Space Charge and Field Waves in an Electron Beam

SIMON RAMO
General Engineering Laboratory, General Electric Company, Schenectady, New York

(Received May 8, 1939)

W. C. Hahn has shown that the basic characteristics of a new type of vacuum tube using a velocity modulated electron beam may be explained by means of waves propagating along the beam. For an "ideal" tube in which the beam was assumed to be of uniform density throughout its length he described the small amplitude, slow "space charge" waves which have axial symmetry. In the following paper a study is made first of the more general slow space charge waves which do not necessarily possess symmetry about the axis. Two cases are considered. First, a very high magnetic focusing field is assumed, so that the motion of electrons in any but the axial direction may be neglected. Then the magnetic focusing field is assumed to be completely absent, and waves having components of velocity of electrons in all directions are treated. Also in the following, attention is given to the fast "field waves" which may exist in the idealized tube under certain conditions. The waves have been termed "space charge" waves and "field waves" because, for the former type, the phase velocities are close to beam velocity and the wave energy is mainly in the electrons. In the case of the field waves, the phase velocities are large compared to beam velocity and the energy is mainly in the electromagnetic field.

INTRODUCTION

THE problem of electromagnetic waves in an electron beam has recently become important because such waves are excited and utilized in a new type of electron tube[1, 2] which may be used to generate, amplify, or detect ultra-high frequency signals. W. C. Hahn[1] has shown that the basic characteristics of this so-called velocity modulation type of vacuum tube may be explained by means of waves propagating along the electron beam of the tube. In his analysis an idealized tube was assumed in which the electron beam and its coaxial perfectly conducting shield were assumed to be infinitely long, the beam consisting of a uniform density of electrons, ρ_0, all traveling at the same constant velocity, v_0, in the absence of waves. This condition was made a possible one by certain additional assumptions: Sufficient positive ions were supposed to be contained in the beam to nullify the average current and average space charge due to the electrons. The heavy positive ions were assumed not to depart, as did the very much lighter electrons, from their drift velocity in the event of passage of a wave. Hence, with these assumptions the positive ions did not enter into the wave motion but simply aided in establishing convenient steady (zero signal) conditions.

For this ideal tube Hahn described the small signal, slow "space charge" waves which have axial symmetry and pointed out that faster waves would be expected. In the following paragraphs a study is made of the more general* slow space charge waves which do not necessarily possess symmetry about the axis.

Also in the following, attention is given to the fast "field waves" which may exist in the idealized tube if conditions are proper. The waves have been termed "space charge waves" and "field waves" because, for the former type, the phase velocities are close to beam velocity and the wave energy is mainly in the electrons. In the case of the "field waves" the phase velocities are large compared to beam velocity and the energy is mainly in the electromagnetic field.

The space charge and the field waves will be studied for two cases, the theory being limited to small wave amplitudes, or signals, in each case: (1) For a very high magnetic focusing field. The focusing field will be assumed so large that motion of electrons in any but the axial direction may be neglected. (2) For no magnetic focusing field. In this case wave components of velocity of electrons may exist in all directions. It is found possible to divide both space charge

[1] W. C. Hahn, "Small Signal Theory of Velocity Modulated Electron Beams," Gen. Elec. Rev. **42**, 258 (1939).
[2] W. C. Hahn and G. F. Metcalf, "Velocity Modulation Tubes," Proc. I. R. E., February, 1939.

* The importance of the asymmetrical waves lies in the fact that a starting and utilizing mechanism may be designed for almost any conceivable wave. To determine which type of wave should be started and utilized requires a comparison of their characteristics, especially as regards the potential transconductance of the tube and the optimum drift tube length (reference 1). It is therefore of considerable importance to consider the asymmetrical waves.

and field waves into two classes which can exist independently and which exhibit differences in their characteristics.

High Magnetic Focusing Field

If the magnetic focusing field is sufficiently high the theory need only consider motion of electrons in the axial direction. This suggests that the equations may be most easily set up in terms of the retarded scalar electric and magnetic vector potentials because only the axial component of the vector potential will be required. If the potential functions be substituted into Maxwell's equations by use of the relations:[3]

$$\bar{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$$
$$\bar{H} = \nabla \times \bar{A}, \qquad (1)$$

in which $\bar{E}$ is the electric field vector, $\bar{H}$ is the magnetic field vector, Φ is the scalar electric potential and $\bar{A}$ is the vector magnetic potential, then the equations reduce to the well-known wave equations for Φ and $\bar{A}$:

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi = -\rho$$
$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\bar{A} = -\frac{\rho\bar{v}}{c}, \qquad (2)$$

in which c is the velocity of light, ρ and $\bar{v}$ are charge density and velocity, respectively, and Heaviside-Lorentz or rational units are used throughout. Eqs. (2) imply that the divergence of $\bar{A}$ has been determined by[3]

$$\nabla \cdot \bar{A} = -\frac{1}{c}\frac{\partial \Phi}{\partial t}. \qquad (3)$$

Since only the wave part of the solution is desired, it will be convenient to denote the scalar electric potential, the charge density and the velocity by

$$\Phi_1 \epsilon^{i(\omega t - \gamma z)}, \quad \rho_1 \epsilon^{i(\omega t - \gamma z)}, \quad \text{and} \quad v_z \epsilon^{i(\omega t - \gamma z)},$$

respectively. Then using cylindrical coordinates, Eq. (2) becomes

$$\frac{\partial^2\Phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi_1}{\partial \theta^2} + (k^2 - \gamma^2)\Phi_1 = -\rho_1, \qquad (4)$$

in which $k = \omega/c$.

It is easy to express ρ_1 in terms of Φ_1. From Appendix A, we obtain

$$\rho_1 = \frac{e\rho_0}{m}\frac{\gamma^2 - k^2}{(\omega - \gamma v_0)^2}\Phi_1, \qquad (6A)$$

so that Eq. (4) now becomes

$$\frac{\partial^2\Phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi_1}{\partial \theta^2} + (k^2 - \gamma^2)\left[1 - \frac{e\rho_0}{m(\omega - \gamma v_0)^2}\right]\Phi_1 = 0, \qquad (5)$$

whose solution appropriate about the origin, is[4]

$$\Phi_1 = B_n J_n(Tr)\epsilon^{in\theta}, \qquad (6)$$

where B_n is an arbitrary constant and

$$T = \left\{(k^2 - \gamma^2)\left[1 - \frac{e\rho_0}{m(\omega - \gamma v_0)^2}\right]\right\}^{\frac{1}{2}}. \qquad (7)$$

In the space between the beam and the conducting boundary the charge density is zero so that we may write directly for the electric scalar potential in this region*

$$\Phi_2 = C_n[I_n(\tau r) + D_n K_n(\tau r)], \qquad (8)$$

in which I_n and K_n are modified Bessel functions of the first and second kind, respectively,[5] and

$$\tau = (\gamma^2 - k^2)^{\frac{1}{2}}. \qquad (9)$$

The constant D_n is determined by writing that the tangential electric field must equal zero at the surface of the perfectly conducting cylinder where $r = bR$. This requirement is satisfied by equating Φ_2 to zero at this radius.

[3] See for instance *Introduction to Theoretical Physics*, a text by J. C. Slater and N. H. Frank (McGraw-Hill Book Co., 1933), Chap. XXI

[4] See for instance, *Electrical and Optical Wave Motion*, a text by H. Bateman (Cambridge, 1915), Chapter III.

* The modified Bessel functions are appropriate here for the slow space charge waves because for these waves γ is very nearly equal to $\omega/v_0 = \gamma_0$, the propagation constant of the beam. For $\gamma \approx \gamma_0$, τ will be appropriately real.

[5] See for instance, *Theory of Bessel Functions*, a text by G. N. Watson (Cambridge, 1922).

Thus

$$D_n = -\frac{I_n(\tau Rb)}{K_n(\tau Rb)}. \quad (10)$$

Two more conditions remain to be applied at the edge of the beam, where $r=b$. These two conditions will serve to determine the ratio C_n/B_n and also the value of γ in terms of the given parameters. Continuity of the tangential electric field is obtained by continuity of the potentials. This gives

$$\frac{C_n}{B_n} = \frac{I_n(\tau b) + D_n K_n(\tau b)}{J_n(Tb)}. \quad (11)$$

For continuity of tangential magnetic fields Eq. (1) discloses that since only z components of $\bar{A}$ exist then the only component of $\bar{H}$ is the azimuthal component, H_θ. Continuity of this component requires continuity of $\partial A_z/\partial r$ which by Eq. (3) leads to continuity of $\partial\Phi/\partial r$. Hence

$$\frac{C_n}{B_n} = -\frac{(\tau b)}{(Tb)} \frac{I'(\tau b) + D_n K_n'(\tau b)}{J_n'(Tb)}, \quad (12)$$

in which the primes indicate derivatives of the Bessel functions with respect to their arguments.

A comparison of (11) and (12) yields

$$(Tb)\frac{J_n'(Tb)}{J_n(Tb)} = (\tau b)\frac{I_n'(\tau b) + D_n K_n'(\tau b)}{I_n(\tau b) + D_n K_n(\tau b)}, \quad (13)$$

which for $n=0$ is the same as the equation for the determination of γ derived by Hahn[1] for the case of an infinite magnetic focusing field (except for an easily recognizable difference in notation).

With sufficient time and a complete set of tables, it is possible to determine the values of γ for the various order of waves and then compare the $n=0$, 1, 2, etc. waves for transconductances, optimum drift tube length and distribution over the cross section. Some idea of the relative usefulness of the various waves may be obtained more quickly if the tube parameters are specialized somewhat by making $R=1$. With this selection of parameters the solution (6) holds over the entire tube cross section. Eq. (8) is no longer significant and the boundary condition at the conductor yields simply

$$J_n(Tb) = 0. \quad (14)$$

For each value of n there will be a series of roots to this equation; the mth root of the nth order wave will be designated as (p_{nm}). Eq. (7) then yields

$$\frac{p_{nm}^2}{b^2} = (k^2 - \gamma^2)\left[1 - \frac{e\rho_0}{m(\omega - \gamma v_0)^2}\right]. \quad (15)$$

Space charge waves

For the slow space charge waves γ is very nearly equal to $\gamma_0 = \omega/v_0$ and if this be substituted in (15) there results

$$[\omega - \gamma v_0]^2 = -e\rho_0 \Big/ m\left[\frac{p_{nm}^2}{b^2(k^2 - \gamma_0^2)} - 1\right] \quad (16)$$

and finally

$$\gamma = \gamma_0[1 \pm \delta], \quad (17)$$

in which

$$\delta = \left\{\frac{e\rho_0(\gamma_0^2 - k^2)b^2}{m\omega^2[p_{nm}^2\gamma_0^2 - b^2k^2]}\right\}^{\frac{1}{2}} \quad (18)$$

and will usually be found so small compared to unity that the substitution of γ_0 for γ at times in its derivation may be considered justified.

It is now possible to write the ratio of conduction current modulation density $(\rho_0 v_z + v_0 \rho_1)$ to the velocity modulation (v_z). If the former is denoted by ξ_z, Eq. (2A) may be altered to*

$$G_w = \xi_z / v_z = -\rho_0 / \pm\delta. \quad (19)$$

This shows that the transconductance of the velocity modulation tube, whether due to the symmetrical wave of zero order or waves of higher order will have essentially the same general characteristics. For example, the waves may occur in pairs and if the starting mechanism is such as to introduce velocity modulation into the beam but no conduction current modulation at some point along its length,[1] then there will be another point farther along the beam at which the conduction current modulation present in the beam will be a maximum. The distance between these points will be that for which $\gamma_0\delta l$ is an odd multiple of $\pi/2$ radians, l being the distance between the two points.[1] Since from Eq. (18) the value of δ is seen to decrease as

* This ratio will be termed "wave transconductance" and will be denoted by G_w.

p_{nm} increases, the wave transconductance will increase with p_{nm}, as will also the optimum drift tube length.

An important consideration in comparing the potential practical application of the waves of various n's and m's is their distribution over the cross section. For $n=0$, $m=1$ the wave will vary with the $J_0(Tr)$ function between $r=0$ and $r=b$, $(T=p_{01}/b)$ and so will have maximum amplitude at $r=0$, and will decrease to zero at $r=b$. The (0,2) wave will have one reversal of phase between $r=0$ and $r=b$, advance to a negative maximum then decrease to zero again at $r=b$. The (0,3) wave will have two reversals, the (0,4) wave three reversals, etc. The* waves of $n>0$ will start at zero amplitude at the origin, have m maximum points and $(m-1)$ reversals of phase between $r=0$, and $r=b$ before returning to zero at $r=b$.

Field waves

There are also solutions to Eq. (15) for which the value of γ is far removed from γ_0. If γ is very much less than γ_0^2 then $(\omega-\gamma v_0)^2$ is very nearly equal to ω^2. The quantity $e\rho_0/m$ which we shall call ω_0^2 is recognized as the square of the familiar natural angular frequency of oscillation of a plasma of electron charge density ρ_0. Thus we know that ω_0^2/ω^2 is appreciably less than unity for electron beam tubes, making $[1-\omega_0^2/\omega^2]$ a positive quantity. Eq. (15) is thus seen to have solutions for which γ^2 is smaller than k^2 these values of γ^2 being given approximately by†

$$\gamma^2=k^2-\frac{p_{nm}^2}{b^2[1-\omega_0^2/\omega^2]}. \tag{20}$$

Equation (20) indicates also that there is a cut-off frequency corresponding to each value of p_{nm}^2/b^2 and ω^2 below which the waves will not propagate, since then γ^2 becomes negative and γ becomes imaginary. If it is remembered that $k=\omega/c$, the cut-off frequency, ω_c, is found from (20) to be given by

$$\omega_c^2=\omega_0^2+c^2p_{nm}^2/b^2. \tag{21}$$

At cut-off, $\gamma=0$ and the wave velocity is infinite. As the frequency approaches infinity the wave velocity decreases to c.

* These statements may be verified by a glance at curves or tables of the J_n functions.

† This result makes use of the approximation $(\omega-\gamma v_0)^2=\omega^2$ which amounts to neglecting γ^2/γ_0^2 with respect to unity. This is consistent with the approximation that has been made throughout in neglecting the relativity correction to mass. It is evidently not difficult to omit these approximations and obtain more precise expressions for very high beam velocities if necessary.

Zero Magnetic Focusing Field

Axial waves

When the magnetic focusing field does not restrict the motion to the axial direction the equations must contain additional variables. Before attempting a general solution, however, it seems pertinent to inquire whether the wave just studied may still exist without the restricting action of the magnetic focusing field. It would be of practical value to learn that there is a space charge wave which in the complete absence of focusing field will still possess only the axial components of velocity and conduction current modulation.

To answer this it may first be noted that in the foregoing analysis every equation is still valid. However it is necessary that additional equations be written if the radial and azimuthal velocities are to be zero without the restricting influence of the magnetic focusing field. To meet these added restraints the force on the electron in the azimuthal and radial directions must be zero. Now the force on the electron is

$$\bar{F}=\bar{E}e+e(\bar{v}\times\bar{H})/c. \tag{22}$$

If there are no components of velocity other than in the axial direction, then as before $A_\theta=A_r=0$. From Eq. (3)

$$A_z=(k/\gamma)\Phi_1, \tag{23}$$

so that substitution in Eq. (1) shows the electric and magnetic fields to be

$$\begin{aligned} E_r&=-\frac{\partial\Phi_1}{\partial r}, & H_r&=\frac{1}{r}\frac{ink}{\gamma}\Phi_1,\\ E_\theta&=-\frac{in}{r}\Phi_1, & H_\theta&=-\frac{k}{\gamma}\frac{\partial\Phi_1}{\partial r},\\ E_z&=\frac{i(\gamma^2-k^2)}{\gamma}\Phi_1, & H_z&=0. \end{aligned} \tag{24}$$

If these are substituted into Eq. (22) the condition that the force components in the azimuthal and radial directions be zero results in

$$0=-\frac{ein}{r}\Phi_1+\frac{e}{c}v_0H_r=\frac{ein}{r}\Phi_1\left[-1+\frac{kv_0}{c\gamma}\right],$$
$$0=-e\frac{\partial\Phi_1}{\partial r}-\frac{e}{c}v_0H_\theta=e\frac{\partial\Phi_1}{\partial r}\left[-1+\frac{kv_0}{c\gamma}\right] \tag{25}$$

(if, as usual, cross-product modulation terms are neglected). These equations are both satisfied when

$$\gamma=kv_0/c. \tag{26}$$

Thus this "axial" wave, if it is to exist, must be a very fast wave for the ratio v_0/c is well below unity in all practical cases making γ much less than k.

We have already seen that waves for which γ is less than k may occur above certain cut-off frequencies for any given tube geometry and electron density. There are obviously series of discrete frequencies above the cut-off point for which

$$\gamma=kv_0/c,$$

which frequencies are found by substituting this value of γ in Eq. (20). Since the purely axial wave may exist at discrete frequencies for no magnetic field and at any frequency (above a certain cut-off frequency) for infinite magnetic field, it appears that at some finite magnetic field bands of frequencies will exist for which these waves are possible.

Division of possible waves

Let us turn now to the more general case in which radial and azimuthal velocities are possible. A convenient way to set up the equations so that the various possible waves are disclosed most easily is suggested by the following line of thought. In a region bounded by a cylinder, free of charge and in which the dielectric constant is uniform, it is convenient to express the various components of the electric and magnetic fields in terms of E_z and H_z, which satisfy the equations

$$(\nabla^2+k_1^2)E_z=0,$$
$$(\nabla^2+k_1^2)H_z=0, \tag{27}$$

whose solutions can immediately be written in cylindrical coordinates. It is convenient then to speak of "E type" waves for which H_z is zero and the "H type" waves for which E_z is zero.[6] Now, consider an observer who moves with the average velocity of the beam, v_0. As is shown in Appendix B, this moving observer would set up equations which are identical with those already solved for the case of a cylindrical boundary by Rayleigh[7] and more recently by others.[6] The dielectric constant would be a fictitious one dependent upon electron density and frequency.

The moving observer might accordingly divide the waves which he would predict into the E and H types, all other field, velocity, and current components being expressed in terms of E_z and H_z. Now by use of the Lorentz[8] transformations, all these expressions could be transformed into relations appropriate for a stationary observer, H_z and E_z being invariant under the transformation, and the problem could be considered solved. However, now that we are assured that all the phenomena may be thus expressed in terms of E_z and H_z, it will be well to discard the moving observer and regard him as of only momentary value in indicating a simple line of attack.

In Appendix C it is shown that E_z and H_z obey the equations

$$(\nabla^2+k_E^2)E_z=0,$$
$$(\nabla^2+k_H^2)H_z=0, \tag{28}$$

in which

$$k_E^2=\frac{\omega^2[c^2\omega_b^2-v_0^2\omega_0^2]+c^2\omega_0^2[\omega_0^2-2\omega\omega_b]}{c^4(\omega_b^2-\omega_0^2)} \tag{29}$$

and

$$k_H^2=k^2\left[1-\frac{\omega_0^2}{\omega^2}\right]. \tag{30}$$

These equations have solutions of the form

$$J_n[(k_E^2-\gamma^2)^{\frac{1}{2}}r]\epsilon^{in\theta} \tag{31}$$

and

$$J_n[(k_H^2-\gamma^2)^{\frac{1}{2}}r]\epsilon^{in\theta}.$$

The components of the first two of Eqs. (1B)

[6] See for instance "Hyper-Frequency Wave Guides" by Carson, Mead and Schelkunoff, Bell Sys. Tech. J., **5**, 15 (1936).

[7] Rayleigh, Phil. Mag., Vol. 43 (1897).

[8] See for instance, *Electricity and Magnetism*, a text by J. H. Jeans (Cambridge, 1927), page 604.

may be rearranged to give

$$E_\theta\left[\gamma^2+k^2-\frac{\omega_0^2}{c^2}\right]=-\frac{n}{r}\frac{\omega_0^2 v_0}{\omega_b c^2}E_z-ik\frac{\partial H_z}{\partial r},$$

$$H_r\left[\gamma^2+k^2-\frac{\omega_0^2}{c^2}\right]=\frac{n}{r}\left[\frac{\omega_0^2}{\omega_b c}-k\right]E_z+i\gamma\frac{\partial H_z}{\partial r},$$

$$E_r\left[\gamma^2+k^2-\left(k+\gamma\frac{v_0}{c}\right)\frac{\omega_0^2}{\omega_b c}\right]=\frac{kn}{r}H_z-i\left[\gamma-\alpha\frac{v_0}{c}\right]\frac{\partial E_z}{\partial r}, \quad (32)$$

$$H_\theta\left[\gamma^2+k^2-\left(k+\gamma\frac{v_0}{c}\right)\frac{\omega_0^2}{\omega_b c}\right]=-\frac{\gamma n}{r}H_z+i\left[\frac{\omega_0^2}{\omega_b c}-k\right]\frac{\partial E_z}{\partial r}.$$

These expressions, together with those for the velocities (8C) (9C) (10C), the currents (13C) (14C) (15C), and the charge density (21C) constitute the necessary relations for all the wave quantities in terms of E_z and H_z.

The H type wave

Consider now the wave for which E_z is zero. The single boundary condition requires that E_θ vanish at $r=b$ or that

$$J_n'[(k_H^2-\gamma^2)^{\frac{1}{2}}b]=0. \quad (33)$$

Designating these roots by q_{nm}, we have from (30)

$$\gamma^2=k^2\left[1-\frac{\omega_0^2}{\omega^2}\right]-\frac{q_{nm}^2}{b^2}. \quad (34)$$

Only values of γ which are less than k are evidently possible. Thus the H type of wave is always a fast wave with a velocity of propagation exceeding c and a cut-off frequency given by

$$\omega_0^2=\omega_0^2+c^2q_{nm}^2/b^2. \quad (35)$$

The $n=0$ or symmetrical H waves, which might be designated as the H_{0m} waves will have only the following components: H_z, H_r, E_θ; v_θ and ξ_θ, all other field, velocity, current, and charge density components being zero. The $n>0$ waves will however have H_z, H_r, E_θ, E_r, H_θ, ξ_θ, ξ_r, v_r, and v_θ components in general, only the v_z and ξ_z components being missing.

The E type wave

When $H_z=0$ the boundary condition at $r=$b requires that

$$J_n[(k_E^2-\gamma^2)^{\frac{1}{2}}b]=0, \quad (36)$$

so that

$$k_E^2-\gamma^2=p_{nm}^2/b^2. \quad (37)$$

For the space charge waves $\omega_b=\omega-\gamma v_0$ is very small compared to ω and approximations based on this fact permit (37) to be solved for γ, and give the approximate relation

$$\gamma=\gamma_0\left\{1+\frac{\omega_0^2b^2}{c^2p_{nm}^2}\pm\frac{b^2\omega_0^2}{c^2p_{nm}^2}\times\left[1+\frac{p_{nm}^2c^2}{b^2}\left(\frac{c^2}{\omega^2\omega_0^2}\frac{p_{nm}^2}{b^2}+\frac{1}{\omega^2}+\frac{1}{\omega_0^2}\right)\right]^{\frac{1}{2}}\right\}. \quad (38)$$

In the case of the fast field waves, ω_b is very nearly equal to ω (unless the beam velocity is very high) and k_E^2 is given approximately by

$$k_E^2=\frac{\omega^2-\omega_0^2}{c^2}=k^2\left[1-\frac{\omega_0^2}{\omega^2}\right], \quad (39)$$

which substituted in (37) yields

$$\gamma^2=k^2\left[1-\frac{\omega_0^2}{\omega^2}\right]-\frac{p_{nm}^2}{b^2}. \quad (40)$$

The cut-off frequency of these waves is given by

$$\omega_0^2=\omega_0^2+\frac{c^2p_{nm}^2}{b^2}. \quad (41)$$

For both the space charge and the field waves only the following components may be present in the E_{0m} type waves: E_z, E_r, H_θ, v_r, v_z, ξ_r, ξ_z, and ρ_1. The E_{nm} waves in which $n>0$ will in general have all components.

Acknowledgment

The author is indebted to W. C. Hahn for many valuable discussions of both the theory originally worked out by the latter as well as of the present study.

Appendix A

From continuity we have

$$-\frac{\partial \rho_1}{\partial t}=\frac{\partial}{\partial z}(\rho_0 v_z+v_0\rho_1) \qquad (1A)$$

or

$$\rho_1=\frac{\gamma\rho_0}{\omega-\gamma v_0}v_z. \qquad (2A)$$

(This neglects modulation cross-products of ρ and $\bar{v}$ and thus limits the theory to small signals.) But

$$m d\bar{v}/dt=eE_z, \qquad (3A)$$

in which E_z is the amplitude of the z modulation component of $\bar{E}$, e is the charge and m is the mass of the electron. Now

$$\frac{dv_z}{dt}=\frac{\partial v_z}{\partial t}+\frac{\partial v_z}{\partial z}\frac{dz}{dt}=i[\omega-\gamma v_0]v_z, \qquad (4A)$$

for small signals and from Eqs. (1) and (3), E_z is seen to be given by

$$E_z=i[\gamma\Phi_1-kA_z]=i[\gamma-k^2/\gamma]\Phi_1. \qquad (5A)$$

The above equations then yield

$$\rho_1=\frac{e\rho_0}{m}\frac{\gamma^2-k^2}{(\omega-\gamma v_0)^2}\Phi_1. \qquad (6A)$$

Appendix B

The moving observer would write, in his own units and system of moving axes,

$$\nabla\times\bar{E}=-\frac{1}{c}\frac{\partial\bar{H}}{\partial t}, \qquad \nabla\cdot\bar{E}=\rho_1,$$
$$\nabla\times\bar{H}=\frac{1}{c}\left[\frac{\partial\bar{E}}{\partial t}+\xi\right], \qquad \nabla\cdot\bar{H}=0. \qquad (1B)$$

Now in the force equation (22), terms of type $\bar{v}x\bar{H}$ will not occur since cross-products of wave components are being discarded. Thus

$$\xi=\rho_0\bar{v}=\rho_0 e\bar{E}/i\omega m \qquad (2B)$$

or

$$\xi=-\frac{e\rho_0}{\omega^2 m}\frac{\partial\bar{E}}{\partial t}. \qquad (3B)$$

From continuity,

$$-i\omega\rho_1=\nabla\cdot\xi=\rho_0 e\nabla\cdot\bar{E}/i\omega m \qquad (4B)$$

or

$$\rho_1=+e\rho_0\nabla\cdot\bar{E}/\omega^2 m. \qquad (5B)$$

Thus substitution in (1B) yields

$$\nabla\cdot\bar{E}(1-e\rho_0/m\omega^2)=0,$$
$$\nabla\cdot\bar{H}=0,$$
$$\nabla\times\bar{E}=-\frac{1}{c}\frac{\partial\bar{H}}{\partial t}, \qquad (6B)$$
$$\nabla\times\bar{H}=\frac{1}{c}\frac{\partial E}{\partial t}\left(1-\frac{e\rho_0}{\omega^2 m}\right),$$

which are identical with the equations for a medium of uniform dielectric constant

$$(1-e\rho_0/m\omega^2).$$

Appendix C

Taking the curl of the first of Eqs. (1B) and combining with the second yields

$$\nabla\times\nabla\times\bar{E}=-\frac{1}{c^2}\frac{\partial\xi}{\partial t}-\frac{1}{c^2}\frac{\partial^2\bar{E}}{\partial t^2} \qquad (1C)$$

or

$$-\nabla^2\bar{E}+\nabla(\nabla\cdot\bar{E})=-\frac{i\omega}{c^2}\xi+k^2\bar{E} \qquad (2C)$$

and finally,

$$(\nabla^2+k^2)\bar{E}-i\omega\xi/c^2-\nabla(\rho_1)=0, \qquad (3C)$$

the z component of which is

$$(\nabla^2+k^2)E_z-i\omega\xi_z/c^2+i\gamma\rho_1=0. \qquad (4C)$$

In a similar fashion it is found that

$$\nabla\times\nabla\times\bar{H}=\frac{1}{c}\nabla\times\xi-\frac{1}{c^2}\frac{\partial^2\bar{H}}{\partial t^2} \qquad (5C)$$

and finally

$$(\nabla^2+k^2)H_z+(1/c)(\nabla\times\xi)_z=0. \qquad (6C)$$

ξ_z, ρ_1, and $(\nabla\times\xi)_z$ must now be expressed in terms of E_z and H_z. This may be done as follows:

From the force equation (22)

$$im(\omega-\gamma v_0)v_z=eE_z \tag{7C}$$

or

$$v_z=\omega_0^2E_z/i\rho_0\omega_b \tag{8C}$$

and, similarly

$$v_r=\frac{\omega_0^2}{i\rho_0\omega_b}\left[E_r-\frac{v_0}{c}H_\theta\right], \tag{9C}$$

$$v_\theta=\frac{\omega_0^2}{i\rho_0\omega_b}\left[E_\theta+\frac{v_0}{c}H_r\right], \tag{10C}$$

in which the substitutions,

$$\omega_0^2=e\rho_0/m \tag{11C}$$

and

$$\omega_b=\omega-\gamma v_0, \tag{12C}$$

have been introduced. Now

$$\xi_z=\rho_0v_z+v_0\rho_1=\omega_0^2E_z/i\omega_b+v_0\rho_1, \tag{13C}$$

$$\xi_r=\frac{\omega_0^2}{i\omega_b}\left[E_r-\frac{v_0}{c}H_\theta\right], \tag{14C}$$

$$\xi_\theta=\frac{\omega_0^2}{i\omega_b}\left[E_\theta+\frac{v_0}{c}H_r\right]. \tag{15C}$$

And from the continuity equation

$$-i\omega\rho_1=\nabla\cdot\bar{\xi}=\rho_0\nabla\cdot\bar{v}-i\gamma v_0\rho_1 \tag{16C}$$

or

$$\nabla\cdot\bar{v}=-i\omega_b\rho_1/\rho_0. \tag{17C}$$

But from Eqs. (8C), (9C), and (10C),

$$\nabla\cdot\bar{v}=\frac{\omega_0^2}{i\rho_0\omega_b}\times\left\{\nabla\cdot\bar{E}+\frac{v_0}{c}\left[\frac{1}{r}\frac{\partial H_r}{\partial\theta}-\frac{1}{r}\frac{\partial}{\partial r}(rH_\theta)\right]\right\}. \tag{18C}$$

The binomial in the brackets is recognized as the z component of ($-$curl H) and so by the second and third of Eqs. (1B), (18C) becomes

$$\nabla\cdot\bar{v}=\frac{\omega_0^2}{i\rho_0\omega_b}\left[\rho_1-\frac{v_0}{c^2}i\omega E_z-\frac{v_0}{c^2}(\rho_1v_0+\rho_0v_z)\right], \tag{19C}$$

$$=\frac{\omega_0^2}{i\rho_0\omega_b}\left[\rho_1-\frac{v_0}{c^2}i\omega E_z-\frac{v_0}{c^2}\frac{\omega_0^2}{i\omega_b}E_z\right], \tag{20C}$$

$$\rho_1=\frac{iv_0\omega_0^2}{c^2\omega_b}\frac{\omega_0^2-\omega\omega_b}{\omega_b^2-\omega_0^2}E_z. \tag{21C}$$

This result may now be substituted into (13C) to give

$$\xi_z=iE_z\frac{\omega_0^4c^2-v_0^2\omega_0^2\omega\omega_b-c^2\omega_0^2\omega_b^2}{c^2\omega_b(\omega_b^2-\omega_0^2)}, \tag{22C}$$

from which,

$$k_E^2=-i\omega\xi_z/cE_z+i\gamma\rho_1/E_z+k^2, \tag{23C}$$

$$=\frac{\omega^2[c^2\omega_b^2-v_0^2\omega_0^2]+c^2\omega_0^2[\omega_0^2-2\omega\omega_b]}{c^4[\omega_b^2-\omega_0^2]}, \tag{24C}$$

$$(\nabla\times\xi)_z=\frac{1}{r}\frac{\partial}{\partial r}(r\xi_\theta)-\frac{1}{r}\frac{\partial\xi_r}{\partial\theta}=\rho_0(\nabla\times\bar{v})_z, \tag{25C}$$

$$=\frac{\omega_0^2}{i\omega_b}\left[(\nabla\times\bar{E})_z+\frac{v_0}{c}\frac{1}{r}\frac{\partial}{\partial r}(rH_r)+\frac{v_0}{c}\frac{1}{r}\frac{\partial H_\theta}{\partial\theta}\right]. \tag{26C}$$

But

$$(\nabla\times\bar{E})_z=-i\omega H_z/c \tag{27C}$$

and from $\nabla\cdot\bar{H}=0$ it follows that

$$i\gamma H_z=\frac{1}{r}\frac{\partial}{\partial r}(rH_r)+\frac{1}{r}\frac{\partial H_\theta}{\partial\theta}. \tag{28C}$$

Substitution in (26C) results in

$$(\nabla\times\xi)_z=\frac{\omega_0^2}{i\omega_b}\left[-\frac{i\omega}{c}+\frac{iv_0\gamma}{c}\right]H_z=\frac{\omega_0^2}{c}H_z. \tag{29C}$$

Consequently,

$$k_H^2=k^2+\frac{1}{c}(\nabla\times\xi)_z=k^2[1-\omega_0^2/\omega^2]. \tag{30C}$$