STATISTICAL METHODOLOGY FOR OPTIMAL SENSOR LOCATIONS FOR DAMAGE DETECTION IN STRUCTURES

James L. Beck, Eduardo Chan and Costas Papadimitriou

California Institute of Technology
Division of Engineering & Applied Science
104-44 Thomas Lab
Pasadena, CA 91125
USA

ABSTRACT. *A Bayesian statistical methodology is presented for optimally locating the sensors in a structure for the purpose of extracting the most information about the model parameters which can be used in model updating and in damage detection and localization. This statistical approach properly handles the unavoidable uncertainties in the measured data as well as the uncertainties in the mathematical model used to represent the structural behavior. The optimality criterion for the sensor locations is based on information entropy which is a measure of the uncertainty in the model parameters. The uncertainty in these parameters is computed by the Bayesian statistical methodology and then the entropy measure is minimized over the set of possible sensor configurations using a genetic algorithm. Results presented illustrate how both the minimum entropy of the parameters and the optimal sensor configuration depend on the location of sensors, number of sensors, number and type of contributing modes and the structural parameterization (substructuring) scheme used.*

NOMENCLATURE

$\underline{a}$	:	model parameters
$\underline{a}_0$	:	nominal model parameters
$\hat{\underline{a}}$	:	best model parameters
$\underline{\delta}$	:	sensor locations vector
$\underline{e}$	:	measurement/model error
$P^{(l)}(\underline{a})$	:	normalized covariance matrix of $\underline{a}$ given measurement at DOF l only
$\underline{q}$	:	model output
$Q(\underline{a})$	:	normalized covariance matrix of $\underline{a}$
$\underline{y}$	:	observed output
$a(t)$	:	base acceleration
$\delta(t)$	:	delta function
E_a	:	mathematical expectation over $\underline{a}$
H	:	information entropy
$J(\underline{a})$	:	measure of fit
k_0	:	nominal interstory stiffness
k_i	:	ith interstory stiffness
N	:	number of data points
N_d	:	model degrees of freedom
N_0	:	instrumented degrees of freedom
$p(\underline{a})$	:	probability density of $\underline{a}$
σ^2	:	variance
$\mathbb{M}$	:	class of structural models
$\mathbb{D}$	:	measured dynamic data

1. INTRODUCTION

The goal in a structural model updating methodology is to select the model(s) from a parameterized class of models that best fit measured dynamic data according to some criterion. The identified models can then be used for improved structural response predictions or structural damage detection and localization. The quality of the model updating can be judged by the uncertainty in the model parameters and the prediction error. Specifically, the smaller these uncertainties are, the better are the quality of the model updating and the reliability of the response predictions or damage detection, localization and assessment of severity. The difficulties associated with the inverse problem of model updating have been addressed by several investigators (e.g. [1-4]). The quality of the model updating depends on the class of mathematical models chosen, the measurement error in the data, the number of contributing modes, the number and location of sensors, and the excitation and response bandwidth. Recently, a framework based on a Bayesian statistical methodology has been developed [2,3,5,6,7] to effectively tackle these problems, including the modeling of the uncertainties due to modeling and measurement noise, the issue of identifiability, and the problem of reliably computing the response prediction uncertainty.

The problem that will be addressed in this study is related to the quality of the model parameter estimation in relation to the location of the array of sensors used. Specifically, the following issue will be addressed. Given a specified number of sensors, what are the best degrees of freedom (DOF) to instrument in a structure in order to give the smallest uncertainty when identifying the model parameters using structural response. Udwadia [8] has developed a rational approach to the problem based on Fisher's information matrix for the model parameters. He proposed that the sensor locations that maximize some norm of the Fisher information matrix be taken as the optimal locations. In his examples, he chose as a "norm" the trace of the matrix.

In the present approach, a different methodology is proposed based on the information entropy of the uncertain model parameters [9]. The Bayesian framework proposed by Beck and Katafygiotis [2] is extended to the computation of the optimal sensor locations. The uncertainty in these parameters is computed by the Bayesian statistical methodology and then the entropy is minimized over the set of possible sensor configurations using a genetic algorithm. It is shown that the results of the present approach are equivalent to those proposed using the Fisher information matrix [8], provided the determinant of that matrix is maximized and not its trace. Optimal sensor locations are computed for a nine-story shear building and results are compared to the results of the method proposed in [8]. It is demonstrated that the two methods give qualitatively different results.

2. FORMULATION

The derivation of the uncertainty in the parameters $\underline{a}$ of a parameterized class of structural models chosen to represent structural behavior is based on the work by Beck and Katafygiotis [2]. Due to space limitations, only a brief summary of the derivation is presented in the following. Let $\underline{q}(n;\underline{a}) \in \mathbb{R}^{N_d}$ be the model output (e.g. accelerations) at all N_d DOF of the structural model, then the system output, $\underline{y}$, at the observed DOF is:

$$\underline{y}(n) = S_0\underline{q}(n;\underline{a}) + S_0\underline{e}(n;\underline{a}) \tag{1}$$

where the prediction error $\underline{e}$ accounts for the modeling and measurement error. The selection matrix $S_0 \in \mathbb{R}^{N_0 \times N_d}$ has only one non-zero element (unity) in each row and no more than one non-zero element in each column. Thus,

$$S_0 = \sum_{i=1}^{N_0} E_{if(i)} \tag{2}$$

where $\underline{q}_{f(i)}$ is the model response corresponding to system output y_i, i.e., $f(i) \in \{1,\dots,N_d\}$, $\forall i \in \{1,\dots,N_0\}$ and $i \neq k \Rightarrow f(i) \neq f(k)$, and $E_{ik} \in \mathbb{R}^{N_0 \times N_d}$ has all zero elements except for unity for the element in the i-th row and k-th column. Note also that $S_0^T S_0 = \sum_{i=1}^{N_o}\sum_{j=1}^{N_0} E_{if(i)}^T E_{jf(j)} = \sum_{i=1}^{N_0} E_{f(i)i}E_{if(i)} = \Delta$, where $\Delta = diag\,(\delta_1,\dots,\delta_{N_d})$ and $\delta_i = 1$ if DOF i is observed, otherwise $\delta_i = 0$. Thus, only N_0 of the δ_i's are non-zero.

The uncertainty in $\underline{a}$ is described by a probability density function (PDF) which can be obtained using the class of structural models $\mathbb{M}$, the class of probabilistic models for the prediction error $\underline{e}$, and the observed dynamic data $\mathbb{D}$. Assuming a linear class of models, and assuming that the uncertainty in the components of $\underline{e}(n;\underline{a})$ are modeled by an independent Gaussian probability density function (PDF) with mean zero and variance σ^2, the updated PDF for the model parameters $\underline{a}$ for the class of models $\mathbb{M}$ is given by the asymptotic expression for large N:

$$p(\underline{a}|\mathbb{D}_N, \mathbb{M}) \cong \frac{|Q(\hat{\underline{a}})|}{(2\pi\hat{\sigma}^2)^{\frac{1}{2}N_a}} e^{-\frac{1}{2\hat{\sigma}^2}(\underline{a}-\hat{\underline{a}})^T Q(\hat{\underline{a}})(\underline{a}-\hat{\underline{a}})} \quad (3)$$

where $\mathbb{D}_N$ is the data for the first N discrete times, $Q \in \mathbb{R}^{N_a \times N_a}$ is given by equation (see [2]):

$$Q_{ij}(\underline{a}) \cong \sum_{n=1}^{N} \left[\frac{\partial \underline{q}(n;\underline{a})^T}{\partial a_i} \Delta \frac{\partial \underline{q}(n;\underline{a})}{\partial a_j} \right] \quad (4)$$

It is assumed that the choice of the N_0 observed DOF gives a globally identifiable model [3], i.e. $\hat{\underline{a}}$ is the unique global minimum of

$$J(\underline{a}) = \frac{1}{NN_0} \sum_{n=1}^{N} \left\| \underline{y}(n) - S_0 \underline{q}(n;\underline{a}) \right\|^2 \quad (5)$$

and $\hat{\sigma}^2 = J(\hat{\underline{a}})$ is assumed small. Equivalently, (4) can be written in a more convenient form:

$$Q_{ij}(\hat{\underline{a}}) \cong \sum_{l=1}^{N_d} \delta_l P_{ij}^{(l)}(\hat{\underline{a}}) \quad (6)$$

where

$$P_{ij}^{(l)}(\hat{\underline{a}}) = \sum_{n=1}^{N} \left[\frac{\partial q_l(n;\underline{a})}{\partial a_i} \frac{\partial q_l(n;\underline{a})}{\partial a_j} \right] \quad (7)$$

Note that this expression is a discrete version of an analogous result derived by Udwadia [8]. The expression has been derived without using the result that an efficient unbiased estimator satisfies the Cramer-Rao lower bound. Indeed, $\hat{\underline{a}}$ is simply the most probable $\underline{a}$ based on $\mathbb{D}_N$ and $\mathbb{M}$, whereas in Udwadia's result, Q is evaluated at the 'true' value of $\underline{a}$ and $\hat{\underline{a}}$ is the unbiased estimator of $\underline{a}$. Also, note that when we are doing experimental design for choosing the sensor location, $\mathbb{D}_N$ is not known and so $\hat{\underline{a}}$ is uncertain. Note also that $Q(\hat{\underline{a}})$ depends on $\mathbb{D}_N$ only through $\hat{\underline{a}}$.

Suppose the input (excitation) history $\hat{Z}_N$ is prescribed for the test to identify model parameters $\underline{a}$. By the previous assumptions, the uncertainty in the system output history, $\underline{y}_N$, is modeled by a Gaussian PDF with mean $S_0 \underline{q}(n;\underline{a})$ and variance σ_0^2 for each component, where $\underline{a}_0$ and σ_0^2 are the nominal model parameters and prediction error variance which are chosen by the designer to be representative for the structure and the given classes of models.

By the law of large numbers, as $N \to \infty$:

$$J(\underline{a}_0) = \frac{1}{NN_0} \sum_{n=1}^{N} \left\| \underline{y}(n) - S_0 \underline{q}(n;\underline{a}_0) \right\|^2 \quad (8)$$

$$\to \frac{1}{N_0} \mathsf{E} \left[\left\| \underline{y}(n) - S_0 \underline{q}(n;\underline{a}_0) \right\|^2 \right] \quad (9)$$

$$= \frac{1}{N_0} \mathsf{E} \left[\| S_0 \underline{e} \|^2 \right] \quad (10)$$

$$= \frac{1}{N_0} \mathsf{E} \left[\sum_{i=1}^{N_0} \underline{e}_{f(i)}^2 (n;\underline{a}_0) \right] \quad (11)$$

$$= \frac{1}{N_0} \sum_{i=1}^{N_0} \mathsf{E} \left[\underline{e}_{f(i)}^2 \right] \quad (12)$$

$$= \sigma_0^2 \quad (13)$$

Also, since $\hat{\underline{a}}$ is the maximum likelihood estimate (MLE), we know that $\hat{\underline{a}} \to \underline{a}_0$ as $N \to \infty$, conditional on the nominal model, so $\hat{\sigma}^2 = J(\hat{\underline{a}}) \to J(\underline{a}_0) = \sigma_0^2$. Thus for large N:

$$Q_{ij}(\hat{\underline{a}}) = Q_{ij}(\underline{a}_0) \cong \sum_{l=1}^{N_d} \delta_l P_{ij}^{(l)}(\underline{a}_0) \quad (14)$$

When an experimental design is being done, the best values for σ_0^2 and $\underline{a}_0$ are unknown. However, σ_0^2 is a constant scaling factor and so does not affect the optimal sensor locations. For the uncertainty in $\underline{a}_0$, we could explore the sensitivity of Q about the nominal value or we could prescribe a PDF for $\underline{a}_0$ to represent the designers uncertainty in the model parameters and take expectation over $\underline{a}_0$ in the final result. In the latter case, the numerical integration over $\underline{a}$ involved in computing the expectation can be carried out approximately but efficiently using an asymptotic expansion developed to treat these type of integrals [10].

For large N, $p(\underline{a}|\mathbb{D}_N, \mathbb{M})$ is a Gaussian PDF with mean $\hat{\underline{a}} \cong \underline{a}_0$ and covariance matrix $\sigma_0^2 Q(\hat{\underline{a}})^{-1} \cong \sigma_0^2 Q(\underline{a}_0)^{-1}$. We wish to minimize the uncertainty in $\underline{a}$ over the sensor locations, i.e., over the δ_i's, where exactly N_0 of the δ_i's are unity and the rest are zero. As a measure of the uncertainty in $\underline{a}$, we take its (information) entropy [9]:

$$H(\underline{a}|\mathbb{D}_N, \mathbb{M}) = \mathsf{E}_a\left[-\ln p(\underline{a}|\mathbb{D}_N, \mathbb{M})\right] \quad (15)$$

$$= \frac{1}{2}N_a \ln(2\pi) + \frac{1}{2}N_a + N_a \ln \sigma_0^2 - \ln \det Q(\hat{\underline{a}}) \quad (16)$$

The entropy is well-known to be a unique measure of probabilistic uncertainty, as first shown by Shannon [9]. Thus, minimizing the uncertainty in $\underline{a}$ is equivalent to maximizing the determinant of $Q(\hat{\underline{a}}) \cong Q(\underline{a}_0)$. Note that $Q(\underline{a})$ is always a positive semi-definite symmetric matrix and $Q(\hat{\underline{a}})$ is positive definite since $\underline{a}$ is globally identifiable. Let $\lambda_i, i = 1, \ldots, N_a$, be the eigenvalues of $Q(\hat{\underline{a}})$ so that $\lambda_i > 0$, $\forall i$. The optimal locations for N_0 sensors is given by maximizing:

$$\det Q(\hat{\underline{a}}) = \prod_{i=1}^{N_a} \lambda_i \quad (17)$$

or, equivalently, $\ln[\det Q(\hat{\underline{a}})] = \sum_{i=1}^{N_a} \ln \lambda_i$ over $\underline{\delta} = [\delta_1, \ldots, \delta_{N_d}]^T$.

Udwadia [8] maximized the trace $\mathrm{tr}Q(\underline{a}_0) = \sum_{i=1}^{N_a} \lambda_i$. The choice of maximizing the trace, instead of the determinant or any other measure of the Fisher information matrix, was justified due to its computational ease and the efficiency with which the maximization can be carried out. The choice of maximizing $\det(Q)$ is justified in the present formulation as giving the smallest amount of uncertainty in the parameters of the structure. It will be demonstrated that the use of the trace in place of the determinant results in sensor configurations which are qualitatively different from the optimal sensor configuration obtained by maximizing $\det(Q)$.

3. OPTIMIZATION

It is straightforward to ascertain that there are $\binom{N_d}{N_0} = \frac{N_d!}{N_0!(N_d - N_0)!}$ discrete values for the sensor location vector, $\underline{\delta}$. For a sufficiently large number of model degrees of freedom N_d, an exhaustive search over all possible values of $\underline{\delta}$ may be computationally expensive or even prohibitive. Instead, genetic algorithms can be used that are well-suited for this type of discrete optimization problem [11,12]. A simple genetic algorithm is used in this work to perform the optimization of the objective function.

4. APPLICATION

The methodology is applied to a nine-story uniform shear building represented by a mass-spring model as shown schematically in Figure 1. The stiffnesses and masses of the nominal sructure are chosen to be equal with $k_0/m_0 = 1450$ for each floor so that the fundamental frequency is 1Hz. Classical normal modes are assumed with the modal damping fixed at 5% for all modes. All results correspond to an impulse base acceleration of unit magnitude, that is, the base acceleration $a(t) = \delta(t)$, where $\delta(t)$ is the delta function. It should be noted that the impulse base excitation is chosen to focus on the optimal sensor location for recording seismic response produced by a broadband earthquake excitation.

To study the effects of structural parameterization on the optimal sensor location, results are presented for the following three cases, designated by Case A, Case B, and Case C. In Case A, the uncertainty in the stiffness is assumed to be fully correlated for all stories, that is, only one parameter a is considered with $k_i = ak_0$, $i = 1, \ldots, 9$. In Case B, only three uncertain parameters are considered by dividing the structure into three substructures with $k_1 = k_2 = k_3 = a_1 k_0$, $k_4 = k_5 = k_6 = a_2 k_0$ and $k_7 = k_8 = k_9 = a_3 k_0$. In Case C, nine uncertain parameters are considered, one for each story stiffness, so that $k_i = a_i k_0$, $i = 1, \ldots, 9$.

Although, in principle, the impulse excitation will excite all modes of the structure, parametric studies as a function of the number of modes used are justified since in real applications the information from higher modes will be lost due to the larger measurement noise-to-signal ratio for higher modes, which is primarily due to low energy of the earthquake motion at the frequency range corresponding to the higher modes.

In case A, the optimal location of N_0 sensors are found to be at the N_0 highest floors of the uniform shear building. The optimal sensor locations for

Case B and C are given in Tables 1 and 3, respectively. Values of the sensor locations are reported only for the cases for which the problem is identifiable. The non-identifiability can easily be predicted by observing the condition number of the matrix Q. Specifically, one eigenvalue of Q equals zero for a non-identifiable structure. Usually, due to numerical errors, the eigenvalues are all different from zero and the non-identifiability is predicted by the ratio $|\lambda_{max}/\lambda_{min}|$ which is expected to be very large for non-identifiable or ill-conditioned cases.

Tables 2 and 4 show the variation of the uncertainty in the parameter $\underline{a}$ as a function of the number of sensors placed at the optimal locations for Cases B and C, respectively. Results are given for different number of modes. For any given number of modes, the value of $-\det(Q)$, and thus the uncertainty in the prediction of the value of $\underline{a}$, is seen to reduce as additional sensors are placed in the structure. Increasing the number of sensors extracts more information from the data, which is reflected by the lower values of $-\det(Q)$.

The results from the entropy approach are compared to those obtained by maximizing the trace of the Fisher information matrix [8]. For case A, the results obtained by maximizing the $\det(Q)$ or the $\mathrm{tr}(Q)$ are identical since Q is a scalar in this case. However, in almost all the results obtained for cases B and C, the optimal sensor location measures $\det(Q)$ and $\mathrm{tr}(Q)$ give qualitatively different results. Specifically, $\mathrm{tr}(Q)$ predicts that for cases B and C, all of the N_0 sensors should be placed at the highest N_0 floors of the shear building, independently of the number of modes and the parameterization used. However, the locations of the sensors predicted by $\det(Q)$ is qualitatively different. As an example, consider finding the optimal locations of four sensors for Cases B and C. The results for Case B shown in Table 1 indicate that three of the sensors should be placed on floors 2 to 6, with the exact locations depending on the number of modes contributing significantly to the response, and one sensor should be placed at the ninth floor. Similarly, the results for Case C in Table 3 indicate that for five modes or more, three sensors should be placed at the lowest three floors and one sensor should be placed at the highest floor. Also, comparing the optimal sensor location results for both cases B and C (Tables 1 and 3), it is concluded that the optimal location of the four sensors also depends on the structural parameterization scheme employed.

From the results in Tables 1 and 3, one can conclude in general that for a very small number of sensors, specifically one sensor for case B and up to two sensors for case C, the optimal locations are generally on the lower floors, while for a larger number of sensors, the optimal locations consist of a combination of lower and higher floors. The number of modes has some effect on the sensor location problem, but it is not strong.

Additional studies on the ninth story building demonstrate that the optimal sensor configuration also depends on the the type and location of the excitation. For example, it depends on whether the excitation is broad-band ambient, impulsive or forced harmonic, and on the location of the excitation, such as whether it is base excitation or excitation at other degrees of freedom.

5. CONCLUSIONS

The optimal sensor locations are chosen to correspond to the minimum information entropy of the uncertainty of the model parameters. The uncertainty in these parameters is computed using a Bayesian statistical methodology. A genetic algorithm is especially suited for solving the resulting discrete optimization problem over all possible sets of sensor configurations. The optimal sensor configuration depends on the number of contributing modes, the parameterization scheme employed, and the type and location of excitation (for example, force excitation applied at specific degrees of freedom, base excitation due to earthquakes, or wind-excited ambient excitation). The optimal sensor configuration can be used in conjunction with a system identification technique to provide significantly improved and more reliable estimates of the identified model parameters from test data. In damage detection applications, the optimal sensor locations predicted by the methodology are expected to provide significantly improved estimates of the severity and location of damage.

6. REFERENCES

[1] Mottershead, J.E. and Friswell, M.I. Model Updating in Structural Dynamics: A Survey. *Journal of Sound and Vibration*, 167(2), 347-375, 1993.

[2] Beck, J.L. and Katafygiotis, L.S. Updating Structural Dynamic Models and their Uncertainties - Statistical System Identification. *Journal of Engineering Mechanics, ASCE*, 1998 (in press).

[3] Katafygiotis, L.S. and Beck, J.L. Updating Structural Dynamic Models and their Uncertainties - Model Identifiability. *Journal of Engineering Mechanics, ASCE*, 1998 (in press).

[4] Natke, H.G. and Yao, J.T.P., Eds. *Structural Safety Evaluation Based on System Identification Approaches.* Proc. of Workshop at Lambrecht/Pfatz, Frieder Vieweg and Sohn, Braunschweig, Germany, 1988.

[5] Beck, J.L. Statistical System Identification of Structures. In *Proc. 5th Int. Conf. on Structural Safety and Reliability, ASCE, II*, pp. 1395-1402, 1989.

[6] Beck, J.L., Vanik, M.W. and Katafygiotis, L.S. Determination of Stiffness Changes from Modal Parameter Changes for Structural Health Monitoring. In *Proc. First World Conference on Structural Control*, Pasadena, CA, 1994.

[7] Vanik, M.W. and Beck, J.L. A Bayesian Probabilistic Approach to Structural Health Monitoring. *Proc. 16th International Modal Analysis Conference*, Santa Barbara, CA, 1998.

[8] Udwadia, F.E. Methodology for Optimal Sensor Locations for Parameter Identification in Dynamic Systems. *Journal of Engineering Mechanics, ASCE*, 120(2), 368-390, 1994.

[9] Jaynes, E.T. Where Do we Stand on Maximum Entropy? In Levine, R.D., and Tribus, M. (Eds.), *The maximum Entropy Formalism.* MIT Press, Cambridge, MA, 1978.

[10] Papadimitriou, C., Beck, J.L and Katafygiotis L.S. Asymptotic Expansions for Reliability and Moments of Uncertain Systems. *Journal of Engineering Mechanics, ASCE*, 123, December 1997.

[11] Goldberg, D.E. *Genetic Algorithms in Search, Optimization, and Machine Learning.* Reading, Addison Wesley, MA, 1989.

[12] Chan, E. *Optimal Design of Building Structures Using Genetic Algorithms.* EERL Report No. 97-06, Caltech, Pasadena, California, 1997.

Table 1: Optimal sensor locations for Case B

	Number of Sensors							
modes	1	2	3	4	5	6	7	8
1		3 6	3 6 9	3 5 6 9	3 4 6 8 9	3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
2	4	3 9	3 4 9	3 4 8 9	2 3 4 5 9	2 3 4 5 8 9	2 3 4 5 7 8 9	2 3 4 5 6 7 8 9
3	9	3 9	3 4 9	3 4 5 9	2 3 4 8 9	2 3 4 5 8 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
4	3	3 9	3 4 9	3 4 6 9	3 4 5 6 9	2 3 4 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
5	3	3 9	3 4 9	3 4 5 9	3 4 5 6 9	2 3 4 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
6	3	3 9	3 4 9	2 3 6 9	2 3 4 6 9	2 3 4 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
7	3	3 9	3 6 9	2 3 6 9	2 3 5 6 9	2 3 4 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
8	3	3 9	3 6 9	3 4 6 9	2 3 4 6 9	2 3 4 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
9	3	3 9	2 3 9	2 3 6 9	2 3 5 6 9	2 3 4 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9

Table 2: Information Entropy measure -ln[det(Q)] for Case B

	Number of Sensors								
modes	1	2	3	4	5	6	7	8	9
1	10.4	-21.3	-22.6	-23.2	-23.7	-24.1	-24.4	-24.6	-24.7
2	-26.5	-28.7	-29.9	-30.6	-31.0	-31.4	-31.7	-31.9	-32.0
3	-27.2	-29.5	-30.5	-31.2	-31.8	-32.3	-32.6	-32.9	-33.1
4	-27.4	-30.0	-30.9	-31.6	-32.1	-32.6	-33.0	-33.3	-33.5
5	-27.6	-30.2	-31.2	-31.9	-32.5	-32.9	-33.3	-33.6	-33.8
6	-27.8	-30.3	-31.3	-32.1	-32.6	-33.1	-33.5	-33.9	-34.1
7	-28.3	-30.7	-31.7	-32.5	-33.1	-33.5	-33.9	-34.2	-34.4
8	-28.3	-30.7	-31.7	-32.5	-33.0	-33.5	-33.9	-34.3	-34.5
9	-28.3	-30.7	-31.7	-32.5	-33.1	-33.5	-33.9	-34.2	-34.5

Table 3: Optimal sensor locations for Case C

	Number of Sensors							
modes	1	2	3	4	5	6	7	8
1								1 2 3 4 5 6 7 8
2						1 2 5 6 7 8	1 2 4 5 6 7 8	1 2 4 5 6 7 8 9
3				1 4 5 8	1 2 4 7 8	1 2 3 4 7 8	1 2 3 4 7 8 9	1 2 3 4 5 7 8 9
4		1 9	2 3 9	1 2 3 8	1 2 3 8 9	1 2 3 4 8 9	1 2 3 4 5 8 9	1 2 3 4 5 7 8 9
5	1	1 3	1 2 9	1 2 3 9	1 2 3 4 9	1 2 3 4 7 9	1 2 3 4 5 7 9	1 2 3 4 5 6 7 9
6	1	1 3	1 2 3	1 2 3 9	1 2 3 4 9	1 2 3 4 7 9	1 2 3 4 6 7 9	1 2 3 4 5 6 7 9
7	1	1 3	1 2 3	1 2 3 9	1 2 3 4 9	1 2 3 4 6 9	1 2 3 4 5 6 9	1 2 3 4 5 6 7 9
8	1	1 3	1 2 3	1 2 3 9	1 2 3 4 9	1 2 3 4 6 9	1 2 3 4 5 6 9	1 2 3 4 5 6 7 9
9	1	1 3	1 2 9	1 2 3 9	1 2 3 4 9	1 2 3 4 6 9	1 2 3 4 5 6 9	1 2 3 4 5 6 7 9

Table 4: Information Entropy measure -ln[det(Q)] for Case C

	Number of Sensors								
modes	1	2	3	4	5	6	7	8	9
1	172	139	107	77.2	46.7	-18.1	-9.62	-37.8	-39.4
2	105	44.6	-11.0	-42.8	-49.5	-54.4	-56.7	-59.2	-59.4
3	39.1	-43.1	-57.1	-62.5	-65.1	-66.9	-68.5	-69.7	-70.6
4	-19.6	-62.2	-68.2	-70.7	-72.8	-74.4	-75.8	-76.8	-77.8
5	-59.6	-68.5	-72.7	-75.4	-77.3	-78.7	-79.9	-81.0	-81.7
6	-60.9	-70.0	-74.3	-77.0	-78.8	-80.2	-81.4	-82.4	-83.2
7	-62.2	-71.5	-75.3	-78.1	-79.9	-81.4	-82.5	-83.5	-84.4
8	-63.7	-72.2	-75.9	-78.6	-80.4	-81.8	-83.0	-84.0	-84.7
9	-64.4	-72.3	-76.2	-78.8	-80.5	-81.8	-83.0	-83.9	-84.6