PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
Volume 16 October 15, 1930 Number 10

REFLECTION OF WAVES IN AN INHOMOGENEOUS ABSORBING MEDIUM

By Paul S. Epstein

Norman Bridge Laboratory of Physics, California Institute of Technology

Communicated July 21, 1930

1. *Introductory.*—The problem of wave propagation in a conducting and absorbing medium was approached from the point of view of geometrical optics in a preceding paper.[1] The purpose of it was to answer the question whether it is permissible to apply the ordinary rules for constructing the path of a ray to problems of radio-telegraphy, since certain layers of the atmosphere possess a considerable conductivity. The main result can be stated as follows: Theoretically speaking, the presence of conductivity has an effect on the geometry of the ray. Generalized equations of geometrical optics were obtained, showing that the rays are no longer normal to the wave surfaces. However, the difference is negligible numerically, unless the absorption is so large as to give an appreciable attenuation in a layer of the thickness of one wave-length.

This generalized discussion is instructive yet in another way, as the limitations of geometrical optics appear in it in a peculiarly drastic form. In an inhomogeneous medium, we have in general two waves: the refracted and the reflected one. The methods of geometrical optics permit us to trace directly only one of them which, for absorbing media, is always the refracted wave, as the phenomenon of total reflection does not exist in them. However, circumstances may occur in which the main part of the incident energy is contained in the reflected wave. In these cases, geometrical optics can give us only a very incomplete notion of the path in which the energy flows. It is, therefore, desirable to supplement that treatment by a discussion, what amount of reflection can be expected and what the conditions are in which the reflected rays may not be neglected. This is the purpose of the present paper: with regard to the problems of radio-telegraphy, the result is satisfactory in its simplicity. It can be shown that in a continuous medium, there is no appreciable reflection from a conducting inhomogeneous layer unless the conductivity is small and the conditions approximate those of total reflection. According to the preceding paper it is, then, permissible to neglect the conductivity altogether and to apply ordinary geometrical optics. This is exactly the procedure that has been followed in radio-telegraphic investigations, and our results give its complete justification.

The question of reflection in a continuously changing *transparent* medium was the topic of a considerable amount of discussion.[2] Under normal conditions the reflection is so small that some authors doubted its very existence. It is, therefore, interesting to investigate whether similar conditions prevail in the case of *conducting and absorbing* media. The following treatment is based on considering a law of change of the refractive index, (formula (8), section 3), which is not general, but of sufficient generality to give a good approximation to practically all cases that may occur in applications.[3] It has the advantage of mathematical elegance and rigor. In section 3, the rigorous general solution is found holding both for transparent and for conducting media. This solution is discussed, for a transparent medium, in section 4, and, for a conducting and absorbing medium, in section 5. It should be mentioned that we restrict our investigation to the ordinary form of the wave equation (Eq. (1) in section 2). If the waves are electromagnetic, this form applies directly to one state of polarization only (electric vector normal to the stratification of the medium). There is no question, however, that all the results will be qualitatively true also for the other state of polarization.

2. *Mathematical Formulation of the Problem.*—As mentioned above, the approximation represented by the methods of geometrical optics does not permit us to say anything about the reflected rays. We have, therefore, to fall back on the rigorous equation of wave motion

$$\Delta^2\psi + k^2\epsilon\psi = 0. \tag{1}$$

The notations are the same as (loc. cit.) ($k = 2\pi/\lambda$, ϵ the dielectric constant). For the purpose we have in mind, it will be sufficient to consider the case of a "*stratified medium*," i.e., the case when ϵ depends on one cartesian variable only $\epsilon = \epsilon(z)$. Moreover, we shall consider only cylindrical waves parallel to the direction y, so that ψ depends on x and z only. We can, then, make the substitution

$$\psi = e^{ikpx}\varphi(z), \tag{2}$$

obtaining for φ the plain differential equation

$$\frac{d^2\varphi}{dz^2} + k^2(\epsilon - p^2)\varphi = 0. \tag{3}$$

Of particular interest is the case, when the function $\epsilon(z)$ is such that it has a constant value ϵ_1 in a certain region, then changes, in a continuous way, to another constant value ϵ_2. Without loss of generality, we can state this so: $\epsilon(z) = \epsilon_1$ for large negative values of z, and $\epsilon(z) = \epsilon_2$ for large positive values of z, with a layer of continuous transition in between.

Equation (3), being of the second order, has two particular integrals, P_1 and P_2, which we can choose in such a way that, *for large negative* z, they become asymptotically $\exp(ik\sqrt{\epsilon_1 - p^2}\,z)$ and $\exp(-ik\sqrt{\epsilon_1 - p^2}\,z)$. The corresponding solutions of (1) are

$$\left.\begin{aligned} [\exp(ikpx)P_1]_{z\to-\infty} &= \exp[ik(px + \sqrt{\epsilon_1 - p^2}\,z)], \\ [\exp(ikpx)P_2]_{z\to-\infty} &= \exp[ik(px - \sqrt{\epsilon_1 - p^2}\,z)]. \end{aligned}\right\} \quad (4)$$

They represent two plane waves going to and from the inhomogeneous layer in the lower medium.

On the other hand, we could select two particular integrals, of the same equation (3) in a different way. Let Q_1 and Q_2 be selected so that *for large positive* z they become $\exp(-ik\sqrt{\epsilon_2 - p^2}\,z)$ and $\exp(ik\sqrt{\epsilon_2 - p^2}\,z)$. Then

$$\left.\begin{aligned} [\exp(ikpx)Q_1]_{z\to\infty} &= \exp[ik(px - \sqrt{\epsilon_2 - p^2}\,z)], \\ [\exp(ikpx)Q_2]_{z\to\infty} &= \exp[ik(px + \sqrt{\epsilon_2 - p^2}\,z)]. \end{aligned}\right\} \quad (5)$$

They represent two plane waves in the upper medium. Any solution of equation (3) can be expressed as a linear combination of two particular integrals. We have, therefore, the relations

$$\left.\begin{aligned} Q_1 &= A_{11}P_1 + A_{12}P_2, \\ Q_2 &= A_{21}P_1 + A_{22}P_2, \end{aligned}\right\} \quad (6)$$

(with constant coefficients A), which are known to mathematicians as the "*monodromy group*" or the "*circuit relations*" of equation (3). Multiplying the second relation by $\exp(ikpx)$, we have

$$\exp(ikpx)Q_2 = A_{21}\exp(ikpx)P_1 + A_{22}\exp(ikpx)P_2.$$

This can be interpreted as a relation between the refracted (Q_2), the incident (P_1) and the reflected (P_2) waves. In other words, *this formula* represents the law of reflection, and the ratio A_{22}/A_{21} is the coefficient of reflection

$$R = A_{22}/A_{21}. \quad (7)$$

Mathematically speaking, the above formula gives the *analytical continuation* of function Q_2, represented by a power series, for instance, beyond the region for which this special representation is valid. The law of reflection gives us, therefore, a physical visualization of the rather abstract concept of analytical continuation. Originally, the theory contained in the following sections was worked out by me because of this pedagogical value.[4]

3. *Solution for a Typical Case.*—By equations (6) and (7) the problem of reflection is reduced to that of finding the monodromy group of equation (3). This group is completely known only for one equation: the differential equation of the hypergeometric type. It is, therefore, advisable to approximate $\epsilon(z)$ by a function which will reduce (3) to the hypergeometric form and, at the same time, give a sufficient approximation to typical distributions of the dielectric constant, as they may occur in our physical applications. The most general function satisfying all our requirements is

$$\left.\begin{aligned} \epsilon(z) &= \epsilon_1 + e^{\zeta}[(\epsilon_2 - \epsilon_1)(e^{\zeta} + 1) + \epsilon_3]/(e^{\zeta} + 1)^2 \\ \zeta &= kz/s. \end{aligned}\right\}, \tag{8}$$

When z is large and positive ($z \gg s\lambda/2\pi$), we have $\epsilon = \epsilon_2$, when it is large and negative, $\epsilon = \epsilon_1$. Roughly speaking, the parameter s represents the thickness of the inhomogeneous layer, measured in units $\lambda/2\pi$. Function (8) is, in fact, general enough for our purpose: If $-|\epsilon_2 - \epsilon_1| \leqslant \epsilon_3 \leqslant |\epsilon_2 - \epsilon_1|$, the change from ϵ_1 to ϵ_2 is monotonic. If this condition does not hold the dielectric constant first rises or falls to a maximum or minimum and then goes back to the final value. By a proper adjustment of the constant ϵ_3 the maximum or minimum can be given any value. At the same time, the thickness of the inhomogeneous layer can be varied by adjusting the parameter s, so that the law of change has a considerable adaptability.

In general the distribution is asymmetric with respect to $z = 0$. However, two special cases are of particular interest.

Case I: $\epsilon_3 = 0$,

$$\epsilon = \frac{\epsilon_2 + \epsilon_1}{2} + \frac{\epsilon_2 - \epsilon_1}{2}\,\mathrm{tgh}\,\frac{\zeta}{2}. \tag{9}$$

This is an antisymmetric inhomogeneous layer giving a monotomic transition from ϵ_1 to ϵ_2.

Case II: $\epsilon_2 = \epsilon_1$,

$$\epsilon = \epsilon_1 + \epsilon_3/4\cosh^2\frac{\zeta}{2}. \tag{10}$$

For large negative ζ, the dielectric constant $\epsilon = \epsilon_1$, then it changes to a maximum or minimum $\epsilon = \epsilon_1 + \epsilon_3$, for $\zeta = 0$, and returns symmetrically to its original value $\epsilon = \epsilon_1$ when ζ becomes large and positive.

We transform equation (3) by choosing

$$u = \exp(\zeta) \tag{11}$$

as a new independent variable:

$$\frac{d^2\varphi}{du^2} + \frac{1}{u}\frac{d\varphi}{du} + s^2\left[\frac{\epsilon_1 - p^2}{u^2} + \frac{(\epsilon_2 - \epsilon_1)(u + 1) + \epsilon_3}{u(1 + u)^2}\right]\varphi = 0. \quad (12)$$

Moreover, we introduce the following abbreviations

$$\left.\begin{aligned} a &= -s\sqrt{p^2 - \epsilon_1} = is\sqrt{\epsilon_1 - p^2}, \\ b &= -s\sqrt{p^2 - \epsilon_2} = is\sqrt{\epsilon_2 - p^2}, \\ &d^2 - d - s^2\epsilon_3 = 0 \\ &d = \frac{1}{2} - \sqrt{\frac{1}{4} + s^2\epsilon_3} \end{aligned}\right\} \quad (13)$$

Putting $\varphi = (1 + u)^d u^a f$, we obtain for f the equation

$$u(u + 1)f'' + [(2d + 2a + 1)u + (2a + 1)]f' + (a + b + d) .(a - b + d)f = 0, \quad (14)$$

showing that f is, in fact, a hypergeometric function. Hence, we obtain two solutions of (12) in ascending powers of u:

$$\left.\begin{aligned} P_1 &= u^a(1 + u)^d F(a + b + d, a - b + d, 2a + 1, -u), \\ P_2 &= u^{-a}(1 + u)^d F(-a + b + d, -a - b + d, -2a + 1, -u), \end{aligned}\right\} \quad (15)$$

and two solutions in descending powers:

$$\left.\begin{aligned} Q_1 &= u^{-b}\left(1 + \frac{1}{u}\right)^d F\left(a + b + d, -a + b + d, 2b + 1, -\frac{1}{u}\right), \\ Q_2 &= u^b\left(1 + \frac{1}{u}\right)^d F\left(a - b + d, -a - b + d, -2b + 1, -\frac{1}{u}\right). \end{aligned}\right\} \quad (15')$$

For large negative values of ζ and z, u becomes small and the approximations of P_1, P_2 are $u^a = \exp(ik\sqrt{\epsilon_1 - p^2}\, z)$, $u^{-a} = \exp(-ik\sqrt{\epsilon_1 - p^2}\, z)$, representing the incident and the reflected wave. Similarly, for large positive ζ and z, the approximation of Q_2 is $u^b = \exp(ik\sqrt{\epsilon_2 - p^2}\, z)$, representing the refracted wave. The part of the monodromy group (6), interesting to us, is given by a well-known Gaussian formula[5]

$$\left.\begin{aligned} Q_2 &= A(a - b, -a - b)P_1 + A(-a - b, a - b)P_2, \\ A(\alpha, \beta) &= \frac{\Gamma(\alpha + \beta + 1)\Gamma(\beta - \alpha)}{\Gamma(\alpha + d)\Gamma(\beta + d)} \end{aligned}\right\} \quad (16)$$

This leads to the coefficient of reflection

$$R = \frac{\Gamma(2a)\Gamma(-a - b + d)\Gamma(-a - b - d + 1)}{\Gamma(-2a)\Gamma(a - b + d)\Gamma(a - b - d + 1)}. \quad (17)$$

4. *Discussion for Non-absorbing Media.*—The coefficient R describes the reflected light as to its intensity and phase. R is, in general, a complex number and can be thrown into the form $R = |R| \exp(i\delta)$, where $|R|$ represents the amplitude of the reflected wave and δ its phase. In the case of non-absorbing materials ϵ is always real. The physical meaning of p is $p = n_1 \cos \varphi_1$, where φ_1 is the angle of incidence (counted from the glancing direction) and $n_1 = \sqrt{\epsilon_1}$. The parameter a becomes, according to (12), $a = isn_1 \sin \varphi_1$. The angle of incidence must, necessarily, be real so that a is always imaginary

$$a = ia' = isn_1 \sin \varphi. \tag{18}$$

The factor $\exp(ikpx)$ of equation (2) is the same everywhere, so that p can be interpreted also as $p = n_2 \cos \varphi_2$, where φ_2 is the angle of emergence of the refracted ray. The relation $n_1 \cos \varphi_1 = n_2 \cos \varphi_2$ is, of course, the law of refraction. The parameter b becomes $b = isn_2 \sin \varphi_2$, but φ_2 may be either real or imaginary, so that b, in its turn, may be imaginary or real. Finally, we see from (12) that d may be either real or have the complex form $d = {}^1/_2 - id'$. Accordingly, we have to distinguish the following three cases:

(A) $a = ia'$, b real. Examining expression (17), we notice that the denominator is the complex conjugate of the numerator (both when d is real and when it has the form $d = {}^1/_2 - id'$). This leads to

$$|R| = 1, \tag{19}$$

or to *total reflection.* The condition of total reflection is for a continuous inhomogeneous medium the same as for a discontinuous one: $p^2 \geqslant \epsilon_2$, $\cos \varphi_2 \geqslant 1$. This shows that we can have no total reflection when $\epsilon_1 = \epsilon_2$, as in the case of expression (9) or when ϵ_2 is complex. This confirms the result found (loc. cit.) by methods of geometrical optics.

(B) $a = ia'$, $b = ib'$, d real. Expression (17) can be transformed by means of a relation from the theory of gamma functions:

$$\Gamma(x)\Gamma(1 - x) = \pi/\sin \pi x. \tag{20}$$

Multiplying (17) by its complex conjugate and applying (20), we obtain

$$|R|^2 = \frac{\sin^2 \pi d + \sinh^2 \pi(a' - b')}{\sin^2 \pi d + \sinh^2 \pi(a' + b')}. \tag{21}$$

When s is small, i.e., when the inhomogeneous layer is thin, compared with $\lambda/2\pi$, we may replace the sin and sinh by their arguments. Moreover, a' and b' are proportional to s, while d becomes proportional to s^2. Neglecting fourth powers of s beside s^2, we obtain Fresnel's formula.

$$|R| = \left|\frac{a' - b'}{a' + b'}\right| = \left|\frac{n_1 \cos \varphi_1 - n_2 \cos \varphi_2}{n_1 \cos \varphi_1 + n_2 \cos \varphi_2}\right|.$$

On the other hand, when s is large, we can again neglect $\sin^2 \pi d$ and replace sinh by the exponential. We obtain $|R| = \exp(-2\pi b')$, for $a' > b'$, and $|R| = \exp(-2\pi a')$, for $a' < b'$.

In the particular case $\epsilon_1 = \epsilon_2$, we have $a' = b'$. The numerator is reduced to $\sin^2 \pi d$ and vanishes for integral values of d: this means that the reflected light exhibits the colors of thin or of thick plates.

(*C*) $a = ia'$, $b = ib'$, $d = {}^1/_2 - id'$. The same procedure as before leads to the formula:

$$|R|^2 = \frac{\cosh^2 \pi d' + \sinh^2 \pi(a' - b')}{\cosh^2 \pi d' + \sinh^2 \pi(a' + b')}. \tag{22}$$

This case will not occur unless ϵ_3 is negative. It will apply to $\epsilon_3 < -|\epsilon_2 - \epsilon_1|$, s being sufficiently large, i.e., when the dielectric constant decreases to a minimum in the inhomogeneous layer. Geometrical optics (loc. cit., p. 44) would lead us to expect total reflection whenever ϵ decreases, in any place of the layer, below p^2. We see, however, from formula (22) that total reflection will not take place as long as ϵ_2 remains larger than p^2. The physical explanation of this fact is obvious: Even if we have total reflection ($\epsilon_2 < p^2$), the less dense medium is not optically vacant but contains a standing wave of the type $\exp(-k\sqrt{p^2 - \epsilon_2}z) . \cos kp(x - ct)$, whose amplitude decreases exponentially with z. In the case of ϵ having a minimum, this standing wave reaches over into those parts of the field where ϵ, increasing again, becomes larger than p^2 and an advancing wave is again possible.

5. *Absorbing Medium.*—We shall adapt our discussion to the conditions prevalent in the case of radio waves. These waves originate in layers of the atmosphere which are practically free of conduction and absorption, so that we must regard ϵ as real and $a = ia'$ as purely imaginary. For ϵ_2 and ϵ_3, on the other hand, we shall make the general assumption

$$\epsilon_2 = \kappa_2 + i\Sigma_2, \quad \epsilon_3 = \kappa_3 + i\Sigma_3, \tag{23}$$

where κ measures the refractive power and Σ the conductive power of the medium.

Correspondingly, we write for b and d

$$b = -b_2 + ib_1, \quad d = {}^1/_2 - d_2 - id_1 \tag{24}$$

In the case of small conduction (i.e., when $\Sigma_2/|\epsilon_2 - p^2| \ll 1$ and $\Sigma_3/|\kappa_3| \ll 1$, so that we may neglect the squares of these small quantities) formulas (12) give us, for $\kappa_2 - p^2 > 0$,

$$b_1 = s\sqrt{\kappa_2 - p^2}, \quad b_2 = s\,\Sigma_2/2\sqrt{\kappa_2 - p^2}, \tag{25}$$

and for $\kappa_2 - p^2 < 0$,

$$b_1 = s\Sigma_2/2\sqrt{p^2 - \kappa_2}, \quad b_2 = s\sqrt{p^2 - \kappa_2}. \tag{26}$$

In the same way, for $\kappa_3 + 1/4s^2 > 0$,

$$d_1 = s\Sigma_3/2\sqrt{\kappa_3 + 1/4s^2}, \quad d_2 = s\sqrt{\kappa_3 + 1/4s^2}, \tag{27}$$

and for $\kappa_3 + 1/4s^2 < 0$,

$$d_1 = s\sqrt{-\kappa_3 - 1/4s^2}, \quad d_2 = s\Sigma_3/2\sqrt{-\kappa_3 - 1/4s^2}. \tag{28}$$

On the other hand, we obtain, for very large conduction ($\Sigma_2/|\kappa_2 - p^2| \gg 1$ or $\Sigma_3/|\kappa_3| \gg 1$)

$$\begin{cases} b_1 = b_2 = s\sqrt{\Sigma_2/2}, \\ d_1 = d_2 = s\sqrt{\Sigma_3/2}. \end{cases} \tag{29}$$

We have to evaluate the coefficient of reflection (17) for these cases. Such an evaluation cannot be given in a rigorous way for any value of s, but if s is sufficiently large to make the arguments of the gamma functions large compared with unity, it can be easily effected by means of the approximate formulas applying for large values of the argument x:

$$\begin{cases} \Gamma(x) = \sqrt{2\pi}\, x^{\,x-1/2}e^{-x} & \text{if } Re(x) > 0, \\ \Gamma(x) = \dfrac{\sqrt{2\pi}}{2\sin\pi x}\,(1 - x)^{\,x-1/2}e^{-x+1}, & \text{if } Re(x) < 0. \end{cases} \tag{30}$$

Fortunately, the case in which these formulas apply is precisely that in which we are interested. We wish to learn something about the amount of reflection that is to be expected when dielectric constant and conductivity change continuously and slowly. We shall not give the general expression of the coefficient of reflection following from (17), because its discussion would become too cumbersome and would require too much space. It is sufficient for our purpose to discuss the two special cases mentioned in section 3: $\epsilon_3 = 0, d = 0$ (Case I) and $\epsilon_1 = \epsilon_2, a = b$ (Case II).

Case I.—The coefficient of reflection is reduced to

$$|R| = \frac{\Gamma(-a - b)\Gamma(-a - b + 1)}{\Gamma(a - b)\Gamma(a - b + 1)}. \tag{31}$$

The evaluation, by means of expressions (30), gives for the different cases:

(*Ia*) Σ_2 small, $\kappa_2 - p^2 > 0$; $b_2 \ll b_1$;

$$|R| = \left|\frac{b_1 + a'}{b_1 - a'}\right|^{2b_2} e^{-(a' - b_1)\pi + |a' - b_1|\pi} \tag{32}$$

(*Ib*) Σ_2 small, $\kappa_2 - p^2 < 0$; $b_1 \ll b_2$;

$$|R| = \exp\left(-4b_1 \, arctg \, \frac{a'}{b_2}\right). \tag{33}$$

(*Ic*) Σ_2 very large; $b_1 \gg a'$, $b_2 = b_1$

$$|R| = \exp(-\pi a'). \tag{34}$$

Case II.

$$|R| = \frac{\sin \pi d}{\pi} \Gamma \; (-2a + d)\Gamma(-\,2a - d + 1). \tag{35}$$

(*IIa*) Σ_3 small, $\kappa + 1/4s^2 > 0$; $d_1 \ll d_2$

$$|R| = [\cos^2 \pi d_2 + \sinh^2 \pi d_1]^{1/2} \exp\left[-2\pi a' - d_1\left(\pi - 2 \; arctg \, \frac{4a'}{2d_2 + 1}\right)\right] \tag{36}$$

In the case Σ_3 small, $\kappa + 1/4s^2 < 0$; $d_2 \ll d_1$,

we have to distinguish here two possibilities:

(*IIa'*) $d_1 < 2a'$

$$|R| = \left(\frac{2a' - d_1}{2a' + d_1}\right)^{d_2} \exp[(-2a' + d_1)\pi]. \tag{37}$$

(*IIb*) $d_1 > 2a'$

$$|R| = \left(\frac{d_1 - 2a'}{d_1 + 2a'}\right)^{d_2} \tag{38}$$

(*IIc*) Σ_3 very large; $d_1 \ll a'$, $d_2 = d_1$

$$|R| = \exp(-\pi a'). \tag{39}$$

Physical Discussion of the Results. Conclusion.—Let us begin the discussion with the case of small conductivity. Our formulas (*Ib*) and (*IIb*) refer to the case when refractive index and angle of incidence are such that, neglecting conductivity altogether, we should have total reflection. We see that there is a continuous transition from the limiting case of nonconducting media. In fact, vanishing conductivity corresponds to $b_1 = 0$ and $d_1 = 0$, giving in both cases $|R| = 1$. If the conductivity is finite but small, we have, strictly speaking, not total but partial reflection. However, the coefficient of reflection is still very near to unity and all conditions are nearly the same as in the case of total reflection. On the other hand, if conditions are such that, neglecting conductivity, there would be a refracted and a partially reflected ray, formulas (32) and (37) apply, giving again results very close to those obtained in section 4 for transparent media. As the number $2a'$ or $2b_1$ appears in the exponent the amount of reflection is extremely small unless the angle of incidence is very close to the angle of total reflection ($b_1 = 0$). The case (*IIa*) has no analogy in a transparent medium but here, too, the amount of reflection is very small.

Going over to the case of high conductivity we find that the amount of reflection is always given by the simple expression $\exp(-a'\pi)$. This is in all practical cases a very small number as $a'\pi$ is large, provided that the inhomogeneous layer is a few times thicker than the wave-length, and that the incidence is not a glancing one. This result is particularly striking if we compare it with the conditions in a transparent medium. If the angle of incidence is larger, or slightly smaller, than the angel of total reflection, a non-conducting medium will give either total or considerable reflection. A slightly conducting upper medium will show practically the same behavior. With increase of conductivity the reflection decreases very rapidly (because of (33) and (26)), and when the conductivity is very large, the coefficient of reflection is extremely small. This seems paradoxical, at first sight, since we are accustomed to associate with metallic conduction a high reflectivity. However, a closer examination makes these facts look less surprising: in a continuous medium the reflected ray has its origin at all depths of the discontinuous layer, and the large absorption existing in this layer prevents its getting out of it with a considerable intensity.

Summarizing our discussion, we can say that the reflection is always very insignificant, except in the case when conductivity is small and where we have conditions very near to total reflection. This result indicates that in radiotelegraphy, if rays are reflected at all, their path can be computed neglecting conductivity, as if the medium were transparent.

[1] P. S. Epstein, *Proc. Nat. Acad. Sci.*, **16**, 37 (1930).

[2] The literature of this subject can be found in a paper by J. Wallot, *Ann. Physik*, **60**, 734 (1919).

[3] A special case of this theory was used by me ever since 1919 in my lectures on differential equations, as a physical illustration of the principle of analytical continua-

tion. For working out another special case, I am indebted to Dr. H. P. Robertson. I was told by the late Prof. E. Hilb of Würzburg, to whom I showed this work several years ago, that he had seen a paper by a French author proceeding on somewhat similar lines. However, I was unable to locate that paper.

[4] As an illustration of this theory, I usually demonstrate in my lectures the Wiener experiment (*Ann. Physik*, **49**, 105 (1893); Kohlrausch, *Praktische Physik*, p. 254, Leipzig (1921)) about the propagation of light in a medium composed of a layer of water over one of glycerine. It can be completely understood with the help of our theory of section 4 (with glancing incidence) and it is suitable for the determination of the relative index of refraction of water and glycerine.

[5] C. F. Gauss, *Werke*, **3**, p. 213, Göttingen, **1876**.

THE STRUCTURAL BASIS OF THE INTEGRATION OF BEHAVIOR

BY G. E. COGHILL

THE WISTAR INSTITUTE OF ANATOMY AND BIOLOGY, PHILADELPHIA

Read before the Academy April 28, 1930

The behavior of embryos has attracted the attention of biologists for a long time. William Harvey recorded important observations on the movements of the chick embryo. With the advent of the microscope Schwammerdam and Leeuwenhoek studied intensively the movements of snail embryos in the egg. Following their time there were only casual or isolated observations in this line until 1885 when Preyer published his work on the general physiology of the embryo, in which he gave large place to movements. Since Preyer, reports on the subject have been for the most part fragmentary; but not altogether so. Since 1920 Minkowski of Zürich has studied the movement of young human fetuses very extensively; and others, particularly Yanase, and Bolaffio and Artom, have made very important contributions to the knowledge of human fetal behavior. From all of this work, however, as it stands alone, no unifying principle or law has been deduced to resolve the knowledge of the development of behavior into an intelligible system. This is due, not to lack of acumen on the part of the observers, but to conditions inherent in the species studied by them; that is to say, the species have been high in the scale of evolution or morphological specialization, and have for the most part permitted of very limited time of observation. For some human fetuses the time of study has been limited to a very few minutes.

For the discovery of law or order in the development of behavior it is necessary to turn to a species which is relatively unspecialized morphologically and which can be studied continuously from the beginning of movement till the adult behavior pattern is established. In a complete historical or chronological panorama of the development of behavior of such a type the