A Lagrangian for water waves

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A Lagrangian for strongly nonlinear unsteady water waves (including overturning waves) is obtained. It is shown that the system of quadratic equations for the Stokes coefficients, which determine the shape of a steady wave (discovered by Longuet-Higgins 100 years after Stokes derived his system of cubic equations) directly follows from the canonical system of Lagrange equations. Applications to the investigation of the stability of water waves and to the construction of numerical schemes are pointed out. © *1996 American Institute of Physics.* [S1070-6631(96)03002-7]

I. INTRODUCTION

This paper was motivated by the following two questions.

(1) In 1880, Stokes published his work on the derivation of equations for the coefficients of the Fourier expansion (in the potential plane) for the shape of a steady water wave of finite amplitude (Refs. 1 and 2). These coefficients are now called *Stokes coefficients*, and the equations, derived by Stokes, are cubic in the Stokes coefficients. They have been used by many authors to study various properties of water waves.

A century later Longuet-Higgins[3] discovered a system of quadratic equations for the Stokes coefficients which is equivalent to the original Stokes system of cubic equations. This new quadratic system is simpler and makes it easier to get higher accuracy in numerical calculation of Stokes coefficients. In particular, Saffman[4] demonstrated that the family of Stokes waves bifurcates into new families of steady waves of different forms, and Zufiria[5] found families of non-symmetric waves. (Note that this calculation requires high accuracy, as non-symmetric waves differ from Stokes waves by only about 2%.)

Is there a general principle, which directly leads to this quadratic system?

(2) Without dissipation water waves constitute a conservative mechanical system. In 1967 Zakharov[6] introduced canonical Hamiltonian variables for water waves. However, Zakharov's construction requires that the surface height is a single-valued function of the horizontal coordinates, which is not the case for overturning waves.

Do there exist canonical variables in a general situation, including overturning waves?

In this paper the Lagrangian for 2-D deep water waves of general form is found.[7] This Lagrangian is a fourth order polynomial in the Stokes coefficients, and their first time derivatives. Hamiltonian variables can be obtained in a standard way with the aid of the Legendre transform. However, the solution of the equations in the Legendre transform is not simple; it leads to the representation of the Hamiltonian by a Taylor series in wave amplitudes as was worked out by Zakharov[6] for small amplitude waves, or to the Padé like approximation used by Zufiria.[5] We see that the Lagrangian representation of water waves is simpler than the Hamiltonian one, as the Lagrangian is given exactly by a polynomial, while the Hamiltonian is given by some infinite (Taylor or Padé) expansion.

For a wave of permanent form (when the time derivatives of the Stokes coefficients are given by the speed of the wave and the Stokes coefficients themselves) this Lagrangian becomes a cubic polynomial, and the equations for the Stokes coefficients (given by the variation of the Lagrangian) turn out to be quadratic, equivalent to Longuet-Higgins' equations.[3] We immediately obtain that these quadratic equations are derivable from a potential function, a fact found by Longuet-Higgins[8] from inspection of his quadratic system.

The approach can also be useful for the investigation of the stability of water waves, as the dynamical behavior and the steady waves are described with the aid of the same representation (in terms of the Stokes coefficients).

After the Lagrangian is found, the problem of calculating the evolution of the water surface is reduced to the extremization of a functional of independent generalized coordinates (the principle of least action). This formulation suggests the construction of numerical schemes as follows. First, the functional of action is discretized, and then numerical equations are obtained by variation of the discretized action. In this case—like the case of simplectic numerical integration—the equations of the numerical scheme preserve important properties of the original continuous system (they are also derived from a variational principle).

Thus, the general approach of theoretical physics based on the principle of least action proves to be fruitful for the description of water waves.

II. GENERALIZED COORDINATES

We consider irrotational 2-D flow of incompressible inviscid fluid with a free surface and of infinite depth. A uniform gravity field is directed along the negative direction of the y-axis, and the flow is supposed to be λ-periodic with respect to the x-coordinate. Surface tension is neglected, and without loss of generality the fluid density ρ and the gravity constant g are put equal to 1, and the wavelength $\lambda = 2\pi$.

The domain of the complex $z = x + iy$-plane filled with the fluid can be considered as the conformal image of the lower half-plane of the auxiliary $\zeta = \xi + i\eta$-plane. This conformal transformation is given by the function

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$z(\zeta)=x(\xi,\eta)+iy(\xi,\eta)$, analytical for $\eta\leqslant 0$. The fluid velocity $w=u+iv$ is given by the complex potential $\gamma(z)=\alpha(x,y)+i\beta(x,y)$ (α is the fluid potential, and β is the stream function),

$$w^*=\frac{d\gamma}{dz}=\frac{dc}{d\zeta}\bigg/\frac{dz}{d\zeta},$$

where $c(\zeta)=a(\xi,\eta)+ib(\xi,\eta)=\gamma(z(\zeta))$ is the complex potential expressed in terms of the auxiliary variable $\zeta=\xi+i\eta$, and * denotes complex conjugation.

The surface of the fluid is the image of the real axis in the ζ-plane; the (x,y)-coordinates of the surface are given by the functions

$$X(\xi)=x(\xi,0),\quad Y(\xi)=y(\xi,0),$$

and the surface values of the fluid potential and the stream function are

$$A(\xi)=a(\xi,0),\quad B(\xi)=b(\xi,0).$$

The functions $X(\xi),Y(\xi),A(\xi),B(\xi)$ completely determine the state of the fluid, as an analytic function is determined by its boundary values. In particular, the potential energy of the fluid (per one wavelength in the x-direction) is

$$2\pi\mathscr{U}=\frac{1}{2}\int_0^{2\pi}Y^2X'\,d\xi$$

($'$ denotes differentiation with respect to ξ), its kinetic energy is

$$2\pi\mathscr{K}=\frac{1}{2}\int[(\alpha_x)^2+(\alpha_y)^2]dx\,dy=-\frac{1}{2}\int_0^{2\pi}AB'\,d\xi,$$

and its x-momentum is

$$2\pi\mathscr{P}=\int\beta_y\,dx\,dy=-\frac{1}{2}\int_0^{2\pi}BX'\,d\xi.$$

Actually, the state of the fluid is determined only by the functions $Y(\xi),B(\xi)$, as the real parts of the boundary values of analytic functions can be found from their imaginary parts (here with the aid of the Hilbert transform).

All the functions introduced above depend also on the time t, which was not written explicitly. Using the kinematic boundary condition we shall express the function $B(\xi,t)$ in terms of the function $Y(\xi,t)$ and its time derivative $\dot{Y}(\xi,t)$. [Of course, this dependence is not local: $B(\xi,t)$ depends on the values of the functions $Y(\xi,t)$ and $\dot{Y}(\xi,t)$ at all points ξ from the interval $0<\xi<2\pi$.] The velocity of the surface can differ from the velocity of the fluid at the surface by some vector tangent to the surface:

$$\dot{X}=u+\mu X',\quad \dot{Y}=v+\mu Y' \tag{1}$$

where $\mu=\mu(\xi,t)$ is some (unknown) scalar function. It depends on the choice of the parameterization $x(\xi,\eta),y(\xi,\eta)$. If the coordinates (ξ,η) were markers of fluid particles, then we would have $\mu=0$, and $\dot{X}=u,\dot{Y}=v$. For general parameterization, the vector $(\dot{X},\dot{Y})$ does not coincide with the vector (u,v) of the velocity of the fluid at the surface, but differs from it by a vector parallel to the tangent vector (X',Y'). From the very beginning, we have required that the transformation from the auxiliary (ξ,η)-plane to the physical (x,y)-plane be a conformal map. This requirement defines the function $\mu(\xi,t)$. However, we will not need the function $\mu(\xi,t)$. Indeed, using (1), we find

$$\begin{aligned}B'=\beta_xX'+\beta_yY'&=-(\dot{Y}-\mu Y')X'+(\dot{X}-\mu X')Y'\\&=\dot{X}Y'-\dot{Y}X'\end{aligned} \tag{2}$$

independently of $\mu(\xi,t)$. Therefore, we can find function $B(\xi,t)$ in terms of the functions $Y(\xi,t),\dot{Y}(\xi,t)$. The function $A(\xi,t)$ is determined in terms of the function $B(\xi,t)$, since $A(\xi,t)+iB(\xi,t)$ is the boundary value of an analytic function. Thus, the state of the fluid is determined by the functions $Y(\xi,t),\dot{Y}(\xi,t)$.

The representation of the fluid flow in terms of conformal mapping takes care of all the constraints of 2-D incompressible irrotational hydrodynamics, except for one constraint — the conservation of the fluid's volume. Therefore, we additionally require that

$$2\pi\mathscr{V}\equiv\int_0^{2\pi}YX'\,d\xi=0 \tag{3}$$

(this corresponds to the choice of the zero of the y-axis being at the surface of the undisturbed fluid).

Now we can consider the functional

$$\mathscr{S}=\int_{t_a}^{t_b}(\mathscr{K}-\mathscr{U})dt \tag{4}$$

where the kinetic energy $\mathscr{K}$ and the potential energy $\mathscr{U}$ are expressed as functionals of the functions $Y(\xi,t),\dot{Y}(\xi,t)$ $(0<\xi<2\pi)$; t_a,t_b are some instants of time). The equations of motion are obtained by variation of this functional with respect to the function $Y(\xi,t)$ [with fixed values at the ends of the interval (t_a,t_b)] under the constraint (3): they are given by the condition

$$\frac{\delta\mathscr{S}}{\delta Y(\xi,t)}=0,$$

where the function Y satisfies (3).

We shall express the functional (4) explicitly in terms of the function Y using an expansion in Fourier series. In this way we shall also satisfy the constraint (3), and thereby express the functional $\mathscr{S}$ in terms of (independent) generalized coordinates; then the equations of motion will be obtained by variation of $\mathscr{S}$ with respect to these generalized coordinates without any constraints.

The surface of the flow 2π-periodic in the x-direction can be represented in the form

$$X(\xi,t)+iY(\xi,t)=\xi+\sum_{k=-\infty}^{+\infty}[X_k(t)+iY_k(t)]e^{-ik\xi} \tag{5}$$

$(X_k=X_{-k}^*,Y_k=Y_{-k}^*)$. In the frame of reference where the fluid velocity at $y=-\infty$ is zero, the complex potential $\gamma(x+iy)$ is 2π-periodic with respect to x, and therefore we have the Fourier expansion

$$A(\xi,t)+iB(\xi,t)=\sum_{k=-\infty}^{+\infty}[A_k(t)+iB_k(t)]e^{-ik\xi} \tag{6}$$

$(A_k = A^*_{-k}, B_k = B^*_{-k})$. In other frames of references the right hand side in (6) would contain an additional term $\nu\xi$ where the constant ν is the speed of this frame.

According to the Hilbert transform we have for $k \neq 0$

$$X_k = i\sigma_k Y_k, \quad A_k = i\sigma_k B_k \tag{7}$$

where σ_k is $+1$ for positive k and -1 — for negative k. The relations (7) can be easily understood. For example, the complex potential of the fluid can be restored from its surface values in the following form:

$$c(\xi + i\eta) = \sum_{k=-\infty}^{+\infty} [A_k + iB_k] e^{-ik(\xi + i\eta)}.$$

It is analytic in the lower half-plane if $A_k + iB_k = 0$ for $k<0$. This gives the second relation in (7) if we recall that $A_k = A^*_{-k}, B_k = B^*_{-k}$.

Since the complex potential is defined up to an arbitrary constant, we can assume that $A_0 = B_0 = 0$. We can also choose the origin of the x-axis so that $X_0 = 0$. Let us define $\sigma_0 = 0$. Then for all integers k, including $k=0$, we have

$$\begin{aligned} X_k &= i\sigma_k Y_k, \\ A_k &= i\sigma_k B_k, \end{aligned} \quad \text{where } \sigma_k = \begin{cases} +1, & \text{if } k>0, \\ 0, & \text{if } k=0, \\ -1, & \text{if } k<0. \end{cases} \tag{8}$$

The constraint (3) takes the form

$$Y_0 = -\sum_{k=-\infty}^{+\infty} |k| Y_k Y_{-k}. \tag{9}$$

Thus, the quantities Y_k ($k = \pm 1, \pm 2, \ldots$; note that $k=0$ is not included) can serve as generalized coordinates. The quantity Y_0 is given by (9). The Fourier coefficients B_k are determined according to (2),

$$B_k = \frac{i}{k}\left[-\dot{Y}_k + \sum_{k_1+k_2=k} \dot{Y}_{k_1} Y_{k_2} k_2 (\sigma_{k_1} - \sigma_{k_2}) \right] \quad (k \neq 0) \tag{10}$$

where the summation is over all pairs of integers k_1, k_2 satisfying the relation $k_1 + k_2 = k$ [X, A are determined from (8)].

III. THE LAGRANGIAN

The potential energy $\mathscr{U}$, the kinetic energy $\mathscr{K}$, the momentum $\mathscr{P}$, and the Lagrangian $\mathscr{L} = \mathscr{K} - \mathscr{U}$ are the functionals of the generalized coordinates $Y_k (k = \pm 1, \pm 2, \ldots)$:

$$\begin{aligned} \mathscr{U} &= \frac{1}{2}\sum_{k=-\infty}^{+\infty} Y_k Y_{-k} + \frac{1}{6} \sum_{k_1+k_2+k_3=0} (|k_1| + |k_2| + |k_3|) \\ &\quad \times Y_{k_1} Y_{k_2} Y_{k_3}, \\ \mathscr{K} &= \frac{1}{2}\sum_{k=-\infty}^{+\infty} |k| B_k B_{-k}, \\ \mathscr{P} &= \sum_{k=-\infty}^{+\infty} |k| B_k Y_{-k}, \\ \mathscr{L} &= \frac{1}{2}\sum_{k=-\infty}^{+\infty} |k| B_k B_{-k} - \frac{1}{2}\sum_{k=-\infty}^{+\infty} Y_k Y_{-k} \\ &\quad - \frac{1}{6}\sum_{k_1+k_2+k_3=0} (|k_1| + |k_2| + |k_3|) Y_{k_1} Y_{k_2} Y_{k_3}, \end{aligned} \tag{11}$$

where Y_0 and B_k are given by (9), (10), respectively.

The Lagrangian (11) is invariant with respect to the transformation

$$Y_k \to Y_k e^{ik\theta}$$

(θ is an arbitrary constant phase) and with respect to an arbitrary time shift. In accordance with Noether's theorem these symmetries correspond to the conservation of the momentum $\mathscr{P}$ and the energy $\mathscr{E} = \mathscr{K} + \mathscr{U}$, respectively.

The equations of motion are obtained by variation of the action

$$\mathscr{S} = \int_{t_a}^{t_b} \mathscr{L}[Y_k(t), \dot{Y}_k(t) | k = \pm 1, \pm 2, \ldots] dt \tag{12}$$

(t_a, t_b are initial and final instants); they have the form

$$\frac{\delta \mathscr{S}}{\delta Y_k(t)} \equiv -\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{Y}_k}\right) + \frac{\partial \mathscr{L}}{\partial Y_k} = 0 \quad (k = \pm 1, \pm 2, \ldots) \tag{13}$$

[it is assumed, that $\delta Y_k(t_a) = \delta Y_k(t_b) = 0$].

The formulation of the dynamical equations in the variational form [(12)–(13)] allows us to develop numerical schemes preserving important properties of the original system. In these methods the discretization is made in the functional of action, and the numerical equations are obtained by the variation of the discretized action.

Let us consider one example of such a scheme with a non-constant time step.

Consider a truncated system retaining only coefficients Y_k with $|k| \leqslant K$ (K is a positive integer). Then, according to (10) we find an interesting property of the representation in terms of the coefficients Y_k, namely

$$B_k = 0 \quad \text{if } |k| > K, \tag{14}$$

which means that aliasing does not occur in the calculation of B_k.

Introduce time discretization

$$t_0, t_1, \ldots, t_n, \ldots, \quad \tau_n = t_n - t_{n-1}, \quad Y_k^n = Y_k(t_n)$$

and replace time derivatives $\dot{Y}_k (k \neq 0)$ by finite differences of the first order

$$D^n Y_k = \frac{1}{\tau_n}(Y_k^n - Y_k^{n-1}) \quad (k = \pm 1, \pm 2, \ldots).$$

Approximate the integral of action (12) by the following sum:

$$S = \sum_{n=0}^{N+1} \left\{ \frac{1}{2}\sum_{k=-K}^{+K} |k| B_k^n B_{-k}^n - \frac{1}{2}\sum_{k=-K}^{+K} Y_k^n Y_{-k}^n \right. \\ \left. - \frac{1}{6}\sum_{k_1+k_2+k_3=0} (|k_1| + |k_2| + |k_3|) Y_{k_1}^n Y_{k_2}^n Y_{k_3}^n \right\}$$

($t_0=t_a, t_{N+1}=t_b$), where

$$B_k^n=\frac{i}{k}\left[-(D^nY_k)+\sum_{k_1+k_2=k}(D^nY_{k_1})Y_{k_2}k_2(\sigma_{k_1}-\sigma_{k_2})\right]$$

$$(k\neq 0),$$

$$Y_0=-\sum_{k=-K}^{+K}|k|Y_k^nY_{-k}^n,$$

$$D^nY_0=-2\sum_{k=-K}^{+K}|k|(D^nY_k)Y_{-k}^n.$$

The equations of the numerical scheme have the form

$$\frac{\partial S}{\partial Y_k^n}=0 \quad (k=\pm 1,\pm 2,\ldots,K;\quad n=1,2,\ldots,N). \quad (15)$$

This method allows us to investigate overturning waves (when the surface height is not a single-valued function of the horizontal coordinate). However, to study ‘‘quite distorted geometries’’ one may need many Fourier coefficients Y_k (large K), as was noted by Meiron *et al.*[9] This happens when the conformal map $z(\zeta)$ has singularities close to the real axis (in the upper half-plane $\eta>0$), and the Fourier series (5) converges slowly. This can be compensated for by the possibility of using the fast Fourier transform. Here the property (14) is useful; it follows from this property that one needs to deal with aliasing in the equations (15) only for the calculation of quadratic expressions, although the equations (15) are cubic. The numerical schemes with the variational principle could be important for the correct computer simulation of water waves.

IV. STEADY WAVES

For a wave of permanent form moving with the speed C, the coefficients Y_k depend on time t according to the formula $Y_k=\bar{Y}_k e^{ikCt}$ ($\bar{Y}_k$ is independent of time). The equations for the coefficients Y_k of a steady wave can be found directly from the least action principle (12)–(13).

In the usual formulation of the least action principle the values of the generalized coordinates are fixed at the ends of the time interval (t_a, t_b) (which provides for vanishing of the finite terms in the calculation of the variational derivatives). This can be replaced by the requirement that the generalized coordinates are periodic with respect to time with a fixed period $T=t_b-t_a$.

For a steady wave $\dot{Y}_k=ikCY_k$, and according to (10), we find $B_k=CY_k$ [the quadratic term in (10) vanishes]. Then, from (11) we have $\mathscr{K}=-C^2Y_0/2$, $\mathscr{P}=-CY_0$ [where we have taken advantage of (9)]. We see that the Lagrangian for steady waves is constant in time, and according to (11)–(12),

$$\mathscr{S}=-T\left\{\frac{C^2}{2}Y_0+\mathscr{U}\right\}\quad\left(T=t_b-t_a=\frac{2\pi}{C}\right), \quad (16)$$

where Y_0 is given by (9). The function (16) is a potential function for the equations defining coefficients Y_k ($k=\pm 1,\pm 2,\ldots$) of a steady wave; these equations have the form

$$\frac{\partial\mathscr{S}}{\partial Y_k}=0 \quad (k=\pm 1,\pm 2,\ldots). \quad (17)$$

While differentiating in (17) one should take into account that the coefficient Y_0 is not independent, but is a function of the rest of the coefficients Y_k. However, $\partial\mathscr{U}/\partial Y_0=0$ when (9) is satisfied, and while differentiating potential energy $\mathscr{U}$ one can consider Y_0 as a constant. The equations (17) together with the relation (9) can be written as a system of quadratic equations for all the coefficients Y_k, including Y_0,

$$\frac{\partial\mathscr{U}}{\partial Y_k}=C^2kY_k \quad (k=0,\pm 1,\pm 2,\ldots). \quad (18)$$

The equation (18) with $k=0$ gives the relation (9), and the rest of the equations are the equations (17). The system (18) is equivalent to the Longuet-Higgins system (Longuet-Higgins' coefficients a_k are related to the coefficients Y_k by simple formulae $a_0=2Y_0-c^2$, $a_k=2Y_k$; see Refs. 8 and 5).

V. CONCLUSION

Thus, Longuet-Higgins' system of quadratic equations[3] for the Stokes coefficients (which is equivalent to the original Stokes system of cubic equations[2]) is the most natural system to determine the shape of a steady wave; it logically appears according to the formalism of analytical mechanics. (Why did it take a century to discover this quadratic system?) The presented method explicitly takes into account all the symmetries and constraints of the nonlinear dynamics of the 2-D water waves and allows us to develop numerical schemes which inherit many of these symmetries. (In particular, the matrices arising in the computation of the time evolution or in the calculation of steady waves with the aid of Newton's method, turn out to be Hermitian.) This method can be crucial in some other problems, in particular, the investigation of stability of water waves (especially for families other than the family of the Stokes waves), as the eigenvalue equations can be easily obtained. This approach can also be applied to the problem of waves on the surface of a fluid with constant vorticity, considered by Simmen and Saffman.[10] Recently Zakharov's Hamiltonian approach[6] has been combined with the conformal mapping.[11] This has been applied to prove nonintegrability of the free surface hydrodynamics.[12]

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