NOTE ON THE HOMOGENEITY OF PHYSICAL EQUATIONS.

BY RICHARD C. TOLMAN.

THE manuscript of the preceding article was kindly sent to me by the author so that I could state my position in regard to it.

Mrs. Ehrenfest-Afanassjewa[1] has presented in this paper a very satisfactory and general treatment of the conditions which must be fulfilled if a set of physical equations are to be homogeneous with respect to a given transformation, that is, if they are to remain unchanged when each of the quantities of a particular kind Y_i occurring in the equations is multiplied by a factor ξ_i. Equations (1), (2) and (3) in her article show the method of determining how many of the factors ξ_1, ξ_2, ξ_3, etc., for the different kinds of quantities Y_1, Y_2, Y_3, etc., can be taken arbitrarily and still have the transformation a homogeneous one for all the equations in the set under consideration. She has also shown that the possibility of constructing a miniature (model) universe, in which all the laws of physical behavior would be identical even to numerical coefficients with those in our own universe, depends on the possibility of finding a homogeneous transformation for *all* the fundamental equations of physics in which at least one of the factors ξ_1 can be taken as arbitrary. And with these conclusions I agree entirely.

Mrs. Ehrenfest-Afanassjewa has further pointed out that the C.G.S. system of dimensions prescribes homogeneity for physical equations when a transformation is carried out in which three of the multipliers ξ_l, ξ_m, ξ_t are arbitrary and that my own principle of similitude requires homogeneity with respect to a transformation in which only one of the multipliers is taken arbitrarily. I agree of course entirely with such a form of statement for the principles of dimensional homogeneity and of similitude, as will be seen in my paper comparing these two principles.[2]

Because of this possibility of expressing in such a similar form the requirements of the two principles, Mrs. Ehrenfest-Afanassjewa is inclined to speak of the theory of similitude (cf. § 1) as being merely another system of dimensions differing from the C.G.S. system. Although this method of speaking is perhaps logically possible, it seems to me to be *very undesirable* since I do not believe that the principle of similitude determines what we ordinarily mean by a set of dimensions.

[1] T. Ehrenfest-Afanassjewa, this journal, (1916).

[2] Tolman, PHYS. REV., *6*, 219 (1916).

The dimensions of a quantity may be best regarded, I believe, as a shorthand statement of the definition of that kind of quantity in terms of certain fundamental kinds of quantity, and hence also as an expression of the essential physical nature of the quantity in question. If, for example, we define force as mass times acceleration, the dimensions of force will be $[mlt^{-2}]$ and this may be regarded as a shorthand recapitulation of the definition of force in terms of mass, length and time, and also as an expression of the essential physical nature of force.

The reason, now, why certain physical equations have to be dimensionally homogeneous is because in the cases under consideration the physical nature of the quantities equated has to be the same. If, for example, we know that the centripetal force acting on a particle depends on its mass m, tangential velocity v, and radius of rotation r, we know that the particular combination of m, v and r which determines the centripetal force will have to have the physical nature of a force and hence the dimensions of force. And since the only combination of these quantities which has the right dimensions is $(mv^2)/r$ we know that the equation for centripetal force must be

$$f = k\frac{mv^2}{r},$$

where the numerical constant turns out to have the value unity.

In contradiction to such considerations as the above, I do not believe that the theory of similitude can be regarded as furnishing a system of dimensions or shorthand statements of the physical nature of quantities. If we should try to regard the principle of similitude as determining a system of dimensions, we should be obliged to say, as will be seen from Table II. in my article already referred to (l. c., p. 226), that force, for example, has the dimensions $[l^{-2}]$ and since this is neither an adequate shorthand statement of the definition of force, nor a satisfactory description of the essential physical nature of force I believe we shall do well if we do not speak of the theory of similitude as determining a system of dimensions, but rather realize that there is a real and fundamental difference between the principle of similitude and the principle of dimensional homogeneity.

A few words with regard to this difference in the fundamental nature of the two principles will not be out of place. As I have already pointed out (l. c.) there are a large number of important physical equations which do *not* stand in "intrinsic" agreement with the principle of dimensional homogeneity.[1] Thus in Stefan's law connecting the energy density

[1] These are equations which cannot be derived with the help of the principle of dimensional homogeneity, but, nevertheless, as I have shown (l. c.) often can be treated with the help of the principle of similitude.

u in a hohlraum with the temperature T

$$u = kT^4$$

there is no identity between the physical nature of energy density and of temperature to the fourth power, and only a "formal" agreement with the principle of dimensional homogeneity can be brought about by arbitrarily assigning to the constant k the dimensions $[ml^{-1}t^{-2}T^{-4}]$. This somewhat artificial procedure of assigning dimensions to such a numerical constant may be useful in determining what changes in the numerical value of the constant will be brought about by a change in units of measurement, but it does not change the fact that in equations such as these we have an equality between the numbers on the two sides but no identity in the physical nature of the terms equated, and hence no "intrinsic" agreement with the principle of dimensional homogeneity.

Since there are these important physical equations which do not stand in intrinsic agreement with the principle of dimensional homogeneity, we do not obtain from this principle any information as to the possibility of constructing a miniature universe in which *all* the laws of physics would be the same even to *numerical constants* as those in our own universe. For as will be seen from the work of Mrs. Ehrenfest-Afanassjewa, the possibility of constructing such a universe depends on the possibility of finding a set of multipliers for physical quantities such that all the fundamental equations of physics will undergo a homogeneous transformation, *at least one of the multipliers being chosen arbitrarily*. And all that we have learned from the principle of dimensional homogeneity with regard to this matter, is that if we take all five of the multipliers (ξ_l, ξ_t, ξ_m, ξ_μ and ξ_T) arbitrarily, there are some important relations between physical quantities which will not be transformed homogeneously.

In contradistinction to dimensional considerations, which have given us no information as to the possibility of constructing a miniature universe, the theory of similitude definitely postulates this possibility, and hence the possibility of finding a homogeneous transformation for all the equations of physics, at least one of the multipliers ξ_i being chosen arbitrarily. Thus I agree with Mrs. Ehrenfest-Afanassjewa that *all* the "fundamental" equations of physics must agree with the theory of similitude if my fundamental postulate is really valid. Moreover, this agreement with the principle of similitude must be an "intrinsic" agreement in the sense in which we have already used that word, and this necessity for "intrinsic" agreement arises because in building the miniature universe we are to use by hypothesis the same materials as in our actual universe[1] and yet the laws of physics are to be the same even to numerical coefficients in the miniature universe as in our actual universe. For this reason I agree with Mrs. Ehrenfest-Afanassjewa (cf. § 5), as will be seen from my very first paper,[2] that the fundamental equation for gravitational considerations must also agree with the principle of similitude. And since Newton's law of gravitation does not agree with the principle of similitude, we are forced, as I have already stated, either to the conclusion that the principle of similitude has like the principle of dimensional homogeneity only a restricted validity and a limited field of usefulness, or to the conclusion that Newton's law of gravitation is not a satisfactory statement of the fundamental equation for gravitational considerations. I am inclined of course to this latter alternative, and have pointed out two possibilities for the reconciliation of Newton's law with the theory of similitude, namely, either (1) that the gravitational constant is not a true numerical constant but is dependent on the properties of some unknown gravitational mechanism, or (2) that the force of attraction between the gravitating bodies is not in reality necessarily proportional to ordinary mass, but to some other quantity ("gravitational mass") whose transformation relations would differ from those of ordinary mass. If this were the case it would be merely an accidental coincidence that in the phenomena with which we are already acquainted "gravitational mass" happens to be proportional to "inertial mass." It is this latter of these possible solutions which has been adopted by Nordström in his article[3] "R. C. Tolman's Prinzip der Aehnlichkeit und die Gravitation." Such considerations as these make it very desirable to determine if electrons or rays of light have a weight proportional to their mass, and hence I am inclined to agree with Mrs. Ehrenfest-Afanassjewa's feeling (cf. § 5) that the universal validity of the principle of similitude is to a certain extent a matter for experimental investigation.[4]

[1] It is this necessity of using the same materials for building the miniature universe which forces us to use in this construction a free space in which the velocity of light is unchanged and to use electrons which have the same charge as in the actual universe. This determines the transformation relations for velocity and for electric charge. (Cf. Mrs. Ehrenfest-Afanassjewa, footnote 12.)

[2] Tolman, Phys. Rev., *3*, 244 (1914).

[3] Nordström, *Öfversigt af Finska Vetenkaps-Societens Förhandlingar*, 57, Afd-A, No. 22 (1915).

[4] As to Mrs. Ehrenfest-Afanssjewa's closing question, I presume that it must have been asked in a spirit of pleasantry. Of course the reason why the principle of similitude cannot be used to derive a physical law showing the relation between the mass and radius of a planet is because there is no physical law connecting the mass and radius of a planet. We could construct planets of any mass and radius that we desired. The actual planets which happen to be in our solar system have masses and radii which were determined by past astronomical accidents, whose history is still a sealed book even to believers in the principle of similitude.

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