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TESTS OF RANDOM DENSITY MODELS OF TERRESTRIAL PLANETS

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Abstract. Random density models are analyzed to determine the low degree harmonics of the gravity field of a planet, and therefrom two properties: an *axiality* P_l, the percent of the degree variance in the zonal term referred to an axis through the maximum for degree l; and an *angularity* E_{ln}, the angle between the maxima for two degrees l, n. The random density distributions give solutions reasonably consistent with the axialities and angularities for the low degrees, $l < 5$, of Earth, Venus, and Moon, but *not* for Mars, which has improbably large axialities and small angularities. Hence the random density model is an unreliable predictor for the non-hydrostatic second-degree gravity of Mars, and thus for the moment-of-inertia, which is more plausibly close to $0.365 MR^2$.

Introduction

Bills (1989) argued that the moment-of-inertia of Mars should be close to $0.345\ MR^2$, where M is the planet mass and R is the radius. This was appreciably lower than the $0.365\ MR^2$ favored by Reasenberg (1977) and Kaula (1979), and has considerable significance for stress levels and composition in Mars (Kaula et al., 1989).

The basis for the argument of Bills (1989) was that the equality of the equatorial non-hydrostatic moments-of-inertia, $\delta C = \delta B$, implicit in the higher value 0.365 MR^2, is extremely improbable for random density distributions in a planet. The lower value of $0.345\ MR^2$ corresponds to the most probable condition of $\delta C - \delta B = \delta B - \delta A$. Earth, Venus, and Moon all are close to this condition, as indicated by figure 1, which is a histogram of the quantity

$$f = (\delta C - \delta B) / (\delta C - \delta A) \qquad (1)$$

Figure 1 was generated from 30,000 configurations of 30 masses random in magnitude and location on a sphere. It is essentially identical with figure 2 of Goldreich & Toomre (1969) and figure 2 of Bills (1989). R-Mars on figure 1 corresponds to the $\delta C = \delta B$ assumed by Reasenberg (1977) and Kaula (1979), while B-Mars corresponds to the $\delta B = (\delta C + \delta A) / 2$ of Bills (1989).

In calculating figure 1 and all subsequent figures it is assumed that the 3-axis coincides with the minimum of the second-degree component of the non-hydrostatic gravity field, because any plausible terrestrial planet has sufficient dissipation to cause a drift to rotation about the axis of maximum moment-of-inertia on a time scale small compared to that of the generation of density irregularities (Goldreich and Toomre, 1969). It is then convenient to place the 1-axis through the maximum of the second-degree component.

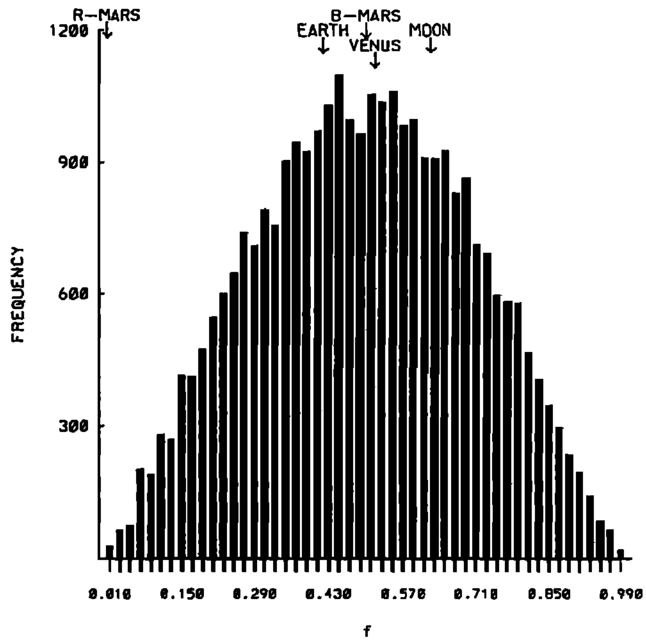

Fig. 1. Probability density distribution of the difference ratio $f = (C - B) / (C - A)$. All of figures 1-7 are from 30,000 random configurations, each containing 30 masses random in magnitude and location on a sphere.

If a property predicted by a physical model (such as a random density distribution on a sphere) is *not* observable, then it is desirable to seek other properties that *are* observable to test the model. Neither the precession nor the non-hydrostatic C_{20} of Mars, alternatives to infer its I/MR^2, are presently observable. But there are other properties of Mars's gravity field which are *both* predicted by a random density distribution *and* observed. If the predictions do not give a reasonable probability for the observed values, then the model is likely to be inapplicable for the non-observed properties.

Analysis

Properties of Mars's gravity field that are *both* predictable *and* observable are, for any spherical harmonic degree $l > 2$:

1. The percent P_l of the variance for degree l that is in the zonal term for a coordinate system that has its

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polar axis through the maximum. We call this property the *axiality*.

And for any pair of spherical harmonic degrees l, n:

2. The angle E_{ln} between the maximum for degree l alone and the maximum for degree n alone. We call this property the *angularity*.

If Mars's gravity field is dominated by a single feature that is near to axial symmetry, such as Tharsis, then for l, n small we should expect the angles E_{ln} to be improbably small according to the random density model, and the percents P_l to be improbably large.

To obtain the axiality P_l, we use the fact that the maximum must be represented solely by the zonal term of spherical harmonics referred to an axis through the maximum:

$$P_l = M_l^2 \,/\, [\,(2l+1)\,\sigma_l^2\,], \qquad (2)$$

where M_l is the maximum value for degree l, located at θ_l, λ_l, $2l+1$ is the square of the normalized zonal harmonic at the pole, and σ_l^2 is the degree variance, the sum of the squares of the normalized coefficients. The maximum M_l is the sum of coefficients times harmonics at its location θ_l, λ_l:

$$M_l = \Sigma_m \left(C_{lm} \cos m\lambda_l + S_{lm} \sin m\lambda_l \right) P_{lm}(\cos\theta_l), \qquad (3)$$

where P_{lm} is the associated Legendre polynomial. If the axes are rotated so that the 3-axis is through the maximum, then the only non-zero term at θ_l, λ_l is the zonal, m = 0. The degree variance σ_l^2 is the sum of the squares of the coefficients,

$$\sigma_l^2 = \Sigma_m \left(C_{lm}^2 + S_{lm}^2 \right), \qquad (4)$$

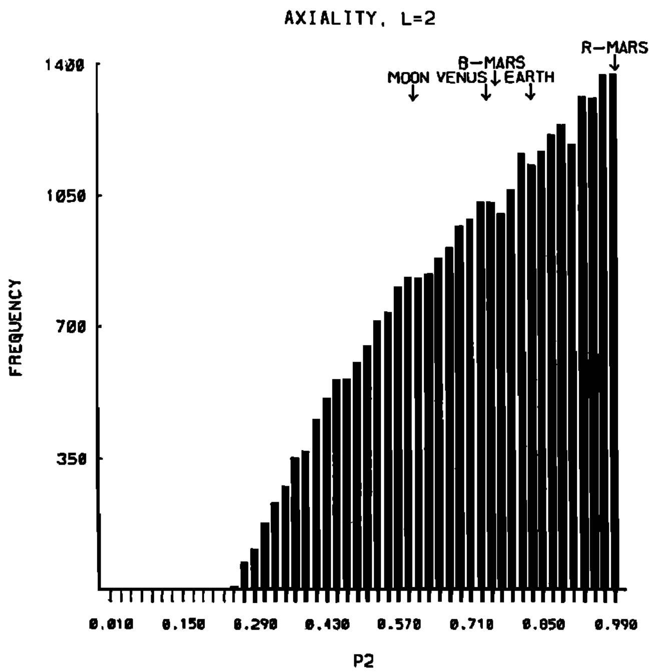

Fig. 2. Probability density distribution of the second-degree axiality P_2, defined by eq. (2), with observed values for Earth, Venus, and Moon, and hypothetical values for Mars: R-Mars, consistent with Reasenberg (1977), and B-Mars, consistent with Bills (1989).

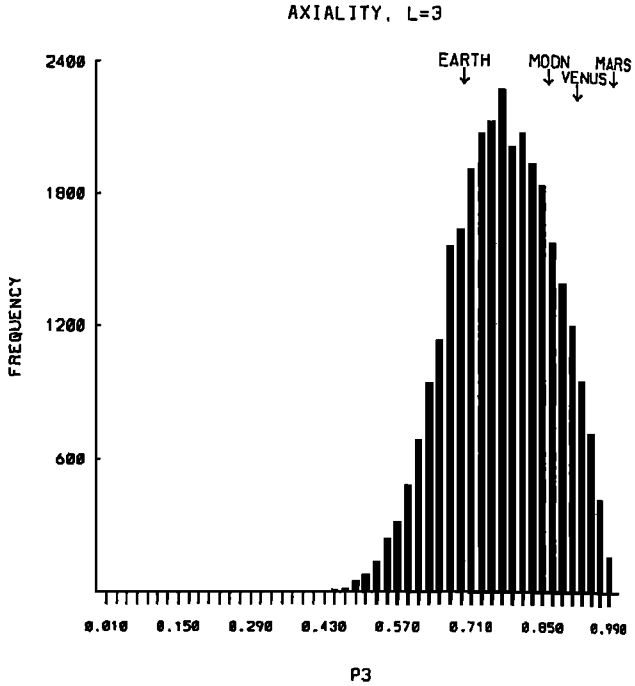

Fig. 3. Probability density distribution of the third degree axiality P_3, with observed values for all planets.

for spherical harmonics normalized so that their mean square is unity.

To obtain the angularity E_{ln} between harmonic degrees l and n, apply the cosine law to the maxima locations θ_l, λ_l and θ_n, λ_n:

$$\cos E_{ln} = \cos\theta_l \cos\theta_n + \sin\theta_l \sin\theta_n \cos(\lambda_l - \lambda_n) \qquad (5)$$

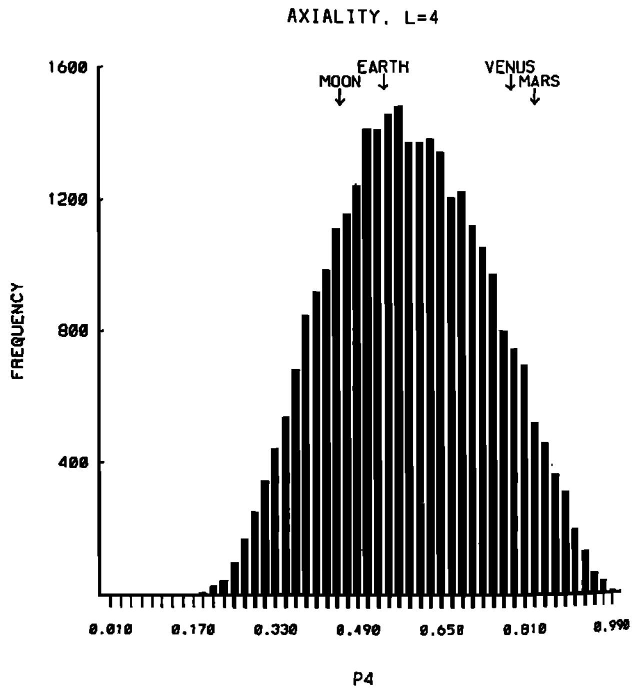

Fig. 4. Probability density distribution of the fourth degree axiality P_4, with observed values for all planets.

The results from 30,000 configurations of 30 random masses for axialities P_2, P_3, and P_4 are given in figures 2 through 4, and for angularities E_{23}, E_{24}, and E_{34} in figures 5 through 7.

The axiality P_2 is related to the function f shown in figure 1. For references axes that are principal axes, only C_{20} and C_{22} are non-zero; using the relationships thereof to the non-hydrostatic moments δA, δB, and δC (Bills, 1989; Kaula et al., 1989), and eqs. (1), (2) above:

$$P_2 = (4 - 4f + f^2) / (4 - 4f + 4f^2) \quad (6)$$

The bunching up of the P_2 histogram toward unity is consistent with dP_2/df going to zero with f.

On the figures are also marked the values for the actual terrestrial planets. The gravity fields used were those of Christensen & Balmino (1979), Bills & Ferrari (1980), Bills et al. (1987), and Marsh et al. (1988).

Conclusions

The figures indicate that the random density model is not inconsistent with observation for Earth, Moon, and Venus, but quite inconsistent for Mars, which has improbably large P_3 and P_4, and improbably small E_{23}, E_{24}, and E_{34}. These probabilities are given in Table 1.

Because the random density model does not agree with observable properties of Mars, it is an unreliable predictor for the unobservable second-degree axiality P_2 of Mars and thence the moment-of-inertia ratio I/MR^2 connected thereto. The second-degree components of the gravity field are quite similar to the higher degree components in their *cause*: there is no special mechanism peculiar thereto, analogous to the Coriolis effect on the magnetic dipole. Where the second-degree components differ is in their *effect*, which is to determine the orientation and, to a minor degree, the rate of rotation. Given the similarity of cause, the high axialities of the third and fourth degree harmonics about axes close to the maximum of the second degree suggest a similar axiality

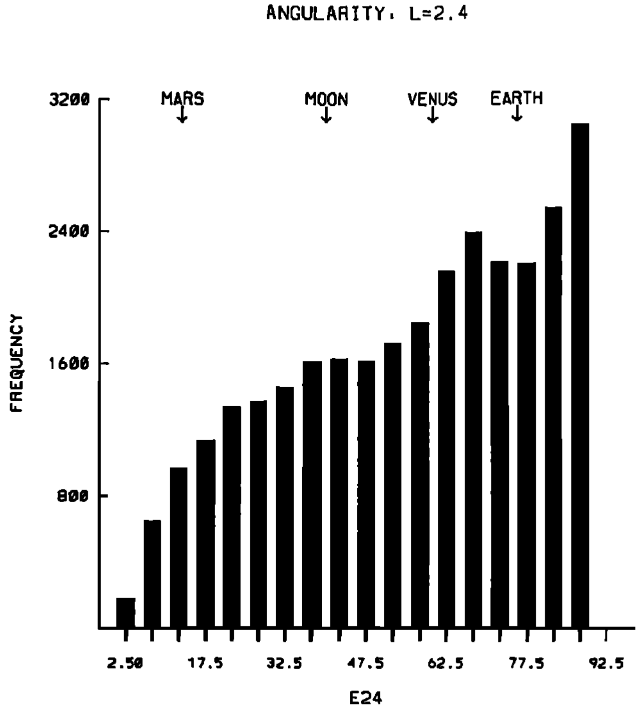

Fig. 6. Probability density distribution of the angularity between the second and fourth degrees E_{24}, with observed values for all planets.

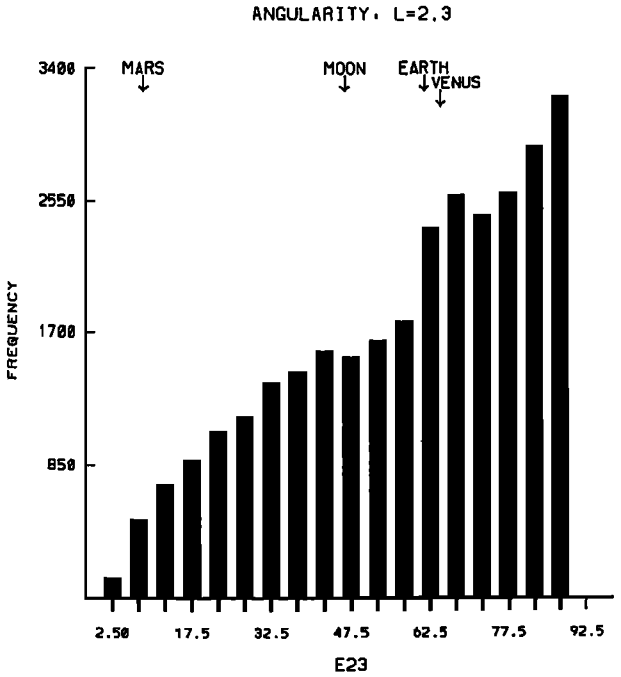

Fig. 5. Probability density distribution of the angularity between the second and third degrees E_{23}, defined by eq. (5), with observed values for all planets.

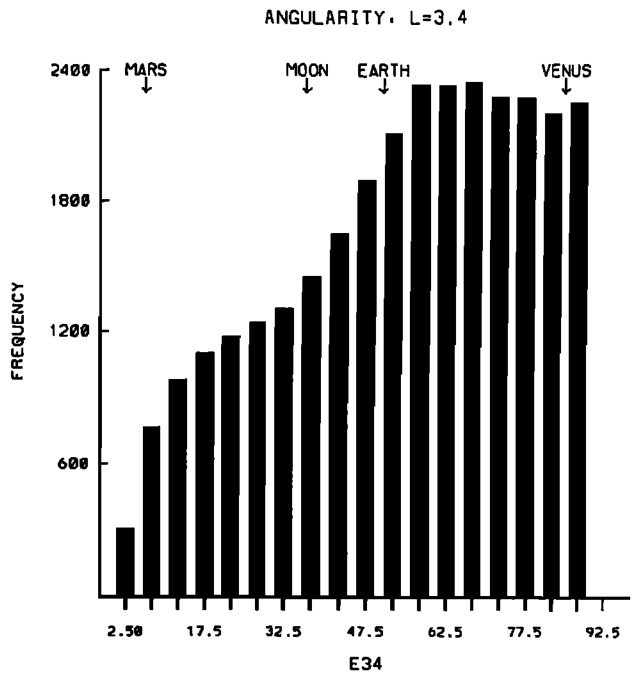

Fig. 7. Probability density distribution of the angularity between the third and fourth degrees E_{34}, with observed values for all planets.

TABLE 1: Magnitudes and Probabilities of More Extreme Values of Observed Martian Properties

Property	Symbol	Magnitude	Probability
Third-Degree Axiality	P_3	0.987	0.004
Fourth-Degree Axiality	P_4	0.827	0.063
Second-Third Angularity	E_{23}	10.78°	0.013
Second-Fourth Angularity	E_{24}	14.84°	0.047
Third-Fourth Angularity	E_{34}	8.42°	0.014

in the second degree, and hence a most probable value for I/MR^2 of 0.365 or slightly less.

Any attempt to predict the second-degree component of the gravity field entails, implicitly if not explicitly, a model to connect the degrees. The most obvious model is axially symmetric support for an axially symmetric load centered on Tharsis, such as a poloidal mantle convection or an elastic shell, as computed by Sleep and Phillips (1985). The simplest axially symmetric load-plus-response is a point mass on the 1-axis. An attempt to fit such a point mass to the observed harmonics, l,m = 2,2 through 4,4, produced a rather poor result: this opposite extreme of a random mass distribution is about as poor. The load-plus-response must extend quite far from the axis.

The apparent consistency of random density distributions with the properties of the other terrestrial bodies probably arises from their being more complex tectonic systems, in which no one feature is dominant.

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