A nonlinear unsteady flexible wing theory

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Dedication This work is dedicated in commemoration of Professor Thomas K. Caughey for his lifetime contributions of great significance to science, engineering, and engineering science, for his role model in truly outstanding scholarship, and for paying a sincere tribute to his most distinguished life and work.

Abstract. This paper extends a previous study by Wu (2001)[1] to continue developing a fully nonlinear theory for calculation of unsteady flow generated by a two-dimensional flexible lifting surface moving in arbitrary manner through an incompressible and inviscid fluid for modeling bird/insect flight and fish swimming. The original physical concept elucidated by Theodore von Kármán and William R. Sears (1938)[2] in describing the complete vortex system of a wing and its wake in non-uniform motion for their linear theory is adapted and applied to a fully nonlinear consideration. The new theory employs a joint Eulerian and Lagrangian description of the lifting-surface movement to facilitate the formulation. The present investigation presents further analysis for addressing arbitrary variations in wing shape and trajectory to achieve a nonlinear integral equation akin to Herbert Wagner's (1925)[3] linear version for accurate computation of the entire system of vorticity distribution.

Key words: nonlinear unsteady flexible wing theory, unsteady camber function, arbitrary trajectory.

1. Introduction

In the world of self-locomotion of aquatic and aerial animals by using lifting surfaces like wings and appended fins, there are several salient features of significance. First, the wings are in general large in aspect-ratio, a feature that would suit for an unsteady lifting-line approach. Secondly, the periodic beating of the wing generally involves change in surface profile shape (or simply *shape*), e.g. from a stretched-straight pronation in downward stroke to a form with an arched camber and spanwise bending in upward supination stroke. Further, in swift maneuvering, the wings may bend and twist asymmetrically to change and turn in orientation and trajectory, e.g. in the beautiful performance of a humming bird using a figure-eight wing beating in keeping its body fixed in front of a flower, and then suddenly fleeting off in a flash. All these features are so strongly nonlinear and time-dependent that a comprehensively valid theory would have to take all these factors fully into account.

Recently, a nonlinear unsteady wing theory has been introduced by Wu ([1] §VI) along this approach to provide a jointly analytical and numerical method for computations of solutions on specific premises. This nonlinear theory has been applied by Stredie (2004)[4] to perform computations of a number of unsteady motions of bodies shedding vortex sheet(s), attaining results of high accuracy (as measured versus relative errors and experiments available) in all the cases pursued, and making valuable contributions to this subject. The present work is a continuation to this series of studies, here addressing further on the issue of arbitrary changes in wing shape and trajectory along the line discussed by Wu[5] with intent to optimize the analytical and computational efforts for attaining solutions efficiently.

2. Wing movement with arbitrary changes in shape and trajectory

To begin with, we recapitulate the nonlinear theory[1,5] of a two-dimensional (actively) flexible lifting surface for modeling aquatic and aerial animal locomotion at high Reynolds number. We opt two-dimensional theory for its simplicity to provide a foundation for further development of unsteady wing theory and for general applications. For proficiency, we seek an existing linear theory that its conceptual structure can best suit for the generalization as intended.

In this respect, we find the simple and clear physical concept crystallized by von Kármán and Sears[2] in providing such a general view on an ingenious restructuring of the vorticity distribution over the wing and its trailing wake that it readily affords powerful generalizations. So it has been extended by Wu (2001, Sect. 6)[1] to account fully for all possible nonlinear effects in theory, and

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bring Herbert Wagner's pioneering work (1925)[3] to more general applications. The principal step is to employ a joint Eulerian and Lagrangian description of the lifting-surface movement for the formulation which we will delineate synoptically next. This useful description of unsteady bodily movement has also been applied by Lighthill [6] to develop a large-amplitude elongated-body theory.

Thus, we consider the irrotational flow of an incompressible and inviscid fluid generated by a two-dimensional flexible lifting surface $S_b(t)$ of negligible thickness, moving with time t through the fluid in arbitrary manner. Its motion can be described parametrically by using a Lagrangian coordinate system (ξ, η) to identify a point $X(\xi,t), Y(\xi,t)$ on the boundary surface $S(t) = S_b(t)+S_w(t)$ consisted of the body surface S_b and a wake surface S_w, with $S(t)$ lying at time $t = 0$ over a stretch of the ξ-axis (at $\eta = 0$) and moving with time $t(\geq 0)$ (see Fig. 1) as prescribed by $z = x + iy = Z(\xi, t)$,

$$Z(\xi, t) = X(\xi, t) + iY(\xi, t) \quad \text{on} \quad S_b(t) + S_w(t), \tag{1}$$

where $S_b(t) : (-1 < \xi < 1)$ and $S_w(t) : (1 < \xi < \xi_m)$, parametrically in ξ, with $\xi = -1$ marking the leading edge and $\xi = 1$ the trailing edge of the wing, from the latter of which a vortex sheet is assumed to be shed smoothly (i.e. under the Kutta condition) to form a prolonging wake S_w, and ξ_m identifies the path $Z(\xi_m, t)$ of the starting vortex shed at $t = 0$ to reach $\xi_m = \xi_m(t)$ at time t. A simple choice for $(\xi + i\eta)$ is the initial material position of $S_b(t = 0)$, taken to be in its stretched-straight shape such that $Z(\xi, 0) = \xi \quad (-1 < \xi < 1,\ \eta = 0)$, lying in an unbounded fluid initially at rest in an inertial frame of reference (see Figure 1). The flexible $S_b(t)$ is assumed to be inextensible ($|Z_\xi| \equiv |\partial Z/\partial\xi| = 1$) and the point ξ on $S_b(t)$ moves with *prescribed* (complex) velocity $W(\xi, t) = U - iV$:

$$W(\xi, t) = \partial\overline{Z}/\partial t = X_t - iY_t \quad (|\xi| < 1,\ t \geq 0;\ \overline{Z} = X - iY), \tag{2}$$

which has a tangential component, $U_s(\xi, t)$, and a normal component, $U_n(\xi, t)$, given by

$$W\partial Z/\partial\xi = (X_\xi X_t + Y_\xi Y_t) - i(X_\xi Y_t - Y_\xi X_t) = U_s - iU_n, \tag{3}$$

and with the same expression for the wake surface $S_w(t)$ for $(1 < \xi < \xi_m)$.

In the spirit of von Kármán and Sears, we adopt for $t > 0$ the following vorticity distribution:

on $S_b(t)$: $\quad \gamma(\xi, t) = \gamma_0(\xi, t) + \gamma_1(\xi, t) \qquad (-1 < \xi < 1)$,

on $S_w(t)$: $\quad \gamma(\xi, t) = \gamma_w(\xi, t) \qquad (1 < \xi < \xi_m)$,

where $\gamma_0(\xi, t)$ is the bound vortex distributed over S_b representing the "quasi-steady" flow past S_b such that the time t in the original prescribed $W(\xi, t)$ is frozen to serve merely as a parameter in evaluating the quasi-steady γ_0 (by steady airfoil theory), and $\gamma_1(\xi, t)$ is the additional bound vortex induced on S_b by the trailing wake vortices $\gamma_w(\xi, t)$ such that γ_1 and γ_w jointly make zero contribution to U_n (but not to U_s) over S_b so as to reinstate the original time-varying normal velocity $U_n(\xi, t)$ on $S_b(t)$.

Thus, we represent the velocity field by a vorticity distribution, $\gamma(\xi, t)$, per unit length spanwise over the body and wake surfaces to give the complex velocity $w(z, t) = u - iv$ of the fluid at a field point z and at time t as

$$w(z, t) = \frac{1}{2\pi i}\int_{-1}^{\xi_m} \frac{\gamma(\xi, t)}{Z(\xi, t) - z}\, d\xi \qquad (z = x + iy \notin S,\ t \geq 0). \tag{4}$$

Applying Plemelj's formula to (4) yields for $w_\pm = \lim w(z(\xi + i\eta), t)$ as $\eta \to \pm 0$ on the two sides of S,

$$u_s^\pm - iu_n^\pm = w_\pm(\xi, t)\frac{dZ}{d\xi} = \pm\frac{1}{2}\gamma(\xi, t) + \frac{1}{2\pi i}\frac{dZ}{d\xi}\int_S \frac{\gamma(\xi', t)}{Z' - Z}\, d\xi', \tag{5}$$

with $Z = Z(\xi), Z' = Z(\xi')$ both on $S = S_b + S_w$. From (5) we have $\gamma(\xi, t) = (u_s^+ - u_s^-)$, and

$$u_n^+(\xi, t) = u_n^-(\xi, t) \quad = \quad Re\left\{\frac{1}{2\pi}\frac{dZ}{d\xi}\int_S \frac{\gamma(\xi', t)}{Z' - Z}\, d\xi'\right\}, \tag{6}$$

$$u_{sm} \equiv \frac{1}{2}(u_s^+ + u_s^-) \quad = \quad Im\left\{\frac{1}{2\pi}\frac{dZ}{d\xi}\int_S \frac{\gamma(\xi', t)}{Z' - Z}\, d\xi'\right\}. \tag{7}$$

Here, (6) shows the continuity of normal velocity $u_n^+ = u_n^- = u_n$ across S and (7) gives the algebraic mean of tangential velocity u_s on S. From (6)-(7) we further deduce the contributions separately made by γ_0, γ_1, and γ_w as:

$$U_n(\xi,t) = Re\left\{\frac{1}{2\pi}\frac{dZ}{d\xi}\int_{-1}^{1}\frac{\gamma_0(\xi',t)}{Z'-Z}\,d\xi'\right\} \qquad (Z = Z(\xi,t)\in S_b), \tag{8}$$

$$U_{1n}(\xi,t) = Re\left\{\frac{1}{2\pi}\frac{dZ}{d\xi}\int_{1}^{\xi_m}\frac{\gamma_w(\xi',t)}{Z'-Z}\,d\xi'\right\} \qquad (Z = Z(\xi,t)\in S_b), \tag{9}$$

$$-U_{1n}(\xi,t) = Re\left\{\frac{1}{2\pi}\frac{dZ}{d\xi}\int_{-1}^{1}\frac{\gamma_1(\xi',t)}{Z'-Z}\,d\xi'\right\} \qquad (Z = Z(\xi,t)\in S_b), \tag{10}$$

$$W_w(\xi,t) = \frac{1}{2\pi i}\frac{dZ}{d\xi}\int_{S_b+S_w}\frac{\gamma(\xi',t)}{Z'-Z}\,d\xi' \qquad (Z = Z(\xi,t)\in S_w), \tag{11}$$

where $W_w(\xi,t) = U_{ws} - iU_{wn}$ is the (complex) flow velocity on the wake.

The problem can now be recast to delineate the course for solution as follows. Equation (8) results from invoking condition that $u_n(\xi,t) = U_n(\xi,t)$, given at S_b, to give an integral equation for γ_0 which is to be solved, with time t frozen, by applying steady airfoil theory. The velocity induced on S_b by wake vorticity γ_w has the normal component U_{1n} given by (9), which is canceled out as is required of γ_1 on S_b according to (10) so that the sum (9)+(10) gives an integral equation for γ_1 in terms of γ_w. This solution for γ_1, which is to be determined under the Kutta condition (on the continuity of vorticity at the trailing edge) may be expressed, in principle, *symbolically* in the form

$$\gamma_1(\xi,t) = \int_{1}^{\xi_m} K(\xi';\xi,t)\gamma_w(\xi',t)\,d\xi' \qquad (|\xi|\le 1). \tag{12}$$

Finally, we apply Kelvin's theorem that the total circulation around $S_b + S_w$ must vanish $\forall t \ge 0$, i.e. $\Gamma_0 + \Gamma_1 + \Gamma_w = \int_{S_b}(\gamma_0+\gamma_1)\,d\xi + \int_{S_w}\gamma_w\,d\xi = 0$ (if it is zero initially), or, symbolically,

$$\Gamma_0 + \int_{1}^{\xi_m}\left\{1 + \int_{-1}^{1}K(\xi';\xi,t)\,d\xi\right\}\gamma_w(\xi',t)\,d\xi' = 0. \tag{13}$$

This is in essence the desired form of "generalized Wagner's integral equation" for wake vorticity γ_w, which has been explicitly given by Wu[1], however in a rather lengthy series expansion form. Its original linear version has been shown by Wagner[3] and by von Kármán and Sears[2] to play a key role in providing accurate solutions for the entire vorticity distributions and hence for the final solution to the linearized problem. For the present nonlinear theory, it is of interest to derive the kernel $K(\xi,\xi',t)$ in closed form for efficient applications to wing movement in arbitrary manner, as will be pursued next.

3. A unified method of solution

Here, the method proposed by Wu[1,5] based conjointly on an analytical-and-numerical scheme is further developed by unifying the original method with a new perturbation scheme in terms of a small residual term of second order for efficient iteration. Accordingly, we start with rewriting (8) as

$$U_n(\xi,t) = \frac{1}{2\pi}\int_{-1}^{1}\{1 + g(\xi,\xi';t)\}\frac{\gamma_0(\xi',t)}{\xi'-\xi}\,d\xi', \tag{14}$$

$$g(\xi,\xi';t) = Re\left\{\frac{dZ}{d\xi}\frac{\xi'-\xi}{Z'-Z}\right\} - 1 \qquad (\forall(\xi,\xi')\in S_b). \tag{15}$$

Here we note that if S_b is a flat wing, then we have

$$Z(\xi) - Z(\xi') = e^{i\theta(t)}(\xi-\xi') \qquad \longrightarrow \qquad g(\xi,\xi',t) = 0, \tag{16}$$

valid for arbitrary inclination angle θ subtended by the flat wing with the x-axis. For wing with small camber, as is usually seen, $g(\xi,\xi';t)$ is found (see (18)) to be regular and quadratic in the camber. We can therefore call $g(\xi,\xi';t)$ the *residual kernel*, and its integral, a *residual term*, which is of form apt for iteration with rapid convergence. This is the principle we shall follow in this unified scheme.

For the body shape, it can assume a general *camber function* $\eta = f(\xi,t)\ \in C^1\ \forall\ \xi[-1,1]$ relative to the chord, which is inclined at angle $\theta(t)$ with the x-axis, so that S_b has the *shape function* (see Fig. 2)

$$Z(\xi,t) = Z_0(t) + e^{i\theta}(\xi + if(\xi,t)) \qquad (-1 \le \xi \le 1), \tag{17}$$

$$g(\xi,\xi',t) = \frac{\Delta f}{\Delta\xi}\left(\frac{\partial f}{\partial\xi} - \frac{\Delta f}{\Delta\xi}\right)\left(1 + \left(\frac{\Delta f}{\Delta\xi}\right)^2\right)^{-1} \qquad (\Delta\xi = \xi - \xi',\ \Delta f = f(\xi) - f(\xi')), \tag{18}$$

which, resulting from (15)-(17), shows g being quadratic in f and $g(\xi,\xi';t) \to 0$ like $\Delta\xi$ as $\Delta\xi \to 0$.

For the body movement, its surface (complex) velocity, $W_b = U_b - iV_b$ comprises in general three components, that due to (i) the rectilinear translation, $W_T = d\overline{Z}_0/dt = -U\exp[-i(\theta+\alpha)]$; (ii) the body rotation, $W_R = (\partial\overline{Z}/\partial\theta)d\theta/dt$ with angular velocity $\Omega = -d\theta/dt$, which is positive for leading edge upward by convention, and (iii) the shape change at rate $W_S = (\partial\overline{Z}/\partial f)\partial f/\partial t$. Here, W_T corresponds to a freestream of speed U attacking the chord at incidence angle $\alpha(t)$, $\alpha > 0$ (or < 0) for pronation (or supination). The total surface velocity, or the sum of the three parts, $W_b = W_T + W_R + W_S$, can be derived directly from the time derivative of the body surface curve so that

$$W_b = \frac{d}{dt}\overline{Z}(\xi,t) = \{-Ue^{-i\alpha} + \Omega(f(\xi,t) + i\xi) - if_t(\xi,t)\}e^{-i\theta}, \tag{19}$$

and by (3), we obtain the normal velocity of the S_b surface as

$$U_n(\xi,t) = -(U\sin\alpha + \Omega\xi - f_t) + (U\cos\alpha - \Omega f)f_\xi, \tag{20}$$

where $f_\xi = \partial f/\partial\xi,\ f_t = \partial f/\partial t$. All the variables and parameters in U_n are supposed to be given.

In determining $U_n(\xi,t)$, the time variations of all the variables and parameters involved have been fully accounted for. However, in using this $U_n(\xi,t)$ to solve (8), or (14), for $\gamma_0(\xi,t)$, the time t is instantly frozen for this quasi-steady lifting flow. Thus, the problem becomes a Riemann-Hilbert singular integral equation[7] for $\gamma_0(\xi,t)$, of which the leading term can be directly inverted for an explicit solution, whereas the residual term with kernel $g(\xi,\xi',t)$ can be resolved efficiently by iteration. Thus, taking iteration once of the leading term of the solution yields

$$\gamma_0(\xi,t) = -\frac{2}{\pi}\sqrt{\frac{1-\xi}{1+\xi}}\int_{-1}^{1}\sqrt{\frac{1+\xi'}{1-\xi'}}\left\{\frac{1}{\xi'-\xi} + G(\xi,\xi',t)\right\}\ U_n(\xi',t)\ d\xi', \tag{21}$$

$$G(\xi,\xi',t) = \frac{1}{\pi^2}\int_{-1}^{1}\sqrt{\frac{1+\eta}{1-\eta}}\frac{d\eta}{\eta-\xi}\int_{-1}^{1}\sqrt{\frac{1-\zeta}{1+\zeta}}\frac{g(\eta,\zeta,t)\ d\zeta}{(\xi'-\zeta)(\zeta-\eta)}, \tag{22}$$

in which the integral expression for $G(\xi,\xi',t)$ (arising from the first iteration) results after interchanging the order of integrals, which is justified by virtue of (16) for $g(\xi,\xi',t)$ (see e.g. Muskhelishvili[7]).

Next, in complete analogy between (10) and (8), we have for γ_1 the solution *formally* as

$$\gamma_1(\xi,t) = \frac{2}{\pi}\sqrt{\frac{1-\xi}{1+\xi}}\int_{-1}^{1}\sqrt{\frac{1+\xi'}{1-\xi'}}\ \left\{\frac{1}{\xi'-\xi} + G(\xi,\xi',t)\right\}\ U_{1n}(\xi',t)\ d\xi', \tag{23}$$

but not yet final because $U_{1n}(\xi,t)$ is still unknown until the wake vorticity $\gamma_w(\xi,t)$ is solved from (9) or otherwise. To proceed, we again rewrite (9), like (14) for (8), as

$$U_{1n}(\xi,t) = \frac{1}{2\pi}\int_{1}^{\xi_m}\{1 + h(\xi,\xi';t)\}\ \frac{\gamma_w(\xi',t)}{\xi'-\xi}\ d\xi' \tag{24}$$

$$h(\xi,\xi';t) \quad = \quad Re\left\{\frac{dZ}{d\xi}\frac{\xi'-\xi}{Z'-Z}\right\}-1 \qquad (-1<\xi<1,\ 1<\xi'<\xi_m). \tag{25}$$

Here we note that $h(\xi,\xi';t)$ differs from $g(\xi,\xi';t)$ by having $\xi\in S_b$ but $\xi'\in S_w$. As a result, unlike $g(\xi,\xi';t)$ being always small for S_b with a small camber, as shown by (18), $h(\xi,\xi';t)$ in general can be finite in magnitude, especially when S_b displaces itself by finite amount at fast rate from a straight trajectory in the space. It is in such general cases that the wake vortices can give rise to finite nonlinear effects on the flow field in addition to the local nonlinear effects due to changes in body shape by (15).

In general, by substituting (24) for U_{1n} in (23), we can derive for the total circulation around the wing due to γ_1, $\Gamma_1=\int_{S_b}\gamma_1(\xi,t)d\xi$, to obtain the following result

$$\Gamma_1(t) \quad = \quad \int_1^{\xi_m}\left\{\sqrt{\frac{\xi+1}{\xi-1}}-1+N_w(\xi,t)+N_b(\xi,t)\right\}\gamma_w(\xi,t)\ d\xi, \tag{26}$$

$$N_w(\xi,t) \quad = \quad \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+\xi'}{1-\xi'}}\ \frac{h(\xi',\xi,t)}{\xi-\xi'}\ d\xi', \tag{27}$$

$$N_b(\xi,t) \quad = \quad \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+\xi'}{1-\xi'}}\ \tilde{G}(\xi',t)\ \frac{1+h(\xi',\xi,t)}{\xi-\xi'}\ d\xi', \tag{28}$$

$$\tilde{G}(\xi',t) \quad = \quad \frac{1}{\pi^2}\int_{-1}^{1}\sqrt{\frac{1+\eta}{1-\eta}}d\eta\int_{-1}^{1}\sqrt{\frac{1-\zeta}{1+\zeta}}\ \frac{g(\eta,\zeta,t)\ d\zeta}{(\xi'-\zeta)(\zeta-\eta)}. \tag{29}$$

Finally, we apply Kelvin's theorem as have just expounded for (13) to obtain for γ_w the equation

$$\Gamma_0(t)+\int_1^{\xi_m}\left\{\sqrt{\frac{\xi+1}{\xi-1}}+N_w(\xi,t)+N_b(\xi,t)\right\}\gamma_w(\xi)\ d\xi=0. \tag{30}$$

This is the *generalized Wagner's integral equation* for wake vorticity γ_w in closed form. It will be called the *nonlinear wake-vorticity integral equation.* Of the different terms in this equation, $\Gamma_0(t)$ has a component in its integral with kernel $g(\xi,\xi';t)$ representing a local nonlinear effect on the flow due to change in body shape. In the wake integral, the term with $N_w(\xi,t)$ represents the nonlinear wake effects primarily due to non-uniformity of the wake vorticity resulting from finite changes in orientation and velocity of body movement. The other term with $N_b(\xi,t)$ represents the nonlinear effects due to change in body shape; so N_b vanishes completely for flat wing (by virtue of (16)). In the linear limit, N_b and N_w both vanish, reducing (30) to Wagner's integral equation.

In practice, it is convenient to start with the motion of S_b prescribed for $t\geq 0$. In a small time interval δt_k at $t=t_k>0$, a new segment of S_w is created (due to body moving forward) in the wake just beyond the trailing edge (at $\xi=1$), namely $\delta z(1,t)=\overline{W}(1,t)\delta t_k$. The wake vorticity shed into this small segment of S_w can be obtained, by analysis and numerics, *accurately* by applying (30). Once the local γ_w of that fluid particle (leaving the trailing edge at $t=t_k$) is determined, its value will remain invariant with the particle, by Helmholtz's theorem, and move on with wake fluid at velocity $W_w(\xi,t)$ of (11) for $t>t_k$ $(k=1,2,\cdots)$. Therefore, the key step in using the nonlinear wake-vorticity integral equation (30) is at the initial time step in which the starting vortex is shed from the trailing edge and at each successive time steps when a new vortex element is shed continuously into the wake.

In conclusion, we have addressed the issue concerning the generation of entire vortex distribution over a flexible wing moving in arbitrary manner, with various nonlinear effects identified that can be applied to self-propulsion. Although the final form (30) is based on iteration of the residual term to first order, higher-order iterations are straightforward, which should converge very rapidly (primarily due to the smallness of the kernel g as stressed by Wu[5]) and can be easily assessed for their contributions to finding the nonlinear effects accurately. These nonlinear effects are expected to play an active and important role in aerial and aquatic animal locomotion.

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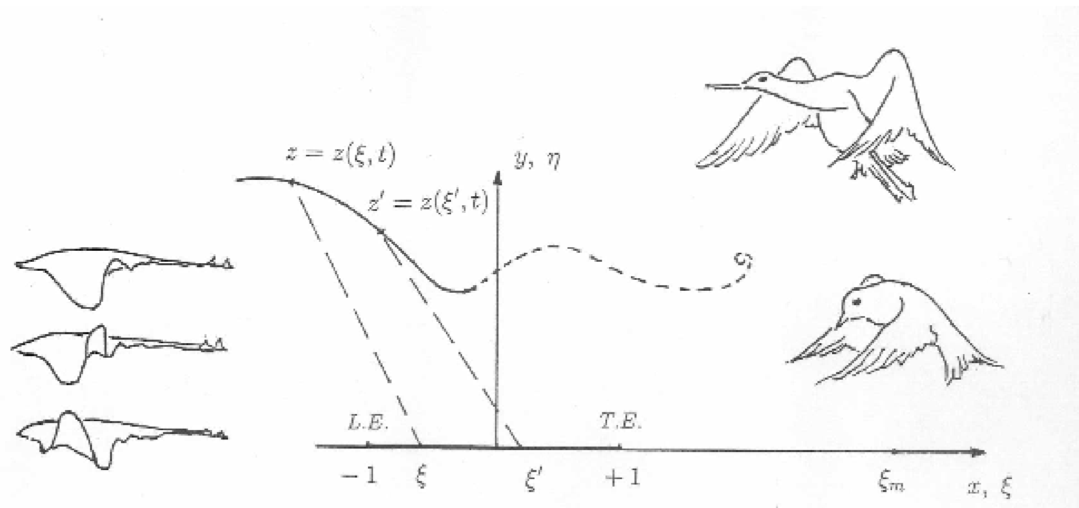

Figure 1: The Lagrangian coordinates (ξ, η) used to describe arbitrary motion of a two-dimensional flexible lifting surface moving along arbitrary trajectory through fluid in an inertial frame fixed with the fluid at infinity.

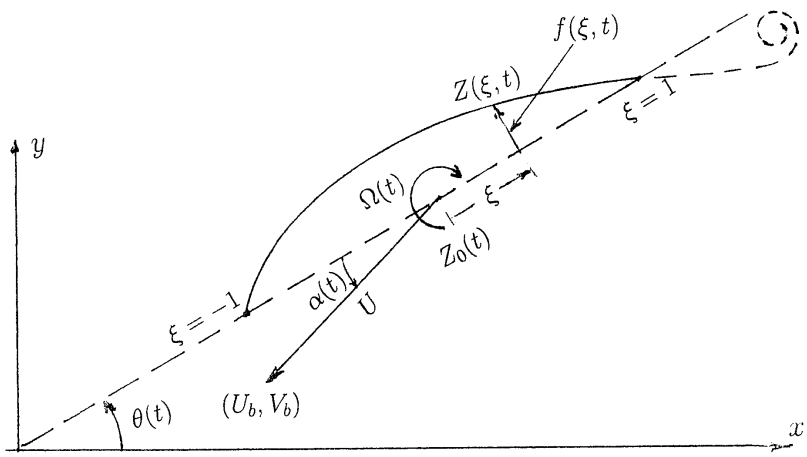

Figure 2: The wing movement is consisted of (i) rectilinear translation with velocity (U_b, V_b) at incidence angle $\alpha(t)$, (ii) rotation with angular velocity $\Omega(t)$), and (iii) time-varying camber function $f(\xi, t)$.