Calif. Inst. of Tech. NsG-56-60

Reprinted from ADVANCES IN THE ASTRONAUTICAL SCIENCES, Volume 8,
Published by Plenum Press, Inc., 227 West 17 St., New York (1962)

pp. 294-302

N63 17698

Code-none

AN "INVARIANT" PROPERTY OF SATELLITE MOTION IN A DISSIPATIVE MEDIUM

Bruce C. Murray and Alan B. Lees†*

17698

A simple expression is derived for near-circular orbits which relates the difference in instantaneous angular momentum of a satellite as observed at two different times from the same station to the dissipative perturbing forces which acted on the satellite during that time interval. This expression is valid for an arbitrarily asymmetric gravity field. Possible applications to the measurement of very short-term atmospheric density variations and to the improvement of knowledge about the earth's shape and gravity field are described.

INTRODUCTION

With the successful demonstration by the first "transit" vehicles of a precise radial-velocity measuring system, the need is developing for precise dynamical expressions which involve the actual instantaneous velocity of the satellite as it moves in the actual terrestrial gravity field where it is subject to unpredictable short-term fluctuations in drag. The object of the present investigation has been the development of several such expressions to exploit the intrinsic value of velocity observations, at least for the case of a near-circular orbit. It is felt that the classical method of deriving orbital relationships, and hence observational relationships, is limited in its applicability to velocity observations because the equations of motion have been integrated twice. As a result of this approach, no description of the position or velocity is complete unless all the secular, long-period, and short-period variations due to the noncentral components of the earth's field are included. It has not proven possible, so far, to formulate precisely by classical methods all the effects of the known departures from a central force of the earth's field. Hence, it is our feeling that theories involving only a single integration of the equations of motion and yielding results applicable directly to velocity observations will be required in order to exploit fully the observational accuracy which apparently is going to be achieved in the next few years.

In this paper, which represents a preliminary approach to the above problem, we will develop the fundamental differential equation of motion suitable for our purposes. The equation will be integrated for the very special but useful case of a near-circular orbit, and some possible applications of that integrated form will be discussed.

DEVELOPMENT OF THE EQUATIONS

In this section will be developed a property of the motion of an earth satellite which finds application both in situations where air drag and noncentral components of the earth's gravity field have a significant effect on observations as well as in those cases where the motion can be considered to take place in a conservative medium.

*Research Fellow in Space Science, Division of Geological Sciences, California Institute of Technology, Pasadena, California. This investigation was initiated while Dr. Murray was still a geophysicist at the Geophysics Research Directorate of the Air Force Cambridge Research Laboratories, Bedford, Massachusetts.

†Director of Engineering Systems Laboratory Division Electronic Specialty Co., Los Angeles, California.

NASA CR-50,235

The dynamical discussions will be framed in three conventional coordinate systems defined as follows:

1. The spherical polar inertial (r, α, δ) system in which r is the length of the vector from the center of mass of the earth to the satellite and α and δ are, respectively, the right ascension and declination of the satellite as viewed by an observer at the center of the earth
2. The Eulerian (r, u, ι, Ω) inertial system in which r is defined as above, u is the argument of the satellite in the instantaneous orbital plane and is measured from the ascending node, ι is the inclination angle of the instantaneous orbital plane to the plane of the equator, and Ω is the angle, measured westward, from the vernal equinox to the ascending node
3. The rotating geocentric system (r, ψ, λ) in which r is again the geocentric radius vector, ψ the geocentric latitude, and λ the longitude. Nutation and precession of the earth are not considered in this discussion.

These various geometrical quantities are illustrated in Fig. 1 from which the geometrical relationship between them may be written down.

$$r = r = r \tag{1}$$

$$\alpha = \lambda + w_e t = \Omega + \tan^{-1} (\cos \iota \tan u) \tag{2}$$

$$\delta = \psi = \sin^{-1} (\sin u \sin \iota) \tag{3}$$

w_e is the sidereal rate of the earth and t is sidereal time. The elements of the system (r, u, ι, Ω) are not expressible uniquely in terms of r, α, and δ alone since the orbital plane depends not only on the position of the satellite but also on the direction of the instantaneous velocity vector. If it were important to have a bi-unique relationship between the two sets of variables then α, and δ could be augmented by a third quantity depending on $\dot{\alpha}$, and $\dot{\delta}$. In the discussion that follows, however, only the $(r, u, \iota, \Omega) \to (r, \alpha, \delta)$ transformations are required, and hence Eqs. (1), (2), and (3) are sufficient.

It is not difficult to show that the elements (r, u, ι, Ω) form a permissible set of Lagrangian generalized coordinates for descending the state of the dynamical system [1] and thus the general equations of motion may be written in terms of these variables. Because the earth has longitudinal variations in its gravitational field and it is also in a state of rotation, the force field in inertial coordinates in which the satellite is moving is partly nonconservative. This raises an initial difficulty since the potential function is customarily defined only for conservative fields. There does not appear to be any problem, however, in these considerations, at least in introducing a pseudopotential function U which is time varying and which, at any time, has a gradient equal to the real force field at that time. Moreover, it is apparent that the diurnal time average of U viewed in the inertial system, which is written $\overline{U}$, is the potential function associated with the conservative force field due to an axially symmetric earth. The gradient of the time-varying residual, U_p, gives the perturbative force field due to longitudinal variations in the earth's field.

Specifically, we can write in the complete tesseral harmonic form,

$$U = \sum_{n=0}^{N} \sum_{m=0}^{n} \frac{1}{r^{n+1}} (A_{nm} \sin m\lambda + B_{nm} \cos m\lambda) P_n^M (\cos \psi) \tag{4}$$

and

$$\overline{U} = \sum_{n=0}^{N} \frac{1}{r^{n+1}} B_{n0} P_n^0 (\cos \delta) \tag{5}$$

$$U_p = \sum_{m=1}^{N} C_m (r,\delta) \cos m\lambda + S_m (r,\delta) \sin m\lambda \tag{6}$$

where

$$C_m (r, \delta) = \sum_{n=m}^{N} \frac{B_{mn}}{r^{n+1}} P_n^m (\cos \delta)$$

$$S_m (r, \delta) = \sum_{n=m}^{N} \frac{A_{mn}}{r^{n+1}} P_n^m (\cos \delta)$$

$\overline{U}$ is, therefore, the conventional zonal harmonics and U_p is the remainder of the set in which the time-dependent coordinate λ appears explicitly.

The Lagrangian L may be defined conventionally in terms of the kinetic energy T of the satellite and the potential function U as

$$\mathsf{L} = T - \overline{U}$$

where, from the geometry of Fig. 1, T has the form

$$T = \tfrac{1}{2} m \, [\dot{r}^2 + r^2 (\dot{u} + \dot{\Omega} \cos \iota)^2] = \tfrac{1}{2} m \, (V_r^2 + V_t^2) \tag{7}$$

and m is the mass of the satellite. The formal Lagrangian equations of motion may be written

$$\frac{d}{dt}\left(\frac{\partial \mathsf{L}}{\partial \dot{r}}\right) - \frac{\partial \mathsf{L}}{\partial r} = \xi_r \tag{8}$$

$$\frac{d}{dt}\left(\frac{\partial \mathsf{L}}{\partial \dot{u}}\right) - \frac{\partial \mathsf{L}}{\partial u} = \xi_u \tag{9}$$

$$\frac{d}{dt}\left(\frac{\partial \mathsf{L}}{\partial \dot{\iota}}\right) - \frac{\partial \mathsf{L}}{\partial \iota} = \xi_\iota \tag{10}$$

$$\frac{d}{dt}\left(\frac{\partial \mathsf{L}}{\partial \dot{\Omega}}\right) - \frac{\partial \mathsf{L}}{\partial \Omega} = \xi_\Omega \tag{11}$$

where the quantities on the right-hand sides of these equations are the generalized, external, nonconservative forces on the satellite. In addition to the perturbations arising from the longitudinal inhomogeneity of the earth, which have been discussed above, the external forces include air drag and a number of other effects of lesser significance, (e.g., lunisolar and planetary perturbations, atmospheric rotation, radiation pressure, particle bombardment). In the treating of the problem presented here, our considerations have omitted the effects mentioned in parentheses. Presumably, they would be introduced as special perturbations in actual application.

At any instant in time, the total nonconservative perturbative forces acting on the satellite may be resolved into three orthogonal components which are

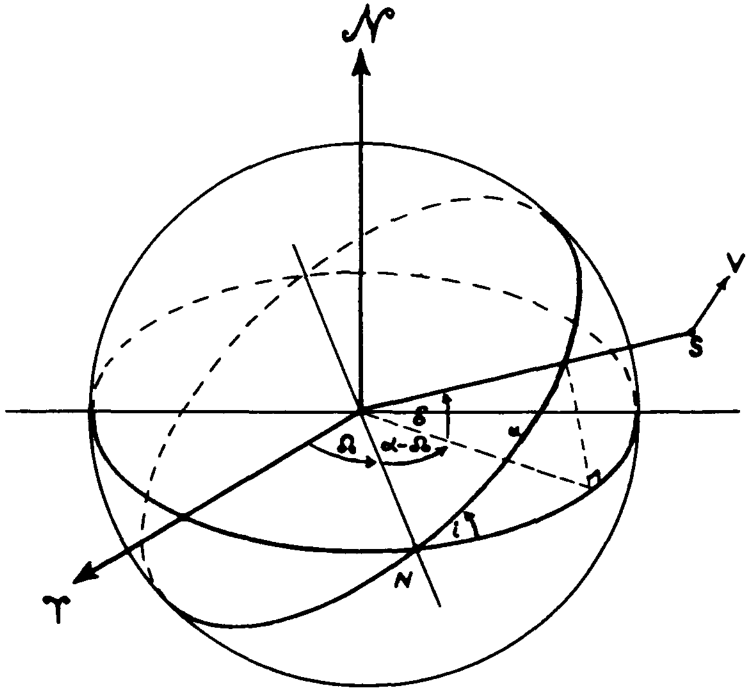

Fig. 1. Instantaneous osculating plane.

(1) along the radius vector, (2) in the instantaneous orbit plane and normal to the radius vector, (3) normal to the orbit plane. Taking the most general point of view, the effect of the atmosphere will contribute to all three of these force components, but by far the most important contribution is in the orbit plane only. We will represent the total aerodynamic force vector by **A** with components A_r, A_u, and $A_\perp$ corresponding, respectively, to the cases (1), (2), and (3) above. A_r is a contributor to the generalized force ξ_r and similarly $r\,A_u$ is a contributor to ξ_u. The normal component, $A_\perp$ contributes both to ξ_i and to ξ_Ω, and by taking resolved components, the contribution to ξ_i is $A_\perp \sin u$ and to ξ_Ω is $A_\perp \cos u$. An analogous discussion may be applied to the force vector **P** arising from the longitudinal gravity field perturbations and having components P_r, P_u, $P_\perp$. Adding together these two perturbation effects and neglecting any others yields for the ξ's

$$\xi_r = A_r + P_r = F_r \tag{12}$$

$$\xi_u = r\,(A_u + P_u) = r F_u \tag{13}$$

$$\xi_i = r \sin u\,(A_\perp + P_\perp) = r F_\perp \sin u \tag{14}$$

$$\xi_\Omega = r \cos u\,(A_\perp + P_\perp) = r F_\perp \cos u \tag{15}$$

where we have written the total perturbative force vector on the satellite as $\mathbf{F}$.

Attention will now be directed only toward the Lagrangian equation (9). The potential function $\bar{U}$ is a function of r and δ (but not α), and since it is recalled from Eq. (3) that δ is a function of u and ι only, say $\delta(u, \iota)$, then $\bar{U}$ may be written $\bar{U}[r, \delta(u, \iota)]$, hence, $U(r, u, \iota)$. Consequently,

$$\frac{\partial \bar{U}}{\partial r} dr + \frac{\partial \bar{U}}{\partial \delta} d\delta = \frac{\partial \bar{U}}{\partial r} dr + \frac{\partial \bar{U}}{\partial u} du + \frac{\partial \bar{U}}{\partial \iota} d\iota \tag{16}$$

and

$$\frac{\partial \bar{U}}{\partial \delta} d\delta = \frac{\partial \bar{U}}{\partial u} du + \frac{\partial \bar{U}}{\partial \iota} d\iota \tag{17}$$

But

$$\dot{u} >> \dot{\iota}$$

hence,

$$\frac{\partial \bar{U}}{\partial \delta} d\delta \simeq \frac{\partial \bar{U}}{\partial u} du \tag{18}$$

Also, the general definition of angular momentum as the partial derivative of the Lagrangian with respect to the velocity allows us to associate the name instantaneous total angular momentum, P_u as follows:

$$P_u = \frac{\partial \mathrm{L}}{\partial \dot{u}} = m r^2 (\dot{u} + \Omega \cos \iota) = m r V_t \tag{19}$$

where $\mathbf{V}$ is the instantaneous velocity and V_t and V_r are the tangential and radial components. Finally, we may rewrite Eq. (9) as

$$dP_u + \frac{1}{\dot{u}} \frac{\partial \bar{U}}{\partial \delta} d\delta = \xi_u \, dt \tag{20}$$

Let us now investigate the formulation of the ξ_u quantity when the perturbative forces are contributed by U_p, the pseudopotential. We are interested in determining its partial derivative with respect to u, i.e., the force. In general, however, the earth's field is expressed in terms of the coordinates (r, ψ, λ) where λ and ψ are the longitude and latitude, $\psi \equiv \delta$ and $\lambda = \alpha - w_e t$. Thus, the conventional form of writing the potential, $U_p(r, \lambda, \psi)$ may be expressed analogously as $U_p(r, \alpha - w_e t, \delta)$, and noting from Eqs. (1) and (2) that α and δ are specific functions of u, ι, and Ω, we have

$$U_p(r, \lambda, \psi) \equiv U_p[r, \alpha(\iota, u, \Omega) - w_e t, \delta(u, \iota)] \tag{21}$$

Taking the partial derivative for variations in u only

$$\frac{\partial U_p}{\partial u} = \frac{\partial U_p}{\partial \lambda} \frac{\partial \alpha}{\partial u} + \frac{\partial U_p}{\partial \delta} \frac{\partial \delta}{\partial u} \tag{22}$$

and since the gravitational part of the generalized force is merely $\partial U_p / \partial u$,

$$\xi_u = \frac{\partial U_p}{\partial \lambda} \frac{\partial \alpha}{\partial u} + \frac{\partial U_p}{\partial \delta} \frac{\partial \delta}{\partial u} + r A_u \tag{23}$$

Finally, we write the drag term of ξ_u as

$$rA_u(t) \tag{24}$$

where $A_u(t)$ is the instantaneous (and time-varying) total aerodynamic force component oppositely-directed to the direction of $r\,du$. The basic equation of motion for the u coordinate now becomes, from Eqs. (20), (23), and (24)

$$dP_u + \frac{1}{\dot{u}} \frac{\partial \bar{U}}{\partial \delta} d\delta = \left(\frac{\partial U_p}{\partial \lambda} \frac{\partial a}{\partial u} + \frac{\partial U_p}{\partial \delta} \frac{\partial \delta}{\partial u}\right) dt + rA_u(t)\,dt \tag{25}$$

By integrating this equation over an arbitrary time interval t_1 to t_2, we may express the change of angular momentum in this interval as

$$\Delta P_u = -\int_{t_1}^{t_2} \frac{1}{\dot{u}} \frac{\partial \bar{U}}{\partial \delta} ds + \int_{t_1}^{t_2} \left(\frac{\partial U_p}{\partial \lambda} \frac{\partial a}{\partial u} + \frac{\partial U_p}{\partial \delta} \frac{\partial \delta}{\partial u}\right) dt + \int_{t_1}^{t_2} rA_u(t)\,dt \tag{26}$$

The left-hand side of this equation is exact since a perfect differential has been integrated. The right-hand side has been subject only to the approximation of Eq. (17), which would be insignificant for any real situation. Of course, a general integration of the right-hand side is not possible without complete knowledge of the real gravity field, of atmosphere drag variations, and of the ephemeris of the satellite. Thus, its only usefulness in this form would be as a general consistency criterion to be applied to satellite observations in conjunction with the gravity and atmospheric models used in the reduction of those observations. In the following section, however, we will show that for those cases where the percentage variation in r is small during the integration a simple, but still accurate, relationship can be obtained for repeated observations from the same station. Equation (26) can also, after rearrangement, and substitution, be evaluated for orbits of moderate to large eccentricity, using numerical integration based on an ephemeris. Such techniques will not, however, be developed in the present investigation.

INTEGRATION OF THE u LAGRANGIAN WHEN VARIATION IN r IS NEGLIGIBLE

In this section we shall consider the three terms on the right-hand side of Eq. (26) separately for the case where the variation in r and, hence, in $\dot{u}$ is small. Also, the previous approximation in Eq. (17) that $\dot{u} \gg \dot{\iota}$ will be repeated here in the form that ι will be considered to be constant or, at least, not to introduce residual terms during the integration over time. The limits of integration will be from t_1 to t_2 where t_1 and t_2 are chosen to correspond to the positions of the satellite when passing over the same place on the earth. Thus, the positions of the satellite at the limits can be written (r, λ, δ) and (r_2, λ, δ). The geocentric radial distance r thus is permitted to change slightly during the interval of integration due to the action of nonconservative forces. λ and δ, however, are assumed to be identical at both upper and lower limits. In practice, small differences in the values of λ and δ, such as would correspond to the distribution of observations from a single station, would not limit the application of the equations.

We may now rewrite the first term on the right-hand side of Eq. (26) as

$$-\frac{1}{\dot{u}} \int_{r_1\delta_1\lambda_1}^{r_2\delta_1\lambda_1} \frac{\partial \bar{U}}{\partial \delta} d\delta \tag{27}$$

Since u is a potential function, this integral would vanish exactly if $r_1 = r_2$. The residual term in the present case differs from 0 only in that $\partial\bar{U}/\partial r$ is a slowly-changing function of r due to the noncentral forces in the earth's field. As long as $(r_1 - r_2)/(r_1 + r_2)$ is small, however, the integral will differ from 0 by an insignificant amount, and it is considered to vanish for the present discussion.

The second integral on the right-hand side will be considered in two parts. First, the second term in the integral can be rewritten as

$$\frac{1}{\dot{u}}\int_{r_1 \delta_1 \lambda_1}^{r_2 \delta_1 \lambda_1} \frac{\partial U_p}{\partial \delta} \simeq 0 \tag{28}$$

because $\delta = \delta\,(u, \iota)$, and

$$\frac{d\delta}{du} = \frac{\partial \delta}{\partial u} + \frac{\partial \delta}{\partial \iota}\,\frac{d\iota}{du} \simeq \frac{\partial \delta}{\partial u} \tag{29}$$

The integral vanishes for the same reasons and with the same approximations as Eq. (27). The first term of the second integral on the right-hand side can be rewritten as

$$\int_{r_1 \delta_1 \lambda_1}^{r_2 \delta_1 \lambda_1} \frac{\partial \bar{U}_p}{\partial \lambda}\left(\frac{da}{du} - \frac{\partial a}{\partial \Omega}\,\frac{d\Omega}{du}\right) dt \tag{30}$$

because $a = a\,(u, \iota, \Omega)$, and

$$\frac{\partial a}{\partial u} = \frac{da}{du} - \frac{\partial a}{\partial}\,\frac{d\Omega}{du} - \frac{\partial a}{\partial \iota}\,\frac{d\iota}{du} \simeq \frac{da}{du} - \frac{\partial a}{\partial}\,\frac{\partial \Omega}{du} \tag{31}$$

but $\partial a/\partial u = 1$, and $\ddot{\Omega}/\dot{\Omega}$ is quite small. Hence, the integral becomes

$$\frac{1}{\dot{u}}\int_{r_1 \delta_1 \lambda_1}^{r_2 \delta_1 \lambda_1} \frac{\partial U_p}{\partial \lambda}\, da - \frac{\dot{\Omega}}{\dot{u}}\int_{t_1}^{t_2} \frac{\partial U_p}{\partial \lambda}\, dt \tag{32}$$

Now, since

$$a = \lambda + w_e t$$

then,

$$da = d\lambda + w_e\, dt$$

and Eq. (32) becomes

$$\frac{1}{\dot{u}}\int_{r_1 \delta_1 \lambda_1}^{r_2 \delta_1 \lambda_1} \frac{\partial U_p}{\partial \lambda}\, d\lambda + \frac{w_e - \dot{\Omega}}{\dot{u}}\int_{t_1}^{t_2} \frac{\partial U_p}{\partial \lambda}\, dt \tag{33}$$

But, the first term also vanishes in the same manner as Eqs. (28) and (27), and the remaining term is merely the very small work done on the satellite by the daily rotation of the longitudinal variations in the earth's field. This nonconservative effect has not been reported yet even for the case of Vanguard I and, hence, is not considered further in this discussion.

The third and final term on the right-hand side of Eq. (26) can be integrated for the case of small variation in r to give

$$\int_{t_1}^{t_2} r\, A_u\,(t)\, dt = \left(\frac{r_1 + r_2}{2}\right) \overline{A_u\,(t_1 t_2)}\,(t_2 - t_1) \tag{34}$$

and Eq. (26) becomes, for the case of low variation in r and for two observations from the same station,

$$\frac{r_1\, V_{t_1} - r_2\, V_{t_2}}{(t_2 - t_1)\,(r_1 + r_2)} = \frac{\overline{A_u\,(t_1 t_2)}}{m} \tag{35}$$

where $\overline{A_u\,(t_1 t_2)}$ is the average aerodynamic force in the u direction during the interval (t_1, t_2). Or if $(V_{t_1} r_2 - V_{t_2} r_1)$ does not differ significantly from 0 (in terms of the accuracy of observations of V_t and r),

$$\frac{\Delta V_t}{\Delta t} = \frac{\overline{A_u\,(t_1 t_2)}}{m} \tag{36}$$

The "invariant" we have obtained is thus simply Newton's second law, generalized for the case of an arbitrarily complex gravitational potential, in which the time rate of change of u momentum is equated to the dissipative forces in the u direction which produce the change. Similar relationships can be derived for the r, Ω, and ι coordinates, but are of less interest in the present investigation because the aerodynamic force components in those directions are so much smaller than that in the u direction.

POSSIBLE APPLICATIONS

The left-hand side of Eq. (35) contains only r and V_t quantities which can be obtained directly from topocentric observations of the position and velocity of the satellite (acquired either from a single station or from simultaneous observations from several stations operating in an intervisible mode). (It is assumed that t, the time of the observation, can always be acquired with adequate accuracy.) Equation (36) shows that only the differences in V_t need be measured accurately. Hence, most kinds of systematic observation errors such as station-position error, systematic range bias, or systematic angular bias have little effect on the measurement of the left-hand side of Eq. (35). There are a number of ways V, the scalar value of instantaneous velocity, can be obtained aside from simple positional differences on a single pass. Probably stellar photography plus Doppler is the most feasible technique for single-station operation, whereas use of range-difference observations, simultaneously acquired from three stations, is perhaps the best over-all approach. In any case, we assume that V and, hence, V_t can be acquired accurately from a suitably instrumented satellite. We also consider two distinct classes of satellites which differ greatly in surface area-to-mass ratio, but are both in near-circular orbits and have spherical shapes. Aerodynamic forces will be dominant in the nonconservative motion of the Echo-type satellite (assuming the effects of radiation pressure are removed), whereas they will be minimized in the case of the low surface-area-to-mass "geodetic" satellite. Hence, the application of Eq. (35) or Eq. (36) directly to observations of an Echo-type satellite should yield values of $A_u\,(t_1 t_2)$ directly, particularly if the altitude is not higher than, say, 300 mi. Then, assuming the size and mass are known, those values can yield average air density over the short period of time between observations (i.e., a few hours in some cases). By combining the results of several stations, the averaging time can be cut down sharply. Presuming there is to be a number of stations in different parts of the world, it appears possible to monitor with high precision the very short-term fluctuations in upper atmosphere density. In view of the recent suggestive correlations of such density variations with solar phenomena, the technique here should yield definitive answers regarding the nature and degree of coupling between the solar and terrestrial atmospheres. Such a detailed record of atmospheric drag would, of course, also be of great value in the computation of the ephemerides of all satellites.

A "geodetic" satellite*, on the other hand, should show very small values for the right-hand side of Eq. (35)—so small, in fact, that the equation approaches being a true invariant for closely-spaced observations and, hence, could be applied directly to the observations as a data smoothing procedure. If observations from different stations are compared, one could include integrals of $\partial U_p/\partial\delta$, $\partial U_p/\partial\lambda$, etc. over part of a revolution. Since the drag effects are negligible, it appears possible, at least, to use this technique to generate a redundant series of simple integral equations from which some coefficients of the tesseral harmonics could be extracted.

REFERENCES

1. Harry Lass, "The Motion of a Satellite of the Moon," Technical Release No. 34-56, Jet Propulsion Lab., C.I.T., pp. 3-7.

*Presumably at an altitude of at least 400 to 500 miles.