*ON THE INVERSION OF THE DENSITY GRADIENT AT THE FRINGE OF THE CONVECTION ZONE**

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Introduction.—It is well known that the total pressure and the temperature increase as one goes inward from the surface to the center of a star. That the density, on the other hand, does not necessarily increase with depth below the surface was pointed out by Hoyle and Schwarzschild (1955)[1] and was borne out quite clearly by the numerical integrations of the solar surface layers by Faulkner, Griffiths, and Hoyle (1963).[2] The question was raised by Tayler and Gough (1963)[3] as to whether the density gradient inversion was real or whether it was due to the particular model of convection adopted by Faulkner *et al.* It is the purpose of this note to show that the inversion is indeed genuine and results from the steep temperature gradient that exists in the outermost layers of the convection zone where convection is not fully efficient and carries only a fraction ($<^1/_2$) of the total energy flux. Also, the electron pressure-temperature plane can be divided into regions where $d\rho/dT$ is negative and positive. The dividing line depends, in an insensitive manner, on the assumed model and efficiency of convection. In the case of the sun it is the hydrogen ionization at about 10^4 °K that causes the opacity to go up sharply and as a result the temperature gradient steepens there by inverting the density gradient. The inversion necessarily results in a Rayleigh-Taylor instability.

Governing Equations.—For hydrostatic equilibrium, we have

$$\frac{dP}{dr} = -g\rho, \tag{1}$$

where P is the sum of the gas pressure and radiation pressure, ρ being the density, and g the acceleration due to gravity.

The temperature gradient is given by the equation of radiative transfer

$$\frac{dT}{dr} = -\frac{3\kappa\rho(F - C)}{4acT^3} = -\frac{3\kappa\rho\alpha F}{4acT^3}, \tag{2}$$

where C is the flux carried by convection and α is the fraction of the total flux F carried by radiation [$\alpha = (F - C)/F$]. Other symbols have their usual meaning. The notation used in this analysis is the same as that adopted by Faulkner *et al.* (1963).[2]

Under conditions of very low density that prevail in stellar atmosphere, the equation of state of an ideal gas holds very well and the electron pressure P_e is given by

$$P_e = R\rho T\theta, \tag{3}$$

R being the gas constant and θ/m_H the number of electrons per unit mass of material (m_H being the mass of the hydrogen atom). The expression for θ is given in the *Appendix.*

The gas pressure can be expressed as

$$P_g = R\rho T\,(X + Y/4 + \theta), \tag{4}$$

and the total pressure is given by

$$P = R\rho T(X + Y/4 + \theta) + {}^1/_3\, aT^4, \tag{5}$$

a being the Stefan-Boltzmann constant. From the combination of equations (5) and (3) we get

$$P = P_e \frac{X + Y/4 + \theta}{\theta} + {}^1/_3\, aT^4, \tag{6}$$

and by logarithmic differentiation we recover after some manipulation of $d\theta$ (see *Appendix*) with the help of the ionization equations

$$dP = P_{p_e} \frac{dP_e}{P_e} + P_T \frac{dT}{T}, \tag{7}$$

where P_{p_e} and P_T are functions of P_e and T. Here we have used the fact that θ is a function of P_e and T, and hence $d\theta$ can be expressed as

$$d\theta = \theta_T \frac{dT}{T} - \theta_{p_e} \frac{dP_e}{P_e}, \tag{8}$$

where θ_T and θ_{p_e} are functions of P_e and T only and are positive definite quantities.

With the aid of equation (3) we have

$$\frac{dP_e}{P_e} = \frac{d\rho}{\rho} + \frac{dT}{T} + \frac{d\theta}{\theta}. \tag{9}$$

Thus,

$$\begin{aligned}
\frac{d\rho}{dT} &= -\frac{\rho}{T} + \frac{\rho}{P_e}\frac{dP_e}{dT} - \frac{\rho}{\theta}\frac{d\theta}{dT} \\
&= -\frac{\rho}{T} + \frac{\rho}{P_e}\left\{\frac{P_e}{P_{p_e}}\frac{dP}{dT} - \frac{P_T}{P_{p_e}}\frac{P_e}{T}\right\} - \frac{\rho}{\theta}\left\{\frac{\theta_T}{T} - \frac{\theta_{p_e}}{P_e}\left(\frac{P_e}{P_{p_e}}\frac{dP}{dT} - \frac{P_T}{P_{p_e}}\frac{P_e}{T}\right)\right\} \\
&= -\frac{\rho}{T}\left(1 + \frac{\theta_T}{\theta}\frac{P_T}{P_{p_e}} + \frac{\theta_{p_e}P_T}{\theta P_{p_e}}\right) + \frac{\rho}{P_{p_e}}\left(1 + \frac{\theta_{p_e}}{\theta}\right)\frac{dP}{dT} \\
&= -\frac{\rho}{T}\left\{\left(1 + \frac{\theta_T}{\theta}\right) + \frac{P_T}{P_{p_e}}\left(1 + \frac{\theta_{p_e}}{\theta}\right)\right\} + \frac{\rho}{P_{p_e}}\left(1 + \frac{\theta_{p_e}}{\theta}\right)\frac{dP}{dT}.
\end{aligned} \tag{10}$$

From equations (1) and (2) we get

$$\frac{d\rho}{dT} = \frac{g\rho}{3\kappa\rho\alpha F/4acT^3} = \frac{4acT^3g}{3\kappa\alpha F},$$

and hence

$$\frac{d\rho}{dT} = -\frac{\rho}{T}\left\{\left(1 + \frac{\theta_T}{\theta}\right) + \frac{P_T}{P_{p_e}}\left(1 + \frac{\theta_{p_e}}{\theta}\right)\right\} + \frac{4acgT^3}{3P_{p_e}\kappa\alpha F}\left(1 + \frac{\theta_{p_e}}{\theta}\right). \tag{11}$$

The right-hand side of equation (11) is a function of P_e, T, F, and g, the density being expressible in terms of P_e and T by employing equation (3). The gradient of the density is therefore composed of a negative contribution, the first term on the right-hand side, and a positive contribution. It is to be emphasized that there are regions in the outermost layers of the convective zone where κ becomes very large and decreases the positive contribution to $d\rho/dT$. However, in these regions, despite the violent superadiabatic temperature gradient, convection is inefficient (because of the very low density and low heat capacity) and only a fraction $[(1 - \alpha) \leq {}^1/_2]$ of the total flux is carried by convection. Thus, α does not decrease sufficiently to compensate for the increase in κ and $d\rho/dT$ does become negative.

It might be questionable whether the equation of hydrostatic equilibrium should be used in the region where there are convective motions. The full equation of motion under steady conditions should read

$$\rho(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p + \rho\mathbf{g} + \text{(viscous terms)}. \tag{12}$$

For the purpose of the present work we have neglected the viscous terms which are probably very small in the outer layers of stars and have defined

$$\beta = \left|\frac{\rho(\mathbf{V}\cdot\nabla)\mathbf{V}}{\rho\mathbf{g}}\right| \tag{13}$$

to transform the above equation to the form

$$\frac{dP}{dr} = -g\rho(1 - \beta). \tag{14}$$

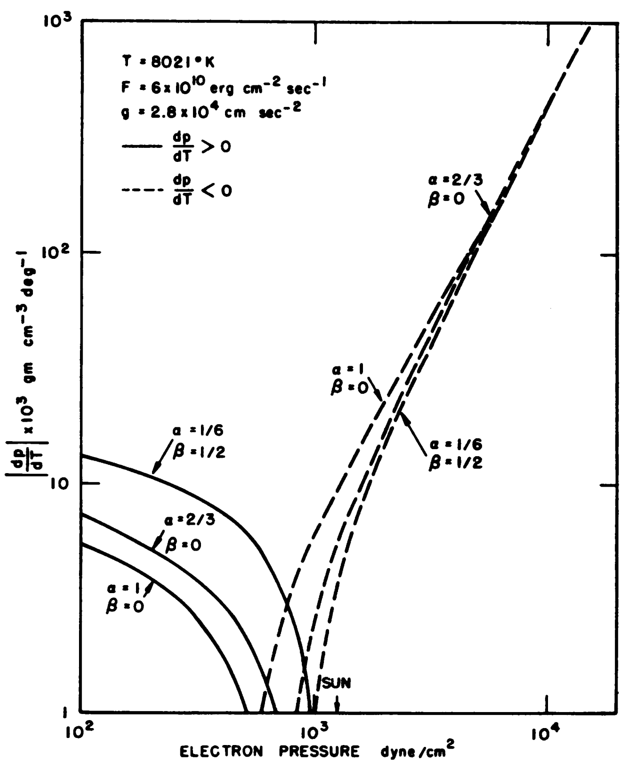

FIG. 1.

This overcomes any possible difficulties that might be raised about the magnitude of the second term on the right side of equation (11) in the presence of convection. Equation (11) then gets modified to the form

$$\frac{d\rho}{dT} = -\frac{\rho}{T}\left\{\left(1 + \frac{\theta_T}{\theta}\right) + \frac{P_T}{P_{p_e}}\left(1 + \frac{\theta_{p_e}}{\theta}\right)\right\} + \frac{4ac\rho g T^3(1-\beta)}{3P_{p_e}\kappa\alpha F}\left(1 + \frac{\theta_{p_e}}{\theta}\right). \quad (15)$$

Results and Discussion.—Our analysis is valid so long as α does not come very close to zero (i.e., $C \rightarrow F$) in which case the second term on the right side will become very large and $d\rho/dT$ will become positive, but it is then questionable whether the equation of radiative transfer (2) should be used to determine the temperature gradient. In fact, the density inversion is only to be expected in the regions of excessive superadiabatic temperature gradient which are inadequate to transport an appreciable fraction of the total flux because of the low density and low heat capacity obtained in this region. More specifically, the superadiabatic gradient results from the inefficiency of convection. We take $\alpha = {}^1/_6$ as a reasonable lower limit for the present work. The quantity β, which is a measure of the ratio of turbulent pressure gradient to gravitational force, is also unlikely to exceed ${}^1/_2$.

The results of the numerical evaluation of $d\rho/dT$ are exhibited in Figures 1–3. Figures 1 and 2 show the logarithmic plots of $d\rho/dT$ against the electron pressure

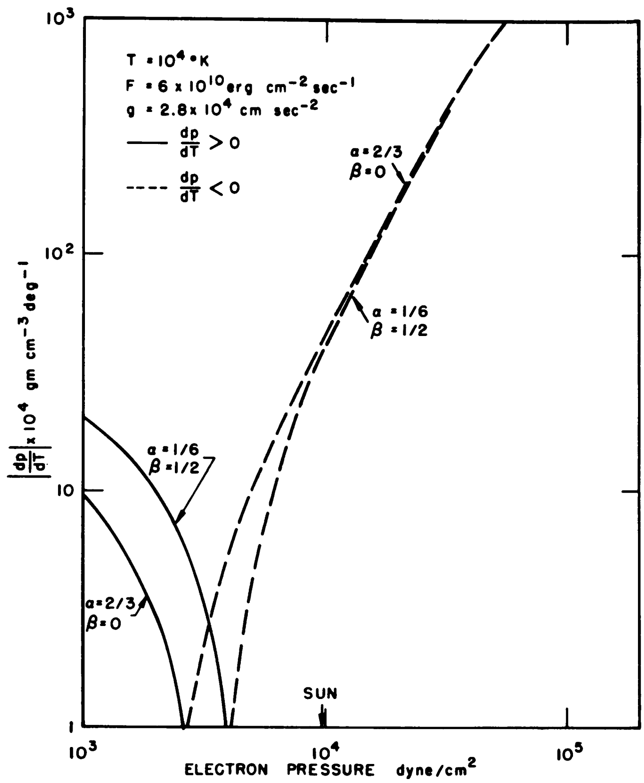

FIG. 2.

corresponding to flux $= 6 \times 10^{10}$ erg cm^{-2} sec^{-1}, $g = 2.8 \times 10^4$ cm sec^{-2}, and temperatures of 8021°K and 10^4°K, respectively. Several sets of parameters α and β are shown. It is clear from the graphs that in all cases we have an inversion of the density gradient. Figure 3 shows the loci $d\rho/dT = 0$ in the $Pe - T$ plane for several sets of parameters. The broken line shows the march obtained in the Sun. As can be seen, the variation of α and β over a reasonable range does not alter the picture much. It is evident that there are regions in the unstable zone where the occurrence of the steep temperature gradient, because of the sharply increasing opacity due to the increasing ionization, causes the density gradient to become negative.

It may be emphasized that the analysis does not depend on any particular model of convection. In fact, for the present work, we do not have to calculate an expression for the convective flux; the parameter α takes care of the amount of flux carried by convection and essentially, the variation of α takes into account the various degrees of the efficiency of convection that may be conceived. It is possible to think of a situation in which the velocity becomes large and consequently β approaches unity, but in such cases the theory of convection becomes vulnerable because of the possible presence of shock wave and energy dissipation.

The inversion of the density gradient is not altogether surprising in the outer layers of stars where the density is very low and as a result the change in the pressure with radius is small compared to the change in the temperature caused principally by the sharp increase in the opacity. These regions of inefficient convection are possible seats of this type of behavior and may have a bearing on the instabilities that are to be seen in the atmospheres of red giants.

Appendix.—The number of electrons per unit mass θ/m_H is given by (see Faulkner *et al.*, 1963)

$$\theta = \frac{Xx}{1+x} + \frac{Y}{4}\frac{y(1+2z)}{(1+y+yz)} + A\left(\frac{u_1}{1+u_1} + \frac{A_2u_2}{1+u_2} + \frac{A_3u_3}{1+u_3} + \frac{A_4u_4}{1+u_4}\right)$$

where X and Y are the mass fraction of hydrogen and helium, respectively, and A, A_2, A_3, and A_4 are the relative abundances of metals, namely:

$$A = \frac{\mathrm{Mg} + \mathrm{Fe}}{\mathrm{H} + 4\mathrm{He}}, \quad A_2 = \frac{\mathrm{Al}}{\mathrm{Mg} + \mathrm{Fe}}, \quad A_3 = \frac{\mathrm{Na}}{\mathrm{Mg} + \mathrm{Fe}}, \quad A_4 = \frac{\mathrm{K}}{\mathrm{Mg} + \mathrm{Fe}}.$$

x, y, z, u_1, u_2, u_3, and u_4 are defined in the following way:

$$x = \frac{\mathrm{H_{II}}}{\mathrm{H_I}}; \quad y = \frac{\mathrm{He_{II}}}{\mathrm{He_I}}; \quad z = \frac{\mathrm{He_{III}}}{\mathrm{He_{II}}}; \quad u_1 = \frac{\mathrm{Mg_{II}}}{\mathrm{Mg_I}}; \quad u_2 = \frac{\mathrm{Al_{II}}}{\mathrm{Al_I}};$$

$$u_3 = \frac{\mathrm{Na_{II}}}{\mathrm{Na_I}}; \quad u_4 = \frac{\mathrm{K_{II}}}{\mathrm{K_I}}.$$

Here, Mg and Fe are treated together because their ionization potentials are very similar. Ionization equilibrium yields the following equations:

$$x, y, z, u_1, u_2, u_3, u_4 = \frac{T^{5/2}}{P_e}\exp\left(21.915 - \frac{15.71}{T}, 23.296 - \frac{28.41}{T}, 21.915 - \frac{62.85}{T}, 23.112 - \frac{8.831}{T}, 20.764 - \frac{6.914}{T}, 21.547 - \frac{5.941}{T}, 20.902 - \frac{5.013}{T}\right).$$

Here, T is measured in units of 10^4°K.

The total differential of θ is given by

$$d\theta = \theta_T \frac{dT}{T} - \theta_{p_e} \frac{dP_e}{P_e},$$

where

$$\theta_T = E_x F_x + G_1 E_y E_y + G_2 E_z F_z + E_{u_1} F_{u_1} + E_{u_2} F_{u_2} + E_{u_3} F_{u_3} + E_{u_4} F_{u_4},$$

$$\theta_{p_e} = +(F_x + G_1 F_y + G_2 F_z + F_{u_1} + F_{u_2} + F_{u_3} + F_{u_4}),$$

$$E_x, E_y, E_z, E_{u_1}, E_{u_2}, E_{u_3}, E_{u_4} = 2.5 + \frac{15.71}{T}, 2.5 + \frac{28.41}{T}, 2.5 + \frac{62.87}{T},$$
$$2.5 + \frac{8.831}{T}, 2.5 + \frac{6.914}{T}, 2.5 + \frac{5.941}{T},$$
$$2.5 + \frac{5.013}{T},$$

$$F_x, F_y, F_z, F_{u_1}, F_{u_2}, F_{u_3}, F_{u_4} = \frac{Xx}{(1+x)^2}, \frac{Y}{4}\frac{y(1.808 + 5.808z)}{(1 + y + yz)^2},$$
$$\frac{Y}{4}\frac{yz(5.808 + 4y)}{(1 + y + yz)^2}, \frac{Au_1}{(1 + u_1)^2}, \frac{AA_2u_2}{(1 + u_2)^2},$$
$$\frac{AA_3u_3}{(1 + u_3)^2}, \frac{AA_4u_4}{(1 + u_4)^2},$$

$$G_1 = \frac{1 + 2z}{1.808 + 5.808z},$$

$$G_2 = \frac{2 + y}{5.808 + 4y}.$$

The total differential of the pressure is given by

$$dP = P_{p_e} \frac{dP_e}{P_e} + P_T \frac{dT}{T},$$

where

$$P_{p_e} = P_e \left(S + \frac{\theta_{p_e}}{\theta} (1 - S)\right),$$

$$P_T = P_e \frac{\theta_T}{\theta} (1 - S) + \frac{4}{3} aT^4,$$

and

$$S = \frac{\theta + X + Y/4}{\theta}.$$

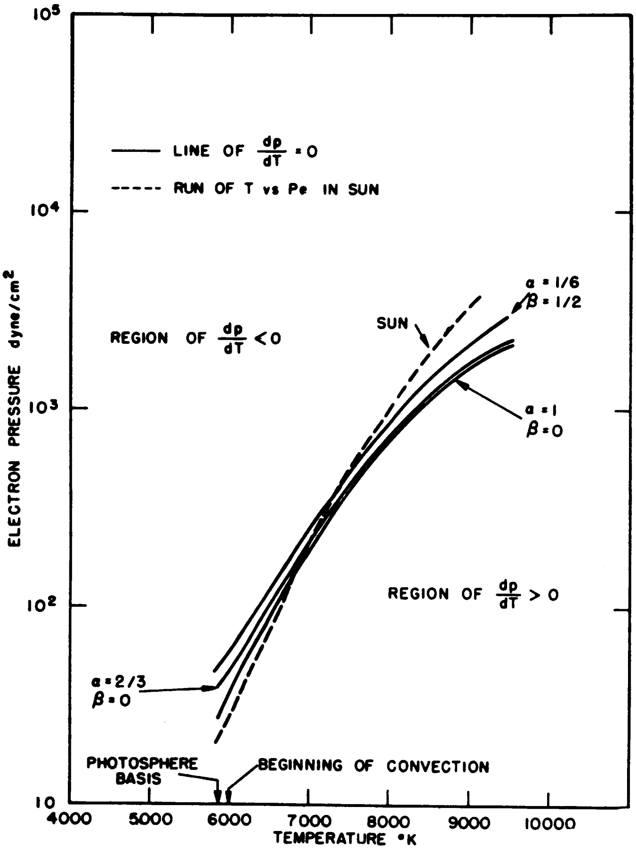

FIG. 3.

The opacity is given by

$$\kappa = 0.40\theta + \frac{X}{1+x}\left\{\frac{P_e}{T^{5/2}}\exp\left(-6.378 + \frac{0.8704}{T}\right) + \frac{\exp\,(15.94 - 11.61/T)}{1 + 0.01T^3}\right\}$$
$$+ \frac{Y}{4(1+y+yz)}\left\{\frac{\exp\,(15.94 - 23.22/T)}{1 + 0.003T^3} + y\exp\left(14.5 - \frac{46}{T}\right)\right\}.$$

The contributions to the opacity are, respectively, from free electrons H^-, H_I, He_I, and He_{II}.

It should be stated that the expression we have adopted for the opacity is approximate. However, in the region under consideration, the changes in the opacity because of ionization are so large that they outweigh any approximations involved in the expression for κ.

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