Almost Classically Damped Continuous Linear Systems

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The dynamic response of a general class of continuous linear vibrating systems is analyzed which possess damping properties close to those resulting in classical (uncoupled) normal modes. First, conditions are given for the existence of classical modes of vibration in a continuous linear system, with special attention being paid to the boundary conditions. Regular perturbation expansions in terms of undamped modeshapes are then utilized for analyzing the eigenproblem as well as the vibration response of almost classically damped systems. The analysis is based on a proper splitting of the damping operators in both the field equations and the boundary conditions. The main advantage of this approach is that it allows application of standard modal analysis methodologies so that the problem is reduced to that of finding the frequencies and mode shapes of the corresponding undamped system. The approach is illustrated by two simple examples involving rod and beam vibrations.

Introduction

In many instances, the dynamic response of a vibrating system can be well predicted by a linear model. When damping effects are neglected, the equations of motion of these systems can be reduced to a set of uncoupled ordinary differential equations, each representing a single-degree-of-freedom linear oscillator. However, introduction of linear viscous damping allows this uncoupling only if the damping distribution bears a special relationship with the mass and stiffness distributions of the system (Caughey, 1960; Caughey and O'Kelly, 1965). If it does, the system is often said to be "classically damped" and to possess "classical normal modes." The terminology "proportional damping" is also used but this is easily confused with the very special case of Rayleigh damping (Rayleigh, 1945), where the damping distribution is a linear combination of the mass and stiffness distributions.

The existence of classical normal modes provides a simple physical interpretation of the behavior of a vibrating system as a linear superposition of characteristic modes of vibration of the system. It also leads to computational advantages when only a few modes of vibration contribute to the response of the system because of a limited bandwidth excitation. When the conditions for classical modes are not met exactly (e.g., Natsiavas, 1993), the system response may be obtained by one of several different methods: by direct numerical integration of the equations of motion; or by properly converting these equations to first-order state-space form (e.g., Foss, 1958; Meirovitch, 1967); or by approximating the damping in some way so that classical modes exist (Bellos and Inman, 1989).

The present work focuses on the response of linear continuous systems which do not possess classical normal modes, but the damping properties are close to those meeting the conditions leading to normal modes. These systems are called "almost classically damped" and are of practical significance. As a result, various aspects of their response have already been examined by others, but with emphasis placed on discrete systems. For example, Chung and Lee (1986) extended a perturbation methodology developed earlier by Meirovitch and Ryland (1979) in order to obtain the eigensolution of these discrete systems. Also, Cronin (1976) presented a perturbation analysis for the response of such systems under harmonic excitation, while Udwadia and Esfandiari (1990) and Udwadia and Kumar (1994) presented an iterative approach for general forcing functions. Other relevant work in discrete systems include the contributions of Knowles (1985), Nicholson (1987), Shahruz and Ma (1988), and Natsiavas and Beck (1994). Various aspects of damped continuous linear systems have also been examined (e.g., Caughey and O'Kelly, 1965; Pan, 1966; Plaut and Infante, 1972; Inman and Andry, 1982; Bergman and Nicholson, 1985; Yang, 1996). More references and information about damping effects can also be found in the books by Snowdon (1968) and Nashif et al. (1985).

The objectives of the present work are, first, to present conditions for the existence of classical modes for continuous linear vibrating systems and, second, to present a perturbation approach for the approximate solution of almost classically damped continuous linear systems under arbitrary forcing functions. For the first objective, the work of Caughey and O'Kelly (1965) is extended by being specific about the concept of "compatible" internal and boundary operators for both the stiffness and damping characteristics. For the second objective, an analysis is developed, based on a proper decomposition of the internal and boundary damping operators. The approach uses classical modal analysis so that the solution is obtained by utilizing only the real modes and frequencies of the corresponding undamped system. This is especially efficient in cases where the solution of a classically damped system has been obtained and the effect of small changes in the damping distribution is sought. Finally, the approach is illustrated by two simple examples.

Problem Statement and Definitions

Boundary Value Problem for Forced Vibrations. The forced vibration behavior of a linear elastic viscously damped continuous body, such as a stretched rod, a torsional shaft, a bending beam, a membrane or a plate, is commonly governed by an equation of motion of the form

$$m(\mathbf{r})\ddot{u} + L_1\dot{u} + L_2u = f(\mathbf{r}, t) \quad \forall \mathbf{r} \in S \tag{1}$$

where the body occupies $\bar{S} = S \cup \partial S$. Here S is a bounded domain in $\mathbb{R}^{n_s}$ (n_s = 1, 2 or 3) with boundary ∂S. Typically, $u(\mathbf{r}, t)$ is a (generalized) displacement at position $\mathbf{r}$ in the body

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at time t, while $\dot{u}(\mathbf{r}, t)$ and $\ddot{u}(\mathbf{r}, t)$ denote the corresponding velocity and acceleration; $m(\mathbf{r})$ is the mass density of the body; $f(\mathbf{r}, t)$ is an external body force and L_1 and L_2 are time-invariant linear differential spatial operators of orders n_1 and n_2, respectively (typically $n_1 \le n_2$), which model the internal damping forces and elastic restoring forces, respectively, within the body.

The appropriate boundary conditions for (1) are of the form

$$D_N^{(i-1)} u = O \quad \forall \mathbf{r} \in \partial S_i$$

$$\text{and} \quad B_i u + C_i \dot{u} = g_i(\mathbf{r}, t) \quad \forall \mathbf{r} \in \partial S_i' \tag{2}$$

for $i = 1, \ldots, n_b$ ($=1/2n_2$, typically), expressing the fact that for each i, on some parts of the boundary (∂S_i), a geometric condition will hold, while on the other parts ($\partial S_i'$), a generalized force condition will hold (so $\partial S_i \cup \partial S_i' = \partial S$, $\partial S_i \cap \partial S_i' = \phi$). $D_N^{(i)}$ denotes the derivative of order i normal to the boundary surface at $\mathbf{r} \in \partial S$, with $D_N^{(0)} u$ being simply u itself. Each B_i and C_i is a time-invariant linear differential spatial operator of order $(n_2 - i)$ and $(n_1 - i)$, respectively, which models the deformation-induced internal and external generalized restoring forces and damping forces, respectively, at the boundary. Each $g_i(\mathbf{r}, t)$ represents the resultant of any external generalized forces applied at the boundary of the body at time t.

An important additional assumption is that the B_i and C_i are *compatible* boundary operators for L_2 and L_1, respectively, in the sense that they are related through "integration by parts" (or a generalized Green's formula) by

$$\langle L_1\psi, \phi\rangle - \langle \psi, L_1\phi\rangle = \sum_{i=1}^{n_b} [(D_N^{(i-1)}\psi, C_i\phi) - (C_i\psi, D_N^{(i-1)}\phi)] \tag{3}$$

and

$$\langle L_2\psi, \phi\rangle - \langle \psi, L_2\phi\rangle = \sum_{i=1}^{n_b} [(D_N^{(i-1)}\psi, B_i\phi) - (B_i\psi, D_N^{(i-1)}\phi)] \tag{4}$$

for all $\psi, \phi \in V$, a linear space of suitably smooth functions on $\bar{S}$, such as $C^{n_3}(S) \cap C^{n_3-1}(\bar{S})$, where $n_3 = \max(n_1, n_2)$, and which satisfy the geometric boundary conditions in (2). Here $C^{n_3}(S)$ denotes the set of functions on S which are continuously differentiable up to order n_3. The body and boundary inner products in (3) and (4) are defined by

$$\langle \psi, \phi\rangle = \int_S \psi(\mathbf{r})\phi(\mathbf{r})d\mathbf{r} \tag{5}$$

and

$$(\psi, \phi) = \int_{\partial S} \psi(\mathbf{r})\phi(\mathbf{r})dS(\mathbf{r}). \tag{6}$$

The compatibility conditions (3) and (4) are types of reciprocity principles which are to hold in the absence of external applied body and boundary forces. Condition (3) can be interpreted as stating that the virtual power from the virtual velocity ϕ together with the body and boundary damping forces set up by the virtual velocity ψ must be equal to the virtual power with the roles of ϕ and ψ swapped. Similarly, condition (4) can be interpreted as stating that the virtual work from the virtual displacement ϕ together with the body and boundary restoring forces set up by the virtual displacement ψ must be equal to the virtual work with the roles of ϕ and ψ swapped.

Equations (1) and (2), together with specified initial conditions $u(\mathbf{r}, 0)$ and $\dot{u}(\mathbf{r}, 0)$ on S, are assumed to give a well-posed boundary value problem on $S \times [0, \infty)$ with solution $u \in V$, $\forall t \in [0, \infty)$. The goal is to find an approximate solution when the system is almost classically damped.

Undamped System. We start with the *undamped homogeneous* version of Eqs. (1) and (2) where L_1 and f are zero on S and each C_i and g_i is zero on $\partial S_i'$. We define a linear subspace U_0 of V consisting of all those functions in V which satisfy this simplified version of the boundary conditions (2). From Eq. (4),

$$\langle L_2\psi, \phi\rangle = \langle \psi, L_2\phi\rangle \quad \forall\, \psi, \phi \in U_0$$

so that L_2 is a self-adjoint linear operator on U_0 with respect to the inner product defined by (5). We assume, therefore, as is typical for vibration problems, that $m^{-1}L_2$ has a discrete spectrum of distinct positive eigenvalues $\{\omega_n^2 : n \in \mathbf{Z}^+\}$ and a corresponding complete set of orthonormal real eigenfunctions $\{\phi_n(\mathbf{r}) : n \in \mathbf{Z}^+\}$ in U_0 so that $\forall\, \psi \in V$:

$$\psi(\mathbf{r}) = \sum_{n=1}^{\infty} \langle \psi, m\phi_n\rangle \phi_n(\mathbf{r}) \quad \forall \mathbf{r} \in S \tag{7}$$

where the orthonormality guaranteed by the self-adjointness of L_2 gives.

$$\langle \phi_n, m\phi_p\rangle = \delta_{np}, \tag{8}$$

the Kronecker delta, and

$$L_2\phi_n = \omega_n^2 m\phi_n \quad \text{on } S, \tag{9}$$

$$D_N^{(i-1)}\phi_n = 0 \quad \text{on } \partial S_i \quad \text{and} \quad B_i\phi_n = 0 \quad \text{on } \partial S_i' \quad \forall i = 1, \ldots, n_b. \tag{10}$$

In general, the eigenfunction expansion in (7) will only converge to ψ on the boundary ∂S if ψ and the ϕ_n satisfy the same boundary conditions (e.g., Courant and Hilbert, 1989, Chapter 5). In this case, it is typical to have uniform convergence on $\bar{S}$ and no Gibbs phenomenon appears near the boundary ∂S. This behavior is advantageous in applications because then the eigenfunction series converges faster, that is, a smaller number of terms is required in (7) to give an approximation of the function ψ to within a specified accuracy.

Almost Classically Damped System. We now consider two special cases of the full damped linear system given by Eqs. (1) and (2):

Definition: The damped system is *classically damped* if the body and boundary damping operators satisfy

$$L_1\phi_n = \beta_n m\phi_n \quad \text{on } S \tag{11}$$

for some constants $\beta_n \ge 0$, and

$$C_i\phi_n = 0 \quad \text{on } \partial S_i' \quad \forall\, i = 1, \ldots, n_b. \tag{12}$$

This means that the eigenfunctions ϕ_n, $n \in \mathbf{Z}^+$, of L_2 on U_0 are also eigenfunctions of L_1 on U_0. For this to be possible, it is necessary that L_1 be a positive semidefinite self-adjoint linear operator on U_0 and that $m^{-1}L_1$ and $m^{-1}L_2$ commute on U_0. These conditions are also sufficient for (11) to hold (Caughey and O'Kelly, 1965). We shall hereafter denote the body and boundary damping operators for a classically damped system by L_{10} and C_{i0}, $i = 1, \ldots, n_b$, respectively.

The eigenfunctions of a classically damped system are identical to the undamped eigenfunctions, $\phi_n(\mathbf{r})$, $n \in \mathbf{Z}^+$. To show this, let $\hat{\phi}_n$ denote an eigenfunction corresponding to eigenvalue ρ_n, then $\hat{\phi}_n(\mathbf{r})e^{\rho_n t}$ is a solution of Eqs. (1) and (2) with $f = 0$ on S and each $g_i = 0$ on $\partial S_i'$, so

$$(\rho_n^2 m + \rho_n L_{10} + L_2)\hat{\phi}_n = 0 \quad \text{on } S$$

and $\forall i = 1, \ldots n_b$,

$$D_N^{(i-1)}\hat{\phi}_n = 0 \text{ on } \partial S_i, \quad (B_i + \rho_n C_{i0})\hat{\phi}_n = 0 \quad \text{on } \partial S_i'.$$

In view of Eqs. (9)–(12), $\hat{\phi}_n = \phi_n$ is the solution if ρ_n satisfies

$$\rho_n^2 + \beta_n\rho_n + \omega_n^2 = 0. \tag{13}$$

The two roots of (13) corresponding to a given ϕ_n are a complex conjugate pair in the underdamped case, that is, when $\beta_n < 2\omega_n$.

Definition. The damped system defined by equations (1) and (2) is *almost classically damped* if

$$L_1 = L_{10} + \epsilon L_{11}, \quad C_i = C_{i0} + \epsilon C_{i1} \quad \forall i = 1, \ldots, n_b \tag{14}$$

where ϵ is a small parameter and L_{10} and the C_{i0} are the classically damped operators defined by (11) and (12), respectively.

In this case, the damping operators are perturbed slightly from the classically damped case, so it is possible to analyze the system using an approximate perturbation approach, as shown in the subsequent sections. Although ϵ is assumed to be small, the underlying classical damping need not be small. It is assumed that the compatibility condition (3) is satisfied by the classical-damping operators L_{10} and C_{i0}, $\forall i$, and so by linearity, the damping operators L_{11} and C_{i1}, $\forall i$, also satisfy this condition.

Approximate Solution of the Eigenproblem

Let $\hat{\phi}_p$, $p \in \mathbf{Z}^+$, be an eigenfunction of the almost classically damped system, with corresponding eigenvalue s_p, then

$$u_p(\mathbf{r}, t) = \hat{\phi}_p(\mathbf{r})e^{s_p t}$$

is a solution of Eqs. (1) and (2) with $f = 0$ on S and each $g_i = 0$ on $\partial S_i'$, so

$$(s_p^2 m + s_p L_1 + L_2)\hat{\phi}_p = 0 \quad \text{on } S \tag{15}$$

and $\forall i = 1, \ldots n_b$

$$D_N^{(i-1)}\hat{\phi}_p = 0 \text{ on } \partial S_i \quad \text{and}$$

$$(B_i + s_p C_i)\hat{\phi}_p = 0 \quad \text{on } \partial S_i'. \tag{16}$$

Since ϵ is small in (14), we expect that $\hat{\phi}_p$ and s_p will remain close to the eigenquantities ϕ_p and ρ_p of the classically damped system, which satisfy Eqs. (9) through (13) when L_1 and C_i are replaced by L_{10} and C_{i0} in Eqs. (11) and (12). We therefore use the asymptotic expansions (Erdelyi, 1956)

$$\hat{\phi}_p(\mathbf{r}) \sim \phi_p(\mathbf{r}) + \epsilon\chi_p(\mathbf{r}) + \epsilon^2\psi_p + \ldots \tag{17}$$

$$s_p \sim \rho_p + \epsilon\sigma_p + \epsilon^2\tau_p + \ldots. \tag{18}$$

In the underdamped case, each s_p is complex-valued, and so, therefore, are the quantities in these expansions, except ϕ_p. Corresponding to each complex conjugate eigenvalue pair, ρ_p and $\bar{\rho}_p$, for the underlying classically damped system, there are perturbed eigenvalues s_p and $\bar{s}_p$, respectively, with associated eigenfunctions $\hat{\phi}_p(\mathbf{r})$ and $\bar{\hat{\phi}}_p(\mathbf{r})$. It is therefore only necessary to determine the perturbed eigenvalue and eigenfunction for one of the two complex roots of Eq. (13). In the overdamped case, however, it is necessary to determine separately a perturbed eigenvalue and eigenfunction for each of the real-valued roots of Eq. (13).

For suitably small ϵ, the second-order expansions are expected to be sufficient to capture the essential features of the eigenquantities. By substituting (17) and (18) into (15) and (16), taking Eqs. (9)–(14) into account, we get from the *first-order* terms in ϵ

$$\mathcal{L}_p\chi_p = \zeta_p \equiv -m\sigma_p(2\rho_p + \beta_p)\phi_p - \rho_p L_{11}\phi_p \quad \text{on } S \tag{19}$$

and $\forall i = 1, \ldots, n_b$,

$$D_N^{(i-1)}\chi_p = 0 \quad \text{on } \partial S_i,$$

$$\mathcal{B}_{ip}\chi_p = \zeta_{ip} \equiv -\rho_p C_{i1}\phi_p \quad \text{on } \partial S_i'. \tag{20}$$

Likewise, the *second-order* terms in ϵ give

$$\mathcal{L}_p\psi_p = \eta_p \equiv -(2m\rho_p\sigma_p + \sigma_p L_{10} + \rho_p L_{11})\chi_p - m(\sigma_p^2 + 2\rho_p\tau_p + \beta_p\tau_p)\phi_p - \sigma_p L_{11}\phi_p \quad \text{on } S \tag{21}$$

and $\forall i = 1, \ldots, n_b$,

$$D_N^{(i-1)}\psi_p = 0 \text{ on } \partial S_i,$$

$$\mathcal{B}_{ip}\psi_p = \eta_{ip} \equiv -(\sigma_p C_{i0} + \rho_p C_{i1})\chi_p - \sigma_p C_{i1}\phi_p \quad \text{on } \partial S_i'. \tag{22}$$

In these equations, the body and boundary operators $\mathcal{L}_p$ and $\mathcal{B}_{ip}$ are defined by

$$\mathcal{L}_p \equiv \rho_p^2 m + \rho_p L_{10} + L_2 \tag{23}$$

$$\mathcal{B}_{ip} \equiv B_i + \rho_p C_{i0}, \quad \forall i = 1, \ldots, n_b. \tag{24}$$

Using Eqs. (9)–(13), we find

$$\mathcal{L}_p\phi_q = m\pi_{pq}\phi_q \quad \text{on } S$$

$$\mathcal{B}_{ip}\phi_q = 0 \text{ on } \partial S_i', \quad \forall i = 1, \ldots, n_b \tag{25}$$

where π_{pq} is defined by

$$\pi_{pq} = \rho_p^2 + \rho_p\beta_q + \omega_q^2 \tag{26}$$

so that $\pi_{pp} = 0$, from Eq. (13). Using the compatibility Eqs. (3) and (4), we can show that $\forall\psi, \phi \in V$,

$$\langle \mathcal{L}_p\psi, \phi\rangle - \langle \psi, \mathcal{L}_p\phi\rangle = \sum_{i=1}^{n_b} [(D_N^{(i-1)}\psi, \mathcal{B}_{ip}\phi) - (\mathcal{B}_{ip}\psi, D_N^{(i-1)}\phi)]. \tag{27}$$

To determine the first-order terms χ_p and σ_p in (17) and (18) from Eqs. (19) and (20) in a way that avoids convergence problems on the boundary, we first employ the decomposition

$$\chi_p(\mathbf{r}) = \hat{\chi}_p(\mathbf{r}) + \mu_p(\mathbf{r}) \quad \text{on } \bar{S} \tag{28}$$

where μ_p is chosen as any function in V which satisfies the following boundary conditions for each $i = 1, \ldots, n_b$,

$$D_N^{(i-1)}\mu_p = 0 \quad \text{on } \partial S_i, \quad \mathcal{B}_{ip}\mu_p = \zeta_{ip} \quad \text{on } \partial S_i'. \tag{29}$$

Then, from Eq. (19), $\hat{\chi}_p \in V$ is found to satisfy the transformed equations

$$\mathcal{L}_p\hat{\chi}_p = \zeta_p - \mathcal{L}_p\mu_p \quad \text{on } S \tag{30}$$

and $\forall i = 1, \ldots, n_b$,

$$D_N^{(i-1)}\hat{\chi}_p = 0 \quad \text{on } \partial S_i, \quad \mathcal{B}_{ip}\hat{\chi}_p = 0 \quad \text{on } \partial S_i'. \tag{31}$$

Since $\hat{\chi}_p$ satisfies the same boundary conditions as the complete orthonormal set $\{\phi_q(\mathbf{r}), q \in \mathbf{Z}^+\}$ (see Eqs. (10) and (25)), its eigenfunction expansion

$$\hat{\chi}_p(\mathbf{r}) = \sum_{q=1}^{\infty} \alpha_{pq}\phi_q(\mathbf{r}) \quad \text{on } \bar{S} \tag{32}$$

where

$$\alpha_{pq} = \langle \hat{\chi}_p, m\phi_q\rangle \tag{33}$$

can also be expected to converge on the boundary ∂S (e.g., Courant and Hilbert, 1989, Chapter 5). To determine χ_p, it remains to determine the α_{pq}, $q \in \mathbf{Z}^+$.

Using Eqs. (25)–(27) and (29)–(31), we get

$$\langle \mathcal{L}_p\hat{\chi}_p, \phi_q\rangle = \pi_{pq}\alpha_{pq} \tag{34}$$

$$\langle \mathcal{L}_p\mu_p, \phi_q\rangle = \pi_{pq}\mu_{pq} + \rho_p\zeta_{pq} \tag{35}$$

if we define

$$\mu_{pq} = \langle \mu_p, m\phi_q \rangle \tag{36}$$

$$\hat{\zeta}_{pq} = \sum_{i=1}^{n_b} (C_{i1}\phi_p, D_N^{(i-1)}\phi_q)_{\partial S_i'}. \tag{37}$$

But from Eq. (19)

$$\langle \mathcal{L}_p\chi_p, \phi_q \rangle = \langle \zeta_p, \phi_q \rangle$$
$$= -\sigma_p(2\rho_p + \beta_p)\delta_{pq} - \rho_p\phi_{pq} \tag{38}$$

if we define

$$\phi_{pq} = \langle L_{11}\phi_p, \phi_q \rangle. \tag{39}$$

Using Eq. (30) to relate Eqs. (34), (35), and (38),

$$\pi_{pq}\alpha_{pq} = -\sigma_p(2\rho_p + \beta_p)\delta_{pq} - \rho_p\phi_{pq} - \pi_{pq}\mu_{pq} - \rho_p\hat{\zeta}_{pq}. \tag{40}$$

When $p = q$, this equation determines the first-order eigenvalue correction constant

$$\sigma_p = -\rho_p \frac{\hat{\zeta}_{pp} + \phi_{pp}}{2\rho_p + \beta_p}, \tag{41}$$

while for $p \neq q$, it determines the Fourier coefficients

$$\alpha_{pq} = -\rho_p(\hat{\zeta}_{pq} + \phi_{pq})/\pi_{pq} - \mu_{pq}. \tag{42}$$

The coefficients α_{pp} will be determined later.

Next, in order to determine the second-order terms ψ_p and τ_p in (17) and (18) from Eqs. (21) and (22), we again employ a decomposition

$$\psi_p(\mathbf{r}) = \hat{\psi}_p(\mathbf{r}) + \upsilon_p(\mathbf{r}) \text{ on } \bar{S} \tag{43}$$

where υ_p is chosen as any function in V which satisfies for each $i = 1, \ldots, n_b$, the conditions

$$D_N^{(i-1)}\upsilon_p = 0 \quad \text{on } \partial S_i, \quad \mathcal{B}_{ip}\upsilon_p = \eta_{ip} \quad \text{on } \partial S_i'. \tag{44}$$

Then, from Eq. (21), $\hat{\psi}_p \in V$ satisfies

$$\mathcal{L}_p\hat{\psi}_p = \eta_p - \mathcal{L}_p\upsilon_p \quad \text{on } S \tag{45}$$

and $\forall i = 1, \ldots, n_b$,

$$D_N^{(i-1)}\hat{\psi}_p = 0 \quad \text{on } \partial S_i, \quad \mathcal{B}_{ip}\hat{\psi}_p = 0 \quad \text{on } \partial S_i'. \tag{46}$$

Therefore

$$\hat{\psi}_p(\mathbf{r}) = \sum_{q=1}^{\infty} \beta_{pq}\phi_q(\mathbf{r}) \quad \text{on } \bar{S} \tag{47}$$

where

$$\beta_{pq} = \langle \hat{\psi}_p, m\phi_q \rangle. \tag{48}$$

Using Eqs. (25)–(27) again and following a similar argument as before, we find

$$\langle \mathcal{L}_p\hat{\psi}_p, \phi_q \rangle = \pi_{pq}\beta_{pq} \tag{49}$$

$$\langle \mathcal{L}_p\upsilon_p, \phi_q \rangle = \pi_{pq}\upsilon_{pq} - \hat{\eta}_{pq}, \tag{50}$$

if we define

$$\upsilon_{pq} = \langle \upsilon_p, m\phi_q \rangle \tag{51}$$

$$\hat{\eta}_{pq} = \sum_{i=1}^{n_b} (\eta_{ip}, D_N^{(i-1)}\phi_q)_{\partial S_i'}. \tag{52}$$

But from Eq. (21),

$$\langle \mathcal{L}_p\psi_p, \phi_q \rangle = \langle \eta_p, \phi_q \rangle$$
$$= -(\sigma_p^2 + 2\rho_p\tau_p + \beta_p\tau_p)\delta_{pq} - \theta_{pq}, \tag{53}$$

where we define

$$\theta_{pq} = \sigma_p\phi_{pq} + \langle (2m\rho_p\sigma_p + \sigma_pL_{10} + \rho_pL_{11})\chi_p, \phi_q \rangle. \tag{54}$$

Simplified expressions for $\hat{\eta}_{pq}$ and θ_{pq} defined in Eqs. (52) and (54) are given in Appendix A.

Using Eq. (45) to relate Eqs. (49), (50), and (53),

$$\pi_{pq}\beta_{pq} = -(\sigma_p^2 + 2\rho_p\tau_p + \beta_p\tau_p)\delta_{pq}$$
$$- \theta_{pq} - \pi_{pq}\upsilon_{pq} + \hat{\eta}_{pq}. \tag{55}$$

When $p = q$, this equation determines the second-order eigenvalue correction constant

$$\tau_p = \frac{\hat{\eta}_{pp} - \theta_{pp} - \sigma_p^2}{2\rho_p + \beta_p}, \tag{56}$$

while for $p \neq q$, it determines the Fourier coefficients

$$\beta_{pq} = (\hat{\eta}_{pq} - \theta_{pq})/\pi_{pq} - \upsilon_{pq}. \tag{57}$$

Finally, α_{pp} and β_{pp} can be determined by normalizing the eigenfunction $\hat{\phi}_p$ so that

$$\langle \hat{\phi}_p, m\hat{\phi}_p \rangle = 1.$$

Employing Eqs. (8), (17), (28), (32), (33), and (48), the last condition yields

$$\alpha_{pp} = -\mu_{pp} \tag{58}$$

$$\beta_{pp} = -\upsilon_{pp} + \tfrac{1}{2}\sum_{q=1}^{\infty} (\mu_{pq}^2 - \epsilon_{pq}^2) \tag{59}$$

where

$$\epsilon_{pq} = \begin{cases} 0, & p = q \\ (\hat{\zeta}_{pq} - \rho_p\phi_{pq})/\pi_{pq}, & p \neq q. \end{cases}$$

In summary, the above analysis gives an asymptotic approximation for the solution of the eigenproblem of almost classically damped continuous linear systems to second order in the small damping parameter ϵ in Eq. (14). The analysis breaks down when multiple eigenvalues occur in the underlying classically damped system, since then the quantity π_{pp} appearing in Eqs. (42) and (57) can be zero for $p \neq q$. A general procedure for treating this case can be found in Courant and Hilbert (1989). The analysis also breaks down at Eqs. (41) and (56), when $2\rho_p + \beta_p = 0$ for some $p \in \mathbf{Z}^+$, which corresponds to a critically damped system, that is, when $\beta_p = 2\omega_p$.

Approximate Solution for Forced Vibrations

Let $u(\mathbf{r}, t)$ be the general solution of (1) and (2) so that $u \in V$ for each time $t \in [0, \infty)$. We again use an asymptotic expansion in terms of the small damping parameter ϵ in Eq. (14),

$$u(\mathbf{r}, t) \sim \sum_{p=0}^{\infty} u_p(\mathbf{r}, t)\epsilon^p \quad \text{on } \bar{S} \times [0, \infty) \tag{60}$$

where $u_0(\mathbf{r}, t)$ is the solution for the classically damped case ($\epsilon = 0$). This gives a regular perturbation problem which will give a good approximation for the vibration response over some finite time interval, but not as $t \to \infty$ (Kevorkian and Cole, 1981). Substitution of this expression and Eq. (14) into the field Eq. (1) yields

$$\mathcal{L}u_p \equiv m\ddot{u}_p + L_{10}\dot{u}_p + L_2u_p = f_p(\mathbf{r}, t) \quad \text{on } S \times (0, \infty) \tag{61}$$

with

$$f_p = \begin{cases} f(\mathbf{r}, t), & p = 0 \\ -L_{11}\dot{u}_{p-1}(\mathbf{r}, t), & p \in \mathbf{Z}^+. \end{cases} \tag{62}$$

Likewise, substitution of (60) and (14) into the boundary conditions (2) gives $\forall t \in (0, \infty)$ and $\forall i = 1, \ldots, n_b$

$$D_N^{(i-1)} u_p = 0 \quad \text{on } \partial S_i,$$

$$\mathcal{B}_i u_p \equiv B_i u_p + C_{i0}\dot{u}_p = g_{ip}(\mathbf{r}, t) \quad \text{on } \partial S_i' \tag{63}$$

with

$$g_{ip} = \begin{cases} g_i(\mathbf{r}, t), & p = 0 \\ -C_{i1}\dot{u}_{p-1}(\mathbf{r}, t), & p \in \mathbf{Z}^+. \end{cases} \tag{64}$$

In order for the eigenfunction expansion (7) to be valid over the closure $\bar{S}$ of the domain S, the boundary conditions (63) must be put in homogeneous form. For this, we first employ the decomposition

$$u_p(\mathbf{r}, t) = v_p(\mathbf{r}, t) + h_p(\mathbf{r}, t) \quad \text{on } \bar{S} \times [0, \infty) \tag{65}$$

where h_p is chosen as any real-valued function in V for each t, which is twice-differentiable with respect to t almost everywhere on $(0, \infty)$ and satisfies the boundary conditions

$$D_N^{(i-1)} h_p = 0 \quad \text{on } \partial S_i,$$

$$\mathcal{B}_i h_p = g_{ip} \quad \text{on } \partial S_i', \quad \forall i = 1, \ldots, n_b. \tag{66}$$

Hence, $v_p \in V$ for each t, it is twice-differentiable with respect to t almost everywhere on $(0, \infty)$, and it satisfies

$$\mathcal{L} v_p = f_p - \mathcal{L} h_p \quad \text{on } S \tag{67}$$

$$D_N^{(i-1)} v_p = 0 \quad \text{on } \partial S_i, \quad \mathcal{B}_i v_p = 0 \quad \text{on } \partial S_i',$$
$$\forall i = 1, \ldots, n_b. \tag{68}$$

The functions v_p can be expressed in the form

$$v_p(\mathbf{r}, t) = \sum_{n=1}^{\infty} q_{pn}(t)\phi_n(\mathbf{r}) \quad \text{on } \bar{S} \tag{69}$$

where

$$q_{pn} = \langle v_p, m\phi_n \rangle. \tag{70}$$

We can obtain a differential equation for the $q_{pn}(t)$ by first employing the compatibility conditions (3) and (4):

$$\begin{aligned} \langle \mathcal{L} v_p, \phi_n \rangle &= \frac{d^2}{dt^2}\langle m v_p, \phi_n \rangle + \frac{d}{dt}\langle L_{10} v_p, \phi_n \rangle + \langle L_2 v_p, \phi_n \rangle \\ &= \ddot{q}_{pn} + \frac{d}{dt}\left\{ \langle v_p, L_{10}\phi_n \rangle + \sum_{i=1}^{n_b} [(D_N^{(i-1)} v_p, C_{i0}\phi_n) \right. \\ &\quad \left. - (C_{i0} v_p, D_N^{(i-1)}\phi_n)] \right\} \\ &\quad + \left\{ \langle v_p, L_2\phi_n \rangle + \sum_{i=1}^{n_b} (D_N^{(i-1)} v_p, B_i\phi_n) - (B_i v_p, D_N^{(i-1)}\phi_n) \right\}. \end{aligned} \tag{71}$$

By using Eqs. (9) to (12), (68), and (70), this expression can be reduced to

$$\langle \mathcal{L} v_p, \phi_n \rangle = \ddot{q}_{pn} + \beta_n \dot{q}_{pn} + \omega_n^2 q_{pn}. \tag{72}$$

In a similar fashion, except that Eq. (66) is used in place of (68), it can be shown that

$$\langle \mathcal{L} h_p, \phi_n \rangle = \ddot{h}_{pn} + \beta_n \dot{h}_{pn} + \omega_n^2 h_{pn} - \hat{g}_{pn}(t) \tag{73}$$

where we define

$$h_{pn} = \langle h_p, m\phi_n \rangle \tag{74}$$

$$\hat{g}_{pn} = \sum_{i=1}^{n_b} (g_{ip}, D_N^{(i-1)}\phi_n)_{\partial S_i'}. \tag{75}$$

Finally, define

$$f_{pn}(t) = \langle f_p, \phi_n \rangle. \tag{76}$$

A differential equation for each $q_{pn}(t)$ can then be obtained from (67), (72), (73), and (76),

$$\ddot{q}_{pn} + \beta_n \dot{q}_{pn} + \omega_n^2 q_{pn} = Q_{pn}(t) \tag{77}$$

where we define

$$Q_{pn}(t) = f_{pn} + \hat{g}_{pn} - (\ddot{h}_{pn} + \beta_n \dot{h}_{pn} + \omega_n^2 h_{pn}). \tag{78}$$

Here f_{pn} and $\hat{g}_{pn}$ for $p \in \mathbf{Z}^+$ depend on $\dot{u}_{p-1}$ through Eqs. (62) and (64), but this dependence can be simplified as shown in Appendix B.

To complete the specification of the coefficients $q_{pn}(t)$, the appropriate initial conditions must be given. From Eqs. (60), (65), and (70),

$$\begin{aligned} q_{pn}(0) &= \langle u_p(\mathbf{r}, 0), m\phi_n(\mathbf{r}) \rangle \\ &= \begin{cases} \langle u(\mathbf{r}, 0), m\phi_n(\mathbf{r}) \rangle - \langle h_0(\mathbf{r}, 0), m\phi_n(\mathbf{r}) \rangle, \ p = 0 \\ -\langle h_p(\mathbf{r}, 0), m\phi_n(\mathbf{r}) \rangle, \ p \in \mathbf{Z}^+ \end{cases} \end{aligned} \tag{79}$$

$$\begin{aligned} \dot{q}_{pn}(0) &= \langle \dot{u}_p(\mathbf{r}, 0), m\phi_n(\mathbf{r}) \rangle \\ &= \begin{cases} \langle \dot{u}(\mathbf{r}, 0), m\phi_n(\mathbf{r}) \rangle - \langle \dot{h}_0(\mathbf{r}, 0), m\phi_n(\mathbf{r}) \rangle, \\ \quad p = 0 \\ -\langle \dot{h}_p(\mathbf{r}, 0), m\phi_n(\mathbf{r}) \rangle, \\ \quad p \in \mathbf{Z}^+ \end{cases} \end{aligned} \tag{80}$$

where, without loss of generality, we can take $h_p(\mathbf{r}, 0)$ and $\dot{h}_p(\mathbf{r}, 0)$ as zero for $p = 0$ and $p \in \mathbf{Z}^+$.

In summary, an asymptotic approximation of the solution of the forced vibration problem to, say, second order in ϵ, may be obtained by using the eigenfunction expansion (69) for $p = 0$, 1, 2 and by taking a sufficient number of terms, N, for the desired accuracy. The coefficients in these expansions for the $v_p(\mathbf{r}, t)$ are determined by solving the initial-value problem (77)–(80) for $q_{pn}(t)$ for $p = 0, 1, 2$ and $n = 1, 2, \ldots, N$.

The functions $h_p(\mathbf{r}, t)$, $p = 0, 1, 2$, involved in Eqs. (78) and in the construction of the $u_p(\mathbf{r}, t)$ from Eq. (65), must be chosen to satisfy the conditions in Eqs. (66). A typical form for these functions can be obtained by generalizing an expression given in Meirovitch (1967) for the undamped problem, namely,

$$h_p(\mathbf{r}, t) = \sum_{i=1}^{n_b} [a_{ip}(\mathbf{r})\lambda_{ip}(t) + b_{ip}(\mathbf{r})\xi_{ip}(t)]$$

where the functions a_{ip}, λ_{ip}, b_{ip}, ξ_{ip} are chosen so that h_p satisfies conditions (66), as illustrated later. Once the $u_p(\mathbf{r}, t)$, $p = 0$, 1, 2, are determined, the solution $u(\mathbf{r}, t)$ can be approximated by the sum of the first three terms in Eq. (60).

Note that the approach developed here avoids direct term-by-term differentiation of the eigenfunction series for each u_p which would be required for $L_1 u$ and $L_2 u$. This operation would be invalid here since each u_p satisfies different boundary conditions than the eigenfunctions, $\phi_n(\mathbf{r})$.

Examples

To illustrate the approach, two simple examples are given in this section.

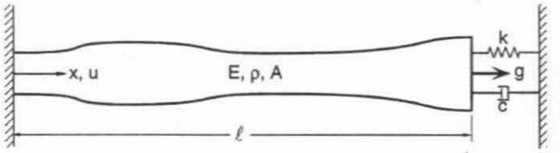

Fig. 1 Rod example

I: Rod Vibrations. Consider the longitudinal vibration of a viscoelastic rod shown in Fig. 1, where Eq. (1) applies with $u(x, t)$ being the longitudinal displacement at time t of a cross section at x and

$$S = (0, l), \quad \bar{S} = [0, l], \quad \partial S = \{0, l\}$$

$$m(x) = \rho(x)A(x), f(x, t)$$

$$= \text{distributed longitudinal load per unit length}$$

$$L_1 = \mu_E(x) - \frac{\partial}{\partial x}\left[\mu_I(x)A(x)\frac{\partial}{\partial x}\right],$$

$$L_2 = -\frac{\partial}{\partial x}\left[E(x)A(x)\frac{\partial}{\partial x}\right].$$

Here $m(x)$, $A(x)$, and $E(x)$ are the mass per unit length, the cross-sectional area and Young's modulus at location x along the rod, while $\mu_E(x)$ and $\mu_I(x)$ are non-negative viscous damping coefficients. From Fig. 1, the appropriate boundary conditions are geometric at $x = 0$ (i.e., ∂S_1) and a force balance at $x = l$ (i.e., $\partial S_1'$), which give

$$D_N^{(0)}u = u = 0 \text{ at } x = 0, \quad B_1u + C_1\dot{u} = g(t) \text{ at } x = l$$

where

$$B_1 = k + E(l)A(l)\frac{\partial}{\partial x}\bigg|_{x=l}, \quad C_1 = c + \mu_I(l)A(l)\frac{\partial}{\partial x}\bigg|_{x=l}$$

and $g(t)$ represents an applied axial force at $x = l$.

Using integration by parts and the appropriate inner product for this one-dimensional example, we get for all $\psi, \phi \in V \subset C^2(0, l) \cap C^1[0, l]$ satisfying the geometric boundary condition at $x = 0$:

$$\langle L_1\psi, \phi\rangle - \langle \psi, L_1\phi\rangle$$

$$= -\int_0^l \left[\phi\ \frac{d}{dx}\left(\mu_I A\ \frac{d\psi}{dx}\right) - \psi\ \frac{d}{dx}\left(\mu_I A\ \frac{d\phi}{dx}\right)\right]dx$$

$$= -\left[\mu_I A\ \frac{d\psi}{dx}\ \phi - \mu_I A\ \frac{d\phi}{dx}\ \psi\right]_0^l$$

$$= \left[\psi\left(c + \mu_I A\ \frac{d}{dx}\right)\phi - \phi\left(c + \mu_I A\ \frac{d}{dx}\right)\psi\right]_{x=l}$$

$$= [\psi C_1\phi - \phi C_1\psi]_{x=l} = (\psi, C_1\phi) - (C_1\psi, \phi).$$

According to (3) with $n_b = 1$, C_1 is therefore confirmed to be a compatible boundary operator for L_1. In a similar manner, B_1 can be confirmed to be a compatible boundary operator for L_2, as defined by Eq. (4).

For the undamped problem, the eigenvalues (frequencies) and eigenfunctions (modeshapes) are determined by

$$-[E(x)A(x)\phi_n'(x)]' = \omega_n^2 m(x)\phi_n(x) \quad \text{in } (0, l)$$

and

$$\phi_n(0) = 0, \quad B_1\phi_n|_{x=l} = E(l)A(l)\phi_n'(l) + k\phi_n(l) = 0$$

where the prime denotes d/dx. These equations define a Sturm-Liouville problem and so there is a countable set of eigenvalues $\{\omega_n^2 : n \in \mathbf{Z}^+\}$ and a complete set of orthonormal eigenfunctions $\{\phi_n : n \in \mathbf{Z}^+\}$ (Courant and Hilbert, 1989, Chapter 5).

For a classically damped system, we require from Eq. (11)

$$\mu_E(x)\phi_n(x) - [\mu_I(x)A(x)\phi_n'(x)]' = L_1\phi_n = \beta_n m(x)\phi_n(x).$$

This will be satisfied, for example, if we choose

$$\mu_E(x) = \alpha m(x), \quad \mu_I(x) = \beta E(x),$$

$$\beta_n = \alpha + \beta\omega_n^2, \quad \forall n \in \mathbf{Z}^+$$

where α and β are non-negative constants, which gives

$$L_{10} = \alpha m + \beta L_2$$

as a classical-damping operator. This is a continuous version of Rayleigh (or ''proportional'') damping. Also, from Eq. (12), we require

$$C_1\phi_n|_{x=l} = c\phi_n(l) + \beta E(l)A(l)\phi_n'(l) = (c - \beta k)\phi_n(l) = 0.$$

The presence of a spring at $x = l$ therefore necessitates a damper at $x = l$ with a particular damping coefficient $c = \beta k$ in order to give classical normal modes. For this value of c, the boundary damping operator becomes

$$C_{10} = \beta k + \beta E(l)A(l)\frac{\partial}{\partial x}\bigg|_{x=l} = \beta B_1.$$

Finally, the eigenvalues ρ_n, $n \in \mathbf{Z}^+$, for the classical-damping case are given by

$$\rho_n^2 + (\alpha + \beta\omega_n^2)\rho_n + \omega_n^2 = 0.$$

Now consider an almost classically-damped system where for a small parameter ϵ:

$$\mu_E(x) = \alpha m(x) + \epsilon\hat{\mu}_1(x), \quad \mu_I(x) = \beta E(x) + \epsilon\hat{\mu}_2(x),$$

$$c = \beta k + \epsilon c_1$$

so

$$L_1 = L_{10} + \epsilon L_{11}, \quad C_1 = C_{10} + \epsilon C_{11}$$

where

$$L_{11} = \hat{\mu}_1(x) - \frac{\partial}{\partial x}\left[\hat{\mu}_2(x)A(x)\frac{\partial}{\partial x}\right]$$

$$C_{11} = c_1 + \hat{\mu}_2(l)A(l)\frac{\partial}{\partial x}\bigg|_{x=l}.$$

Once the various field and boundary operators are defined, the next step involves the evaluation of the functions $\mu_p(x)$, $\upsilon_p(x)$, and $h_p(x, t)$ in Eqs. (28), (43), and (65), respectively, if both the eigenproblem and the forced vibration problem are of interest. The function $\mu_p(x)$ is evaluated by applying the conditions (29) on ∂S,

$$D_N^{(0)}\mu_p = \mu_p = 0 \quad \text{at} \quad x = 0$$

$$\mathcal{B}_{1p}\mu_p = \zeta_{1p} \quad \text{at} \quad x = l,$$

where

$$\mathcal{B}_{1p}\mu_p = B_1\mu_p + \rho_p C_{10}\mu_p = (1 + \beta\rho_p)B_1\mu_p$$

$$= (1 + \beta\rho_p)[k\mu_p(l) + E(l)A(l)\mu_p'(l)]$$

$$\zeta_{1p} = -\rho_p C_{11}\phi_p = -\rho_p[c_1\phi_p(l) + \hat{\mu}_2(l)A(l)\phi_p'(l)].$$

Therefore, $\mu_p(x)$ can be chosen in the linear form,

$$\mu_p(x) = c_p x + d_p,$$

with the conditions at $x = 0$ and $x = l$ giving, respectively, $d_p = 0$ and

$$c_p = -\rho_p \frac{c_1\phi_p(l) + \hat{\mu}_2(l)A(l)\phi_p'(l)}{(1+\beta\rho_p)[kl + E(l)A(l)]}.$$

This gives from Eqs. (36), (37), and (39)

$$\mu_{pq} = c_p \int_0^l x m(x)\phi_q(x)dx$$

$$\hat{\zeta}_{pq} = (C_{11}\phi_p, \phi_q)_{\partial S_1^-} = [C_{11}\phi_p]_{x=l}\phi_q(l)$$

$$= c_1\phi_p(l)\phi_q(l) + \hat{\mu}_2(l)A(l)\phi_p'(l)\phi_q(l)$$

$$= [c_1 - k\hat{\mu}_2(l)/E(l)]\phi_p(l)\phi_q(l)$$

$$\phi_{pq} = \int_0^l \hat{\mu}_1(x)\phi_p(x)\phi_q(x)dx - \int_0^l [\hat{\mu}_2(x)A(x)\phi_p'(x)]'\phi_q(x)dx$$

$$= \int_0^l \hat{\mu}_1(x)\phi_p(x)\phi_q(x)dx + \int_0^l \hat{\mu}_2(x)A(x)\phi_q'(x)dx + k\hat{\mu}_2(l)\phi_p(l)\phi_q(l)/E(l).$$

These parameters, together with function $\mu_p(x)$, permit the evaluation of all the parameters needed to determine the first-order eigenquantities σ_p and α_{pq} from Eqs. (41), (42), and (58).

In a similar fashion, application of condition (44) results in

$$D_N^{(0)}v_p|_{x=0} = v_p(0) = 0, \quad \mathcal{B}_{1p}v_p|_{x=l} = \eta_{1p}|_{x=l}$$

which in turn determine $v_p(x)$ in the form

$$v_p(x) = e_p x$$

where e_p is a known constant. This allows v_{pq} in Eq. (51) to be evaluated. The quantity $(\hat{\eta}_{pq} - \theta_{pq})$ can be evaluated as in Appendix A. The second-order eigenquantities τ_p and β_{pq} can then be determined from Eqs. (56), (57), and (59).

Finally, the functions $h_p(x, t)$, which are needed to determine the solution of the response problem up to some prescribed order in ϵ (e.g., taking $p = 0, 1, 2$) are chosen by applying conditions (66). In the present example, these conditions lead to

$$D_N^{(0)}h_p|_{x=0} = h_p(0, t) = 0$$

$$\mathcal{B}_1 h_p|_{x=l} = g_{1p}(t) = \begin{cases} g(t), & p = 0 \\ -C_{11}\dot{u}_{p-1}(l, t), & p \in \mathbf{Z}^+ \end{cases}$$

where $g(t)$ represents an applied axial force at $x = l$. Here

$$\mathcal{B}_1 h_p|_{x=l} = kh_p(l, t) + E(l)A(l)h_p'(l, t) + \beta k\dot{h}_p(l, t) + \beta E(l)A(l)\dot{h}_p'(l, t)$$

$$C_{11}\dot{u}_{p-1}(l, t) = c_1\dot{u}_{p-1}(l, t) + \hat{\mu}_2(l)A(l)\dot{u}_{p-1}'(l, t).$$

Next, choose h_p in the form

$$h_p(x, t) = \lambda_p(t)x + \xi_p(t).$$

Then, to satisfy the above conditions on h_p, we must have $\xi_p(t) = 0$ and

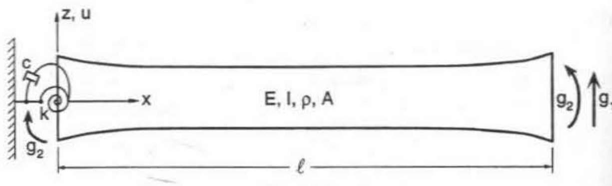

Fig. 2 Beam example

$$\beta\dot{\lambda}_p(t) + \lambda_p(t) = \frac{1}{[kl + E(l)A(l)]} g_{1p}(t)$$

where $g_{10}(t) = g(t)$ and $g_{1p}(t)$, $p \in \mathbf{Z}^+$, can be expressed as

$$g_{1p}(t) = -[c_1 + \hat{\mu}_2(l)A(l)]\dot{\lambda}_{p-1}(t) + \left[c_1 - \frac{k\hat{\mu}_2(l)}{E(l)}\right]\sum_{n=1}^{\infty}\dot{q}_{p-1,n}(t)\phi_n(l).$$

In practice, only a finite sum would be used to approximate $v_p(x, t)$ to sufficient accuracy and so the infinite sum in $g_{1p}(t)$ would be replaced by a finite one. The initial condition for the differential equation for $\lambda_p(t)$ depends on the choice of $h_p(x, 0)$, but this is arbitrary since whatever is chosen, it is corrected for by Eqs. (79) and (80). We take $h_p(x, 0) = 0$, so $\lambda_p(0) = 0$ and each $\lambda_p(t)$ can then be determined uniquely from a first-order linear initial value problem. This completes the determination of the functions $h_p(x, t)$. The functions $h_{pn}(t)$, and so $Q_{pn}(t)$, can be evaluated from Eqs. (74) and (78) and Appendix B. Then, the second-order linear initial value problem given by Eqs. (77), (79), and (80) can be solved to obtain each $q_{pn}(t)$. The function $v_p(x, t)$ can then be approximated by the first N terms in Eq. (69), where N is chosen to give the desired level of accuracy. Each $u_p(x, t)$ is then given by Eq. (65) to finally obtain an approximation of the solution $u(x, t)$ by Eq. (60).

II: Beam Vibrations. Consider the bending vibration of the beam shown in Fig. 2, where Eq. (1) applies with $u(x, t)$ being the transverse displacement at time t of a cross section at x and

$$S = (0, l), \quad \bar{S} = [0, l], \quad \partial S = \{0, l\}$$

$$m(x) = \rho(x)A(x),$$

$$f(x, t) = \text{distributed transverse load per unit length}$$

$$L_1 = \mu_E(x) + \frac{\partial^2}{\partial x^2}\left[\mu_I(x)I(x)\frac{\partial^2}{\partial x^2}\right],$$

$$L_2 = \frac{\partial^2}{\partial x^2}\left[E(x)I(x)\frac{\partial^2}{\partial x^2}\right].$$

Here $m(x)$, $I(x)$, and $E(x)$ are the mass per unit length, the cross-sectional moment of inertia, and Young's modulus at location x along the beam, while $\mu_E(x)$ and $\mu_I(x)$ are non-negative viscous damping coefficients. Using Fig. 2, the boundary conditions (2) are

$$D_N^{(0)}u = u = 0 \text{ and } B_2(0)u + C_2(0)\dot{u} = g_2(0, t)$$

$$\text{at } x = 0$$

$$B_1(l)u + C_1(l)\dot{u} = g_1(l, t) \text{ and } B_2(l)u + C_2(l)\dot{u} = g_2(l, t)$$

$$\text{at } x = l$$

where

$$B_1(l) = -\frac{\partial}{\partial x}\left[E(x)I(x)\frac{\partial^2}{\partial x^2}\right]\bigg|_{x=l},$$

$$C_1(l) = -\frac{\partial}{\partial x}\left[\mu_I(x)I(x)\frac{\partial^2}{\partial x^2}\right]\bigg|_{x=l}$$

$$B_2(0) = -k\frac{\partial}{\partial x}\bigg|_{x=0} + E(0)I(0)\frac{\partial^2}{\partial x^2}\bigg|_{x=0},$$

$$C_2(0) = -c\frac{\partial}{\partial x}\bigg|_{x=0} + \mu_I(0)I(0)\frac{\partial^2}{\partial x^2}\bigg|_{x=0}$$

$$B_2(l) = E(l)I(l)\frac{\partial^2}{\partial x^2}\bigg|_{x=l}, \quad C_2(l) = \mu_I(l)I(l)\frac{\partial^2}{\partial x^2}\bigg|_{x=l}$$

and where $g_1(l, t)$ and $g_2(0, t)$, $g_2(l, t)$ are applied force and moments at the boundaries, while c and k are the damping and stiffness coefficients of the rotational dashpot and spring at $x = 0$.

Using integration by parts and the appropriate inner product, we get for all $\psi, \phi \in V \subset C^4(0, l) \cap C^3[0, l]$ satisfying the geometric boundary condition at $x = 0$,

$$\langle L_1\psi, \phi\rangle - \langle \psi, L_1\phi\rangle = [\psi C_1\phi - \phi C_1\psi]_0^l + [\psi' C_2\phi - \phi' C_2\psi]_0^l.$$

According to (3) with $n_b = 2$, C_1 and C_2 are therefore confirmed to be compatible boundary operators for L_1. In a similar manner, B_1 and B_2 can be confirmed to be compatible boundary operators for L_2.

For the undamped problem, the frequencies and modeshapes are determined by

$$[E(x)I(x)\phi_n''(x)]'' = \omega_n^2 m(x)\phi_n(x) \quad \text{in } (0, l)$$

$$\phi_n(0) = 0, \quad k\phi_n'(0) - E(0)I(0)\phi_n''(0) = 0$$

$$\frac{d}{dx}[E(x)I(x)\phi_n''(x)]|_{x=l} = 0, \quad E(l)I(l)\phi_n''(l) = 0.$$

We assume that this eigenvalue problem gives a countable set of eigenvalues $\{\omega_n^2 : n \in \mathbf{Z}^+\}$ and a complete set of corresponding orthonormal eigenfunctions $\{\phi_n : n \in \mathbf{Z}^+\}$.

For a classically damped system, conditions (11) and (12) must be satisfied. This will be the case if we choose

$$\mu_E(x) = \alpha m(x), \quad \mu_I(x) = \beta E(x), \quad \beta_n = \alpha + \beta\omega_n^2$$

as in Example I, and then

$$C_1\phi_n|_{x=l} = -\beta\frac{d}{dx}[E(x)I(x)\phi_n''(x)]_{x=l} = 0$$

$$C_2\phi_n|_{x=l} = \beta E(l)I(l)\phi_n''(l) = 0$$

$$C_2\phi_n|_{x=0} = -c\phi_n'(0) + \beta E(0)I(0)\phi_n''(0) = (\beta k - c)\phi_n'(0) = 0$$

if the rotational dashpot and spring at $x = 0$ satisfy the condition $c = \beta k$ for classical normal modes. This means that the classical damping operators are

$$L_{10} = \alpha m + \beta L_2, \quad C_{10} = \beta B_1, \quad C_{20} = \beta B_2.$$

Now consider an almost classically damped system with:

$$\mu_E(x) = \alpha m(x) + \epsilon\hat{\mu}_1(x), \quad \mu_I(x) = \beta E(x) + \epsilon\hat{\mu}_2(x),$$
$$c = \beta k + \epsilon c_1,$$

so

$$L_1 = L_{10} + \epsilon L_{11} \quad \text{on } (0, l)$$

$$C_1 = C_{10} + \epsilon C_{11} \quad \text{at } x = l$$

$$C_2 = C_{20} + \epsilon C_{21} \quad \text{at } x = 0, l$$

where

$$L_{11} = \hat{\mu}_1(x) + \frac{\partial^2}{\partial x^2}\left[\hat{\mu}_2(x)I(x)\frac{\partial^2}{\partial x^2}\right]$$

$$C_{11}(l) = -\frac{\partial}{\partial x}\left[\hat{\mu}_2(x)I(x)\frac{\partial^2}{\partial x^2}\right]\bigg|_{x=l}$$

$$C_{21}(0) = -c_1\frac{\partial}{\partial x}\bigg|_{x=0} + \hat{\mu}_2(0)I(0)\frac{\partial^2}{\partial x^2}\bigg|_{x=0}$$

$$C_{21}(l) = \hat{\mu}_2(l)I(l)\frac{\partial^2}{\partial x^2}\bigg|_{x=l}.$$

At this point, the function $\mu_p(x)$ can be evaluated by applying the conditions (29) on the boundary

$$\mu_p(0) = 0, \quad \mathcal{B}_{1p}\mu_p = \zeta_{1p} \equiv -\rho_p C_{11}\phi_p \quad \text{at } x = l$$

$$\mathcal{B}_{2p}\mu_p = \zeta_{2p} \equiv -\rho_p C_{21}\phi_p \quad \text{at } x = 0, l$$

where from (24),

$$\mathcal{B}_{ip} = B_i + \rho_p C_{i0} = (1 + \beta\rho_p)B_i.$$

These conditions can be satisfied by a cubic of the form

$$\mu_p(x) = c_{p3}x^3 + c_{p2}x^2 + c_{p1}x$$

where the coefficients c_{p1}, c_{p2}, and c_{p3} are determined by the nonsingular linear system of equations

$$\frac{\partial}{\partial x}[E(x)I(x)(6c_{p3}x + 2c_{p2})]_{x=l} = \frac{\rho_p}{1 + \beta\rho_p}[C_{11}\phi_p]_{x=l}$$

$$kc_{p1} - 2E(0)I(0)c_{p2} = \frac{\rho_p}{1 + \beta\rho_p}[C_{21}\phi_p]_{x=0}$$

$$E(l)I(l)(6c_{p3}l + 2c_{p2}) = \frac{-\rho_p}{1 + \beta\rho_p}[C_{21}\phi_p]_{x=l}.$$

This allows the parameters μ_{pq} in (36) to be determined. Furthermore, ζ_{pq} and ϕ_{pq} in (37) and (39) are given by

$$\zeta_{pq} = [\phi_q C_{11}\phi_p]_{x=l} + [\phi_q' C_{21}\phi_p]_{x=0}^{x=l}$$

$$\phi_{pq} = \int_0^l \phi_q(x)L_{11}\phi_p(x)dx.$$

These parameters allow the first-order eigenvalue correction σ_p to be computed from (41) and the first-order eigenfunction correction $\chi_p(x)$ to be computed from (28), (32), (42), and (58).

In a similar fashion, $\upsilon_p(x)$ in (43) can be determined in the form

$$\upsilon_p(x) = e_{p3}x^3 + e_{p2}x^2 + e_{p1}x$$

by the conditions (44) that it must satisfy

$$\upsilon_p(0) = 0, \quad \mathcal{B}_{1p}\upsilon_p = \eta_{1p} \quad \text{at } x = l$$

$$\mathcal{B}_{2p}\upsilon_p = \eta_{2p} \quad \text{at } x = 0, l.$$

This allows the parameters v_{pq} in (51) to be determined. The quantity $(\hat{\eta}_{pq} - \theta_{pq})$ can be evaluated as in Appendix A. Therefore, the second-order eigenvalue correction τ_p can be computed from (56) and the second-order eigenfunction correction $\psi_p(x)$ can be computed from (43), (47), (57), and (59).

In order to determine the solution of the response problem up to, say, second order in ϵ, the functions $h_p(x, t)$ $(p = 0, 1, 2)$ in (65) must be specified so that they satisfy the conditions (66),

$$h_p(0, t) = 0, \quad B_1 h_p + \beta B_1 \dot{h}_p = g_{1p} \quad \text{at } x = l$$

$$B_2 h_p + \beta B_2 \dot{h}_p = g_{2p} \quad \text{at } x = 0, l,$$

where $g_{ip}(x, t)$ is given by (64). To satisfy these conditions, try a function of the form

$$h_p(x, t) = \sum_{i=1}^{2} [a_{ip}(x)\lambda_{ip}(t) + b_{ip}(x)\xi_{ip}(t)].$$

It follows that

$$0 = h_p(0, t) = \sum_{i=1}^{2} [a_{ip}(0)\lambda_{ip}(t) + b_{ip}(0)\xi_{ip}(t)]$$

$$g_{1p}(l, t) = \sum_{i=1}^{2} [B_1 a_{ip}(x)(\beta\dot{\lambda}_{ip}(t) + \lambda_{ip}(t)) + B_1 b_{ip}(x)(\beta\dot{\xi}_{ip}(t) + \xi_{ip}(t))]_{x=l}$$

$$g_{2p}(x, t) = \sum_{i=1}^{2} [B_2 a_{ip}(x)(\beta\dot{\lambda}_{ip}(t) + \lambda_{ip}(t)) + B_2 b_{ip}(x)(\beta\dot{\xi}_{ip}(t) + \xi_{ip}(t))] \quad \text{at } x = 0, l.$$

These equations are satisfied if

$$a_{1p}(0) = a_{2p}(0) = 0, \quad b_{1p}(0) = b_{2p}(0) = 0$$

$$B_1 a_{1p}|_{x=l} = B_1 a_{2p}|_{x=l} = 0, \quad B_1 b_{1p}|_{x=l} = 1, \quad B_1 b_{2p}|_{x=l} = 0$$

$$B_2 a_{1p}|_{x=l} = B_2 a_{2p}|_{x=l} = 0, \quad B_2 b_{1p}|_{x=l} = 0, \quad B_2 b_{2p}|_{x=l} = 1$$

$$B_2 a_{1p}|_{x=0} = 0, \quad B_2 a_{2p}|_{x=0} = 1, \quad B_2 b_{1p}|_{x=0} = B_2 b_{2p}|_{x=0} = 0$$

$$\beta\dot{\xi}_{1p}(t) + \xi_{1p}(t) = g_{1p}(l, t), \quad \beta\dot{\xi}_{2p}(t) + \xi_{2p}(t) = g_{2p}(l, t)$$

$$\beta\dot{\lambda}_{2p}(t) + \lambda_{2p}(t) = g_{2p}(0, t).$$

It is sufficient to set $a_{1p}(x) = 0$ on $[0, l]$ and so $\lambda_{1p}(t)$ is not required. Also, we can take $a_{2p}(x)$, $b_{1p}(x)$, and $b_{2p}(x)$ as cubics with the four coefficients in each cubic selected to satisfy the above four conditions on each of these functions. Finally, we can take $h_p(x, 0) = 0$ which gives the initial conditions $\xi_{1p}(0) = \xi_{2p}(0) = \lambda_{2p}(0) = 0$. This leads to a first-order linear initial value problem for each of $\xi_{1p}(t)$, $\xi_{2p}(t)$, and $\lambda_{2p}(t)$. Solution of these three initial value problems completes the determination of the functions $h_p(x, t)$.

The remainder of the procedure follows the description in Example I, where $h_{pn}(t)$, $Q_{pn}(t)$, and $q_{pn}(t)$ are determined successively and $v_p(x, t)$ in (69) is approximated by the first N terms to give the desired level of accuracy. Each $u_p(x, t)$ is then given by (65) to finally obtain an approximation of the beam displacement $u(x, t)$ by (60).

Summary

Conditions are presented for the existence of classical normal modes in continuous linear vibrating systems which have compatible (internal and boundary) stiffness and damping operators. A perturbation analysis is then developed for the response of almost classically damped continuous linear systems. The latter methodology provides approximate but sufficiently accurate analytical solutions for the eigenproblem as well as the response under general forcing functions.

The analysis is based on a decomposition of the internal and boundary damping operators into a part that results in classical normal mode response, with arbitrarily large damping ratios, plus a small perturbation. This permits application of standard modal analysis methodologies and provides the solution of the problem in terms of the frequencies and modes of the corresponding undamped problem only. A special treatment is presented for the series representations of the solutions in terms of the undamped eigenfunctions so that Gibbs phenomenon is avoided at the boundary. This ensures faster convergence of the eigenfunction expansions, that is, a smaller number of terms is required to give an approximation of the solutions to within a specified accuracy.

The methodology presented here is expected to result in computational benefits, especially when performing parametric studies to investigate damping effects for design purposes.

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APPENDIX A

A simplified expression is derived for the difference of $\hat{\eta}_{pq}$ and θ_{pq} which are defined in Eqs. (52) and (54), respectively. This difference is needed to evaluate the second-order eigenvalue and eigenfunction corrections, τ_p and $\psi_p(\mathbf{r})$, given in Eqs. (56), (43), (44), (47), (57), and (59).

Start by defining

$$\gamma_{pq} = \sum_{i=1}^{n_b} (C_{i0}\chi_p, D_N^{(i-1)}\phi_q)_{\partial S_i'} \tag{81}$$

$$\hat{\gamma}_{pq} = \sum_{i=1}^{n_b} (C_{i1}\chi_p, D_N^{(i-1)}\phi_q)_{\partial S_i'}. \tag{82}$$

By substituting Eq. (22) into (52), η_{pq} can be expressed as

$$\hat{\eta}_{pq} = -\sigma_p\gamma_{pq} - \rho_p\hat{\gamma}_{pq} - \sigma_p\hat{\zeta}_{pq}. \tag{83}$$

Also, from Eq. (54),

$$\begin{aligned}\theta_{pq} &= \sigma_p\phi_{pq} + \langle(2m\rho_p\sigma_p + \sigma_p L_{10} + \rho_p L_{11})\chi_p, \phi_q\rangle \\ &= \sigma_p(2\rho_p + \beta_q)(\alpha_{pq} + \mu_{pq}) + \sigma_p\phi_{pq} \\ &\qquad + \rho_p\langle L_{11}\chi_p, \phi_q\rangle - \sigma_p\gamma_{pq}.\end{aligned} \tag{84}$$

Here we use $\langle\chi_p, m\phi_q\rangle = \langle\hat{\chi}_p, m\phi_q\rangle + \langle\mu_p, m\phi_q\rangle = \alpha_{pq} + \mu_{pq}$, from Eqs. (28), (33), and (36), together with the compatibility Eq. (3) for L_{10} and the C_{i0}.

Subtracting (84) from (83),

$$\begin{aligned}\hat{\eta}_{pq} - \theta_{pq} &= -\rho_p\hat{\gamma}_{pq} - \sigma_p(\hat{\zeta}_{pq} + \phi_{pq}) \\ &\qquad - \sigma_p(2\rho_p + \beta_q)(\alpha_{pq} + \mu_{pq}) - \rho_p\langle L_{11}\chi_p, \phi_q\rangle.\end{aligned} \tag{85}$$

This is the desired expression. Both $\hat{\gamma}_{pq}$ and $\langle L_{11}\chi_p, \phi_q\rangle$ can be evaluated using $\chi_p(\mathbf{r}) = \hat{\chi}_p(\mathbf{r}) + \mu_p(\mathbf{r})$ where $\hat{\chi}_p(\mathbf{r})$ is given by the eigenfunction expansion (32) on $\bar{S}$. In practice, only a finite number of terms, N, need be used in (32) to give the desired accuracy. Then $\hat{\gamma}_{pq}$ and $\langle L_{11}\chi_p, \phi_q\rangle$ can be approximated by

$$\begin{aligned}\hat{\gamma}_{pq} &= \sum_{i=1}^{n_b} (C_{i1}\hat{\chi}_p + C_{i1}\mu_p, D_N^{(i-1)}\phi_q)_{\partial S_i'} \\ &= \sum_{n=1}^{N} \alpha_{pn}\hat{\xi}_{nq} + \sum_{i=1}^{n_b} (C_{i1}\mu_p, D_N^{(i-1)}\phi_q)_{\partial S_i'}\end{aligned} \tag{86}$$

$$\begin{aligned}\langle L_{11}\chi_p, \phi_q\rangle &= \langle L_{11}\hat{\chi}_p + L_{11}\mu_p, \phi_q\rangle \\ &= \sum_{n=1}^{N} \alpha_{pn}\phi_{nq} + \langle L_{11}\mu_p, \phi_q\rangle.\end{aligned} \tag{87}$$

APPENDIX B

A simplified expression is derived for the sum $f_{pn}(t) + \hat{g}_{pn}(t)$ required in the differential Eq. (77) for $q_{pn}(t)$ where $p, n \in \mathbf{Z}^+$. (The $p = 0$ case is relatively straightforward.) Substituting Eqs. (62) and (64) into (76) and (75), we get

$$\begin{aligned}f_{pn} + \hat{g}_{pn} &= -\langle L_{11}\dot{u}_{p-1}, \phi_n\rangle - \sum_{i=1}^{n_b} (C_{i1}\dot{u}_{p-1}, D_N^{(i-1)}\phi_n)_{\partial S_i'} \\ &= -\langle\dot{u}_{p-1}, L_{11}\phi_n\rangle - \sum_{i=1}^{n_b} (D_N^{(i-1)}\dot{u}_{p-1}, C_{i1}\phi_n)_{\partial S_i'}\end{aligned}$$

where the compatibility condition (3) for L_{11} and the C_{i1} has been used. Equation (65) can now be used where $v_{p-1}(\mathbf{r}, t)$ is approximated by a finite number of terms, N, in the eigenfunction expansion (69), so that

$$\begin{aligned}f_{pn} + \hat{g}_{pn} &= -\sum_{m=1}^{N} \dot{q}_{p-1,m}(\phi_{nm} + \hat{\zeta}_{nm}) - \langle\dot{h}_{p-1}, L_{11}\phi_n\rangle \\ &\qquad - \sum_{i=1}^{n_b} (D_N^{(i-1)}\dot{h}_{p-1}, C_{i1}\phi_n)_{\partial S_i'}\end{aligned} \tag{88}$$

where the definitions for $\hat{\zeta}_{nm}$ and ϕ_{nm} in (37) and (39) have been used. Equation (88) shows that the $q_{pn}(t)$ must be evaluated sequentially with respect to increasing p since q_{pn} depends on the $q_{p-1,m}$, $m = 1, 2, \ldots, N$.