in $\sigma(r)$ which is better than $\approx 10\%$ with realistic assumptions about small-scale secondary temperature anisotropy. The limiting factor is poor signal-to-noise in the TT lens reconstruction on small angular scales ($\ell \sim 1000$).

We also considered the possibility of using large-scale structure to help delens the CMB. While the forecasts here are not especially encouraging, they are not as grim as the temperature delensing case. At some point in the future, it may be interesting to obtain a modest (say 50%) improvement in $\sigma(r)$ using large-scale structure, as a proof of concept of the delensing idea. (Large-scale structure delensing may also be easier in practice than polarization delensing, since there are fewer systematics which will cross-correlate the two datasets.) Going beyond this would require small-scale polarization measurements and polarization delensing. Another futuristic possibility is a 21-cm experiment with sufficient sensitivity and resolution to reconstruct the lenses out to redshift ~ 10 [63, 64, 68], but this is unlikely to be available for the next few generations of CMB polarization experiments. We conclude that polarization delensing is the only prospect in the foreseeable future for probing the gravity wave signal significantly beyond the noise floor imposed by gravitational lensing.

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References

- [1] E. Komatsu *et al.*, “Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation,” 2010.
- [2] W. J. Percival *et al.*, “Baryon acoustic oscillations in the sloan digital sky survey data release 7 galaxy sample,” *Mon. Not. Roy. Astron. Soc.*, vol. 401, p. 1741, 2009.
- [3] A. G. Riess *et al.*, “A Redetermination of the Hubble Constant with the Hubble Space Telescope from a Differential Distance Ladder,” *Astrophys. J.*, vol. 699, p. 539, 2009.
- [4] R. Kessler *et al.*, “First-year Sloan Digital Sky Survey-II (SDSS-II) Supernova Results: Hubble Diagram and Cosmological Parameters,” *ApJS*, vol. 185, pp. 32–84, 2009.
- [5] A. Vikhlinin *et al.*, “Chandra Cluster Cosmology Project III: Cosmological Parameter Constraints,” *Astrophys. J.*, vol. 692, pp. 1060–1074, 2009.
- [6] D. H. Lyth, “What would we learn by detecting a gravitational wave signal in the cosmic microwave background anisotropy?,” *Phys. Rev. Lett.*, vol. 78, pp. 1861–1863, 1997.
- [7] L. Boyle, P. J. Steinhardt, and N. Turok, “Inflationary predictions reconsidered,” *Phys. Rev. Lett.*, vol. 96, p. 111301, 2006.
- [8] S. Kachru, R. Kallosh, A. Linde, J. Maldacena, L. McAllister, and S. P. Trivedi, “Towards Inflation in String Theory,” *JCAP*, vol. 0310, p. 013, 2003.
- [9] H. C. Chiang *et al.*, “Measurement of CMB Polarization Power Spectra from Two Years of BICEP Data,” *Astrophys. J.*, vol. 711, pp. 1123–1140, 2010.
- [10] M. Kamionkowski, A. Kosowsky, and A. Stebbins, “Statistics of Cosmic Microwave Background Polarization,” *Phys. Rev.*, vol. D55, pp. 7368–7388, 1997.

- [11] M. Zaldarriaga and U. Seljak, “An All-Sky Analysis of Polarization in the Microwave Background,” *Phys. Rev.*, vol. D55, pp. 1830–1840, 1997.
- [12] M. Zaldarriaga and U. Seljak, “Gravitational Lensing Effect on Cosmic Microwave Background Polarization,” *Phys. Rev.*, vol. D58, p. 023003, 1998.
- [13] W. Hu, “Weak lensing of the CMB: A harmonic approach,” *Phys. Rev.*, vol. D62, p. 043007, 2000.
- [14] L. Knox and Y.-S. Song, “A limit on the detectability of the energy scale of inflation,” *Phys. Rev. Lett.*, vol. 89, p. 011303, 2002.
- [15] M. Kesden, A. Cooray, and M. Kamionkowski, “Separation of gravitational-wave and cosmic-shear contributions to cosmic microwave background polarization,” *Phys. Rev. Lett.*, vol. 89, p. 011304, 2002.
- [16] U. Seljak and C. M. Hirata, “Gravitational lensing as a contaminant of the gravity wave signal in CMB,” *Phys. Rev.*, vol. D69, p. 043005, 2004.
- [17] K. Benabed, F. Bernardeau, and L. Van Waerbeke, “CMB B-polarization to map the Large-scale Structures of the Universe,” *Phys. Rev.*, vol. D63, p. 043501, 2001.
- [18] J. Guzik, U. Seljak, and M. Zaldarriaga, “Lensing effect on polarization in the microwave background: Extracting the convergence power spectrum,” *Phys. Rev.*, vol. D62, p. 043517, 2000.
- [19] W. Hu and T. Okamoto, “Mass Reconstruction with CMB Polarization,” *Astrophys. J.*, vol. 574, pp. 566–574, 2002.
- [20] C. M. Hirata and U. Seljak, “Reconstruction of lensing from the cosmic microwave background polarization,” *Phys. Rev.*, vol. D68, p. 083002, 2003.
- [21] G. Efstathiou, C. Lawrence, and J. Tauber, *The Planck Blue Book*, 2005.
- [22] S. Das *et al.*, “The Atacama Cosmology Telescope: A Measurement of the Cosmic Microwave Background Power Spectrum at 148 and 218 GHz from the 2008 Southern Survey,” 2010.
- [23] M. Lueker *et al.*, “Measurements of Secondary Cosmic Microwave Background Anisotropies with the South Pole Telescope,” *Astrophys. J.*, vol. 719, pp. 1045–1066, 2010.
- [24] D. Larson *et al.*, “Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Power Spectra and WMAP-Derived Parameters,” 2010.
- [25] A. Lewis, A. Challinor, and A. Lasenby, “Efficient Computation of CMB anisotropies in closed FRW models,” *Astrophys. J.*, vol. 538, pp. 473–476, 2000.
- [26] R. E. Smith *et al.*, “Stable clustering, the halo model and nonlinear cosmological power spectra,” *Mon. Not. Roy. Astron. Soc.*, vol. 341, p. 1311, 2003.

- [27] A. Lewis and A. Challinor, “Weak Gravitational Lensing of the CMB,” *Phys. Rept.*, vol. 429, pp. 1–65, 2006.
- [28] D. Hanson, A. Challinor, and A. Lewis, “Weak lensing of the CMB,” *Gen. Rel. Grav.*, vol. 42, pp. 2197–2218, 2010.
- [29] A. Challinor and G. Chon, “Geometry of weak lensing of CMB polarization,” *Phys. Rev.*, vol. D66, p. 127301, 2002.
- [30] A. Challinor and A. Lewis, “Lensed CMB power spectra from all-sky correlation functions,” *Phys. Rev.*, vol. D71, p. 103010, 2005.
- [31] J. Dunkley *et al.*, “CMBPol Mission Concept Study: Prospects for polarized foreground removal,” 2008.
- [32] E. F. Bunn, M. Zaldarriaga, M. Tegmark, and A. d. Oliveira-Costa, “E/B decomposition of finite pixelized CMB maps,” *Phys. Rev.*, vol. D67, p. 023501, 2003.
- [33] K. M. Smith, “Pseudo- C_ℓ estimators which do not mix E and B modes,” *Phys. Rev.*, vol. D74, p. 083002, 2006.
- [34] T. Okamoto and W. Hu, “CMB Lensing Reconstruction on the Full Sky,” *Phys. Rev.*, vol. D67, p. 083002, 2003.
- [35] F. Bernardeau, “Weak Lensing Detection in CMB Maps,” *Astron. Astrophys.*, vol. 324, pp. 15–26, 1997.
- [36] F. Bernardeau, “Lens distortion effects on CMB maps,” *Astron. Astrophys.*, vol. 338, pp. 767–776, 1998.
- [37] M. Zaldarriaga and U. Seljak, “Reconstructing projected matter density from cosmic microwave background,” *Phys. Rev.*, vol. D59, p. 123507, 1999.
- [38] M. Zaldarriaga, “Lensing of the CMB: Non-Gaussian aspects,” *Phys. Rev.*, vol. D62, p. 063510, 2000.
- [39] W. Hu, “Angular trispectrum of the cosmic microwave background,” *Phys. Rev.*, vol. D64, p. 083005, 2001.
- [40] W. Hu, “Mapping the Dark Matter through the CMB Damping Tail,” *Astrophys. J.*, vol. 557, pp. L79–L83, 2001.
- [41] A. Cooray and M. Kesden, “Weak Lensing of the CMB: Extraction of Lensing Information from the Trispectrum,” *New Astron.*, vol. 8, pp. 231–253, 2003.
- [42] T. Okamoto and W. Hu, “The Angular Trispectra of CMB Temperature and Polarization,” *Phys. Rev.*, vol. D66, p. 063008, 2002.

- [43] C. M. Hirata and U. Seljak, “Analyzing weak lensing of the cosmic microwave background using the likelihood function,” *Phys. Rev.*, vol. D67, p. 043001, 2003.
- [44] M. H. Kesden, A. Cooray, and M. Kamionkowski, “Lensing Reconstruction with CMB Temperature and Polarization,” *Phys. Rev.*, vol. D67, p. 123507, 2003.
- [45] C. M. Hirata, N. Padmanabhan, U. Seljak, D. Schlegel, and J. Brinkmann, “Cross-correlation of CMB with large-scale structure: weak gravitational lensing,” *Phys. Rev.*, vol. D70, p. 103501, 2004.
- [46] K. M. Smith, O. Zahn, and O. Doré, “Detection of Gravitational Lensing in the Cosmic Microwave Background,” *Phys. Rev.*, vol. D76, p. 043510, 2007.
- [47] C. M. Hirata, S. Ho, N. Padmanabhan, U. Seljak, and N. A. Bahcall, “Correlation of CMB with large-scale structure: II. Weak lensing,” *Phys. Rev.*, vol. D78, p. 043520, 2008.
- [48] W. Hu, N. Sugiyama, and J. Silk, “The Physics of microwave background anisotropies,” *Nature*, vol. 386, pp. 37–43, 1997.
- [49] M. Zaldarriaga, D. N. Spergel, and U. Seljak, “Microwave Background Constraints on Cosmological Parameters,” *Astrophys. J.*, vol. 488, pp. 1–13, 1997.
- [50] R. B. Metcalf and J. Silk, “Gravitational Magnification of the Cosmic Microwave Background,” *Astrophys. J.*, vol. 489, pp. 1–6, 1997.
- [51] R. Stompor and G. Efstathiou, “Gravitational lensing of cosmic microwave background anisotropies and cosmological parameter estimation,” *Mon. Not. Roy. Astron. Soc.*, vol. 302, p. 735, 1999.
- [52] W. Hu, M. Fukugita, M. Zaldarriaga, and M. Tegmark, “CMB Observables and Their Cosmological Implications,” *Astrophys. J.*, vol. 549, p. 669, 2001.
- [53] W. Hu and S. Dodelson, “Cosmic Microwave Background Anisotropies,” *Ann. Rev. Astron. Astrophys.*, vol. 40, pp. 171–216, 2002.
- [54] W. Hu, “Dark Synergy: Gravitational Lensing and the CMB,” *Phys. Rev.*, vol. D65, p. 023003, 2002.
- [55] K. N. Abazajian and S. Dodelson, “Neutrino mass and dark energy from weak lensing,” *Phys. Rev. Lett.*, vol. 91, p. 041301, 2003.
- [56] M. Kaplinghat, L. Knox, and Y.-S. Song, “Determining neutrino mass from the CMB alone,” *Phys. Rev. Lett.*, vol. 91, p. 241301, 2003.
- [57] J. Lesgourgues, L. Perotto, S. Pastor, and M. Piat, “Probing neutrino masses with CMB lensing extraction,” *Phys. Rev.*, vol. D73, p. 045021, 2006.

- [58] L. Perotto, J. Lesgourgues, S. Hannestad, H. Tu, and Y. Y. Y. Wong, “Probing cosmological parameters with the CMB: Forecasts from full Monte Carlo simulations,” *JCAP*, vol. 0610, p. 013, 2006.
- [59] R. de Putter, O. Zahn, and E. V. Linder, “CMB Lensing Constraints on Neutrinos and Dark Energy,” *Phys. Rev.*, vol. D79, p. 065033, 2009.
- [60] A. Amblard, C. Vale, and M. J. White, “Weak Lensing of the CMB by Large-Scale Structure,” *New Astron.*, vol. 9, pp. 687–704, 2004.
- [61] A. Albrecht *et al.*, “Report of the Dark Energy Task Force,” 2006.
- [62] L. Marian and G. M. Bernstein, “Detectability of CMB tensor B modes via delensing with weak lensing galaxy surveys,” *Phys. Rev.*, vol. D76, p. 123009, 2007.
- [63] O. Zahn and M. Zaldarriaga, “Lensing Reconstruction using redshifted 21cm Fluctuations,” *Astrophys. J.*, vol. 653, pp. 922–935, 2006.
- [64] K. Sigurdson and A. Cooray, “Cosmic 21-cm delensing of microwave background polarization and the minimum detectable energy scale of inflation,” *Phys. Rev. Lett.*, vol. 95, p. 211303, 2005.
- [65] K. S. Mandel and M. Zaldarriaga, “Weak Gravitational Lensing of High-Redshift 21 cm Power Spectra,” *Astrophys. J.*, vol. 647, p. 719, 2006.
- [66] K. M. Smith, W. Hu, and M. Kaplinghat, “Weak Lensing of the CMB: Sampling Errors on B-Modes,” *Phys. Rev.*, vol. D70, p. 043002, 2004.
- [67] C. Dvorkin, D. Hanson, and K. M. Smith, “Quadratic and iterative delensing of the CMB B-modes,” *to appear*.
- [68] S. Hilbert, R. B. Metcalf, and S. D. M. White, “Imaging the Cosmic Matter Distribution using Gravitational Lensing of Pregalactic HI,” *Mon. Not. Roy. Astron. Soc.*, vol. 382, p. 1494, 2007.
- [69] K. M. Smith *et al.*, “CMBPol Mission Concept Study: Gravitational Lensing,” 2008.
- [70] C. Dvorkin, W. Hu, and K. M. Smith, “B-mode CMB Polarization from Patchy Screening during Reionization,” *Phys. Rev.*, vol. D79, p. 107302, 2009.

A Appendix: minimum-variance estimators

In this appendix, we derive the minimum-variance delensing estimator and the residual B-mode power spectrum, in the lens reconstruction case (Eqs. (9), (12)) and the large-scale structure case (Eqs. (21), (22)).

Considering first the lens reconstruction case, we assume that the observed B-mode is the sum of contributions from tensor modes, noise, and lensing:

$$\begin{aligned} B_{\ell_1 m_1}^{\text{obs}} &= B_{\ell_1 m_1}^{\text{tens}} + B_{\ell_1 m_1}^{\text{noise}} + B_{\ell_1 m_1}^{\text{len}} \\ &= B_{\ell_1 m_1}^{\text{tens}} + B_{\ell_1 m_1}^{\text{noise}} + \sum_{\ell_2 m_2 \ell m} f_{\ell_1 \ell_2 \ell}^{EB} \begin{pmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{pmatrix} E_{\ell_2 m_2}^* \phi_{\ell m}^*, \end{aligned} \quad (23)$$

and that we have noisy observations $E_{\ell_2 m_2}^{\text{obs}}$, $\phi_{\ell m}^{\text{obs}}$ of the small-scale E-mode and lensing potential, with noise power spectra $N_{\ell_2}^{EE}$ and $N_{\ell}^{\phi\phi}$. We consider a general delensing estimator of the form

$$B_{\ell_1 m_1}^{\text{del}} = B_{\ell_1 m_1}^{\text{obs}} - \sum_{\ell_2 m_2 \ell m} g_{\ell_1 \ell_2 \ell} \begin{pmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{pmatrix} E_{\ell_2 m_2}^{\text{obs}*} \phi_{\ell m}^{\text{obs}*} \quad (24)$$

with weights $g_{\ell_1 \ell_2 \ell}$ to be determined. The form of the second term (in which g does not depend on m, m_1, m_2) is the most general form which is bilinear in E, ϕ and satisfies rotation invariance. A short calculation shows that the power spectrum of the delensed B-mode is given by $C_{\ell}^{B^{\text{del}}} = C_{\ell}^{B^{\text{tens}}} + N_{\ell}^{BB} + C_{\ell}^{B^{\text{res}}}$, where

$$\begin{aligned} C_{\ell_1}^{B^{\text{res}}} &= \frac{1}{2\ell_1 + 1} \sum_{\ell_2 \ell} \left[(f_{\ell_1 \ell_2 \ell}^{EB} f_{\ell_1 \ell_2 \ell}^{EB*} - f_{\ell_1 \ell_2 \ell}^{EB} g_{\ell_1 \ell_2 \ell}^* - g_{\ell_1 \ell_2 \ell} f_{\ell_1 \ell_2 \ell}^{EB*}) C_{\ell_2}^{EE} C_{\ell}^{\phi\phi} \right. \\ &\quad \left. + g_{\ell_1 \ell_2 \ell} g_{\ell_1 \ell_2 \ell}^* (C_{\ell_2}^{EE} + N_{\ell_2}^{EE}) (C_{\ell}^{\phi\phi} + N_{\ell}^{\phi\phi}) \right]. \end{aligned} \quad (25)$$

Now we determine $g_{\ell_1 \ell_2 \ell}$ by minimizing the residual power spectrum $C_{\ell}^{B^{\text{res}}}$. Differentiating (25) with respect to g , we find that the minimum occurs at

$$g_{\ell_1 \ell_2 \ell} = f_{\ell_1 \ell_2 \ell}^{EB} \left(\frac{C_{\ell_2}^{EE}}{C_{\ell_2}^{EE} + N_{\ell_2}^{EE}} \right) \left(\frac{C_{\ell}^{\phi\phi}}{C_{\ell}^{\phi\phi} + N_{\ell}^{\phi\phi}} \right) \quad (26)$$

and the value of $C_{\ell}^{B^{\text{res}}}$ at minimum is given by

$$C_{\ell_1}^{B^{\text{res}}} = \frac{1}{2\ell_1 + 1} \sum_{\ell_2 \ell} |f_{\ell_1 \ell_2 \ell}^{EB}|^2 \left[C_{\ell_2}^{EE} C_{\ell}^{\phi\phi} - \left(\frac{(C_{\ell_2}^{EE})^2}{C_{\ell_2}^{EE} + N_{\ell_2}^{EE}} \right) \left(\frac{(C_{\ell}^{\phi\phi})^2}{C_{\ell}^{\phi\phi} + N_{\ell}^{\phi\phi}} \right) \right]. \quad (27)$$

This completes the derivation of Eqs. (9), (12) and concludes the lens reconstruction case.

Moving on to the large-scale structure case, we assume that the lensing potential is a sum of uncorrelated terms

$$\phi_{\ell m} = \phi_{\ell m}^{\text{lo}} + \phi_{\ell m}^{\text{hi}}, \quad (28)$$

where $\phi_{\ell m}^{\text{lo}}$ has been observed without noise, and that we have a noisy observation $E_{\ell_2 m_2}^{\text{obs}}$ of the small-scale E-mode with noise power spectrum N_{ℓ}^{EE} .

We consider the delensing estimator:

$$B_{\ell_1 m_1}^{\text{del}} = B_{\ell_1 m_1}^{\text{obs}} - \sum_{\ell_2 m_2 \ell m} h_{\ell_1 \ell_2 \ell} \begin{pmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{pmatrix} E_{\ell_2 m_2}^{\text{obs}*} \phi_{\ell m}^{\text{lo}*}. \quad (29)$$

In this case, the residual B-mode power spectrum is given by

$$C_{\ell_1}^{B_{\text{res}}} = \frac{1}{2\ell_1 + 1} \sum_{\ell_2 \ell} \left[f_{\ell_1 \ell_2 \ell}^{EB} f_{\ell_1 \ell_2 \ell}^{EB*} C_{\ell_2}^{EE} C_{\ell}^{\phi} + h_{\ell_1 \ell_2 \ell} h_{\ell_1 \ell_2 \ell}^* (C_{\ell_2}^{EE} + N_{\ell_2}^{EE}) C_{\ell}^{\phi_{10}} - (f_{\ell_1 \ell_2 \ell}^{EB} h_{\ell_1 \ell_2 \ell}^* + h_{\ell_1 \ell_2 \ell} f_{\ell_1 \ell_2 \ell}^{EB*}) C_{\ell_2}^{EE} C_{\ell}^{\phi_{10}} \right]. \quad (30)$$

Solving for the weights $h_{\ell_1 \ell_2 \ell}$ which minimize $C_{\ell}^{B_{\text{res}}}$, we get

$$h_{\ell_1 \ell_2 \ell} = f_{\ell_1 \ell_2 \ell}^{EB} \left(\frac{C_{\ell_2}^{EE}}{C_{\ell_2}^{EE} + N_{\ell_2}^{EE}} \right) \quad (31)$$

and residual B-mode power spectrum

$$C_{\ell_1}^{B_{\text{res}}} = \frac{1}{2\ell_1 + 1} \sum_{\ell_2 \ell} |f_{\ell_1 \ell_2 \ell}^{EB}|^2 \left[C_{\ell_2}^{EE} C_{\ell}^{\phi\phi} - \left(\frac{(C_{\ell_2}^{EE})^2}{C_{\ell_2}^{EE} + N_{\ell_2}^{EE}} \right) C_{\ell}^{\phi_{10}} \right]. \quad (32)$$

This completes the derivation of Eqs. (21), (22) and finishes the large-scale structure case.

B Appendix: fast real-space expressions

The forecasts in this paper require repeated calculations of the lensed B-mode power spectrum (Eq. (8)) and lens reconstruction power spectra in the TT (Eq. 20) and EB (Eq. (17)) cases. We have written expressions for these quantities as harmonic-space sums whose computational cost is $\mathcal{O}(\ell_{\text{max}}^3)$. In this appendix, we will present equivalent real-space expressions which can be evaluated exactly with cost $\mathcal{O}(\ell_{\text{max}}^2)$. The method to convert these expressions to real space was originally proposed in [70]. This optimization is very convenient since forecasts can be done in a few CPU-seconds for a given set of instrumental specifications, permitting rapid exploration of parameter space.

We first note that the F symbol defined previously in Eq. (7) can be equivalently written

$$F_{\ell_1 \ell_2 \ell_3}^s = -\sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(\ell_3)(\ell_3 + 1)(2\ell_3 + 1)}{16\pi}} \times \left[\sqrt{(\ell_2 - s)(\ell_2 + s + 1)} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -s & s + 1 & -1 \end{pmatrix} + \sqrt{(\ell_2 + s)(\ell_2 - s + 1)} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -s & s - 1 & 1 \end{pmatrix} \right]. \quad (33)$$

We also have the identity

$$\int_{-1}^1 d(\cos \theta) d_{s_1 s_1'}^{\ell_1}(\theta) d_{s_2 s_2'}^{\ell_2}(\theta) d_{s_3 s_3'}^{\ell_3}(\theta) = 2 \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ s_1 & s_2 & s_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ s_1' & s_2' & s_3' \end{pmatrix}, \quad (34)$$

where $d_{s_i s_i'}^{\ell_i}$ are Wigner d -functions such that $s_1 + s_2 + s_3 = s_1' + s_2' + s_3' = 0$.

Armed with identities (33), (34), a long but straightforward calculation shows that Eq. (8) for the lensed B-mode power spectrum is equivalent to:

$$C_\ell^{B_{\text{lens}}} = \frac{\pi}{4} \int d(\cos \theta) \left(\zeta_{3,3}^E(\theta) \zeta_+^\phi(\theta) + 2\zeta_{3,1}^E(\theta) \zeta_-^\phi(\theta) + \zeta_{1,1}^E(\theta) \zeta_+^\phi(\theta) \right) d_{22}^\ell(\theta) - \frac{\pi}{4} \int d(\cos \theta) \left(\zeta_{3,-3}^E(\theta) \zeta_-^\phi(\theta) + 2\zeta_{3,-1}^E(\theta) \zeta_+^\phi(\theta) + \zeta_{1,-1}^E(\theta) \zeta_-^\phi(\theta) \right) d_{2,-2}^\ell(\theta), \quad (35)$$

where the correlation functions which appear are defined by

$$\zeta_{3,\pm 3}^E(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{EE} (\ell-2)(\ell+3) d_{3,\pm 3}^\ell(\theta) \quad (36)$$

$$\zeta_{3,\pm 1}^E(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{EE} \sqrt{(\ell-1)(\ell+2)(\ell-2)(\ell+3)} d_{3,\pm 1}^\ell(\theta) \quad (37)$$

$$\zeta_{1,\pm 1}^E(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{EE} (\ell-1)(\ell+2) d_{1,\pm 1}^\ell(\theta) \quad (38)$$

$$\zeta_\pm^\phi(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{\phi\phi} \ell(\ell+1) d_{1,\pm 1}^\ell(\theta). \quad (39)$$

To obtain a fast algorithm for computing $C_\ell^{B_{\text{lens}}}$, we first note that the integral on the RHS of (35) can be evaluated exactly using Gauss-Legendre quadrature with $\lceil (3\ell_{\text{max}} + 1)/2 \rceil$ points, since the integrand is a polynomial of degree $\leq 3\ell_{\text{max}}$. We precompute the correlation functions in Eqs. (36)–(39) at the Gauss-Legendre quadrature points with computational cost $\mathcal{O}(\ell_{\text{max}}^2)$, evaluating the Wigner d -functions by upward recursion in ℓ . The integral can then be done for all values of ℓ with the same $\mathcal{O}(\ell_{\text{max}}^2)$ cost, using the same recursion. This completes the fast algorithm for $C_\ell^{B_{\text{lens}}}$.

Analogously, a fast algorithm for the EB lens reconstruction noise can be obtained using the expression (equivalent to Eq. (17)):

$$N_\ell^{\phi\phi} = \left[\frac{\pi}{4} \ell(\ell+1) \int d(\cos \theta) \left(\zeta_{3,3}^{E'}(\theta) \zeta_+^B(\theta) - 2\zeta_{3,-1}^{E'}(\theta) \zeta_-^B(\theta) + \zeta_{1,1}^{E'}(\theta) \zeta_+^B(\theta) \right) d_{11}^\ell(\theta) - \frac{\pi}{4} \ell(\ell+1) \int d(\cos \theta) \left(\zeta_{3,-3}^{E'}(\theta) \zeta_-^B(\theta) - 2\zeta_{3,1}^{E'}(\theta) \zeta_+^B(\theta) + \zeta_{1,-1}^{E'}(\theta) \zeta_-^B(\theta) \right) d_{1,-1}^\ell(\theta) \right]^{-1} \quad (40)$$

with correlation functions defined by

$$\zeta_{3,\pm 3}^{E'}(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} \frac{(C_\ell^{EE})^2}{C_\ell^{EE} + N_\ell^{EE}} (\ell-2)(\ell+3) d_{3,\pm 3}^\ell(\theta) \quad (41)$$

$$\zeta_{3,\pm 1}^{E'}(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} \frac{(C_\ell^{EE})^2}{C_\ell^{EE} + N_\ell^{EE}} \sqrt{(\ell-1)(\ell+2)(\ell-2)(\ell+3)} d_{3,\pm 1}^\ell(\theta) \quad (42)$$

$$\zeta_{1,\pm 1}^{E'}(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} \frac{(C_\ell^{EE})^2}{C_\ell^{EE} + N_\ell^{EE}} (\ell-1)(\ell+2) d_{1,\pm 1}^\ell(\theta) \quad (43)$$

$$\zeta_\pm^B(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} \frac{1}{C_\ell^{BB} + N_\ell^{BB}} d_{2,\pm 2}^\ell(\theta). \quad (44)$$

For the TT lens reconstruction noise , we use the expression (equivalent to Eq. (20)):

$$N_\ell^{\phi\phi} = \left[\pi\ell(\ell+1) \int d(\cos\theta) (\zeta_{00}^T(\theta)\zeta_{11}^T(\theta) - \zeta_{01}^T(\theta)\zeta_{01}^T(\theta)) d_{-1,-1}^\ell(\theta) + \pi\ell(\ell+1) \int d(\cos\theta) (\zeta_{00}^T(\theta)\zeta_{1,-1}^T(\theta) - \zeta_{01}^T(\theta)\zeta_{0,-1}^T(\theta)) d_{1,-1}^\ell(\theta) \right]^{-1}, \quad (45)$$

with correlation functions defined by

$$\zeta_{00}^T(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} \frac{1}{C_\ell^{TT} + N_\ell^{TT}} d_{00}^\ell(\theta) \quad (46)$$

$$\zeta_{0,\pm 1}^T(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} \sqrt{\ell(\ell+1)} \frac{C_\ell^{TT}}{C_\ell^{TT} + N_\ell^{TT}} d_{0,\pm 1}^\ell(\theta) \quad (47)$$

$$\zeta_{1,\pm 1}^T(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} \ell(\ell+1) \frac{(C_\ell^{TT})^2}{C_\ell^{TT} + N_\ell^{TT}} d_{1,\pm 1}^\ell(\theta). \quad (48)$$

Fast algorithms can also be given for lens reconstruction noise power spectra which arise from TE, TB, and EE estimators, but we omit these cases here since they are not needed in this paper. Indeed, fast algorithms exist for calculating the noise power spectra for many quadratic estimators of statistical anisotropy, provided that the weights which they use are separable.