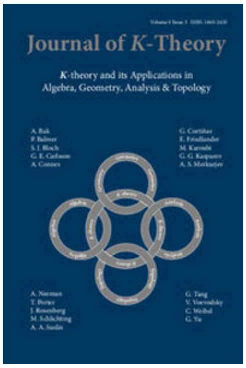

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Comparison of Karoubi's regulator and the p-adic Borel regulator

by

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Abstract

In this paper we prove the p-adic analogue of a result of Hamida [11], namely that the p-adic Borel regulator introduced by Huber and Kings for the K-theory of a p-adic number field equals Karoubi's p-adic regulator up to an explicit rational factor.

Key Words: p-adic regulator, relative Chern character, Karoubi's regulator, local field, Lazard isomorphism

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Contents

Introduction

In a series of papers [16, 17, 18] Karoubi introduced topological and relative K-groups for Banach algebras A and constructed a relative Chern character

$$\mathrm{ch}_n^{\mathrm{rel}} : K_n^{\mathrm{rel}}(A) \to HC_{n-1}^{\mathrm{top}}(A)$$

mapping the relative K-theory to continuous cyclic homology. His construction is quite explicit in nature and works equally well in the classical case of Banach algebras over $\mathbf{R}$ or $\mathbf{C}$ as well as for ultrametric Banach algebras. He also pointed out that one could use his relative Chern character for the construction of regulators.

Regulators are intimately connected with special values of L-functions through conjectures of Beilinson, Bloch-Kato, Perrin-Riou, etc., and a deep understanding of regulators is an important step towards a proof of these (see e.g. [1, 4, 5]).

For the field of complex numbers Hamida [12] showed that one can reconstruct Borel's regulator [3]

$$K_{2n-1}(\mathbf{C}) \to \mathbf{C}$$

from the relative Chern character for $\mathbf{C}$ considered as a Banach algebra and thereby got explicit representatives for the Borel regulator in group cohomology.

More generally, one can prove a comparison result for a suitably generalized version of Karoubi's relative Chern character and the Beilinson regulator for smooth varieties over $\mathbf{Q}$ [25]. Thus one might hope that the relative Chern character gives a unifying frame for the study of both, classical and p-adic, regulators. Since the p-adic analogue of Beilinson's regulator is the rigid syntomic regulator [2] there should also be a comparison result for Karoubi's relative Chern character and the syntomic regulator for smooth varieties over a p-adic field. So far this is not known in general and in this paper we treat the case of a finite extension of $\mathbf{Q}_p$.

Let K be such an extension. In [14] Huber and Kings introduce the so-called p-adic Borel regulator $b_p \colon K_{2n-1}(K) \to K$ and relate it with the syntomic regulator, hence by a result of Nizioł via Bloch-Kato's exponential map also with Soulé's étale regulator. The definition of b_p parallels Borel's construction of his regulator $K_{2n-1}(\mathbf{C}) \to \mathbf{C}$, only that the van Est isomorphism is replaced by the Lazard isomorphism $H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K) \simeq H^{2n-1}(\mathfrak{gl}_N, K)$ between locally analytic group cohomology and Lie algebra cohomology. The p-adic regulator is then induced from the "same" primitive element in Lie algebra cohomology that shows up in the construction of the classical Borel regulator.

Our main result is:

Theorem *Fix $n > 1$. Under the identification $K^{\mathrm{rel}}_{2n-1}(K)_{\mathbf{Q}} \cong K_{2n-1}(K)_{\mathbf{Q}}$ Karoubi's regulator equals $\frac{(-1)^{n-1}}{(n-1)!(2n-2)!} b_p$.*

Note that the constant factor appearing in the comparison is a rational number. This is remarkable since the constructions of both regulators are of a transcendental nature.

To prove the Theorem we use Hamida's explicit description of Karoubi's regulator [13]. We show that Hamida's cocycle is in fact locally analytic and compute its image in Lie algebra cohomology under the Lazard map using the formula for the latter obtained by Huber and Kings [14].

In a forthcoming paper we will generalize the relative Chern character to all smooth separated schemes of finite type over the ring of integers in K and obtain explicit formulas for the cocycles in a slightly different manner (cf. [25, Part II]).

The present paper had its origin in a remark of Karoubi that the p-adic Borel regulator should be directly related to his regulator which led to the preprint [24]. In the meantime Hamida's cocycle was independently studied by Choo and Snaith [7] with the aim of deriving a formula for the regulator well suited for computer computations. Whereas they also obtain the continuity of Hamida's cocycle, for our comparison we need the stronger result of local analyticity.

The paper is organised as follows: The first section recalls the relevant definitions and facts about relative K-theory and Karoubi's construction as well as Hamida's results. In the second section we recall the construction of the p-adic Borel regulator and compute the image of Hamida's cocycle under the Lazard isomorphism. The comparison of the regulators is deduced from this in Section 3.

Since the explicit cocycles involve the integration of a p-adic differential form over the standard simplex, we have included an appendix where the technical questions of integration are discussed. The reader is encouraged to compare this with the method of Choo and Snaith [7] based on nice explicit computations of integrals.

The results presented here are part of my Ph.D. thesis at the Universität Regensburg [25]. I would like to thank Guido Kings for suggesting this topic to me and for several helpful discussions about the Lazard isomorphism. Furthermore I would like to thank Amnon Besser for his interest in my work and some hints concerning the syntomic regulator. I am thankful to the referee for several remarks improving the exposition of the article.

Notation. For $n \in \mathbf{N}_0 = \{0,1,2,\dots\}$ we denote by $[n]$ the finite set $\{0,\dots,n\}$ with its natural order. The category of finite ordered sets with monotone maps as morphisms is the *simplicial category*.

If G is a discrete group we denote by BG the simplicial set $[n] \mapsto B_nG = G^{\times n}$ with faces and degeneracies as in [20, B.12]. If G_* is a simplicial group BG_* denotes the diagonal of the bisimplicial set $([n],[m]) \mapsto B_nG_m$. By abuse of notation we denote the geometric realization by the same symbol and call it the *classifying* *space* of the (simplicial) group.

1. Karoubi's p-adic regulator

1.1. Relative K-theory

Topological K-theory of ultrametric Banach rings was originally introduced by Karoubi and Villamayor in [19] and further studied by Calvo in [6]. In this section we recall the definitions of topological and relative K-theory in the form given in [16].

Let A be an ultrametric Banach ring (cf. [6]), i.e. a ring A equipped with an ultrametric"quasi-norm" $\|.\| : A \to \mathbf{R}_+$ verifying $\|a\| = 0 \iff a = 0$, $\|a\| = \|-a\|$, $\|ab\| \leq \|a\|\|b\|$, $\|a+b\| \leq \max\{\|a\|,\|b\|\}$, for which it is complete. Let $A\langle x_0,\dots,x_n\rangle$ denote the ring of power series $\sum_{I\in\mathbf{N}_0^{n+1}} a_I x^I$ with $a_I \in A$, $x = (x_0,\dots,x_n)$, and $\|a_I\||I|^r \xrightarrow{|I|\to\infty} 0$ for every $r \in \mathbf{N}_0$. $A\langle x_0,\dots,x_n\rangle$ is an ultrametric Fréchet ring where the topology is given by the family of seminorms p_r, $r \in \mathbf{N}_0$, $p_r(\sum a_I x^I) = \sup_I \|a_I\| \cdot |I|^r$.

Let $I_n \subset A\langle x_0,\dots,x_n\rangle$ be the principal ideal generated by $x_0 + \cdots + x_n - 1$ and define $A_n := A\langle x_0,\dots,x_n\rangle / I_n$. A_* defines a simplicial ring with faces ∂_i and degeneracies s_i induced by

$$\partial_i(x_j) = \begin{cases} x_j & \text{if } j < i, \\ 0 & \text{if } j = i, \\ x_{j-1} & \text{if } j > i, \end{cases} \qquad s_i(x_j) = \begin{cases} x_j & \text{if } j < i, \\ x_i + x_{i+1} & \text{if } j = i, \\ x_{j+1} & \text{if } j > i \end{cases}$$

(see the Appendix for details).

By [17, 6.17] the natural map of classifying spaces $\mathrm{BGL}(A) \to \mathrm{BGL}(A_*)$ induces a homotopy fibre sequence

$$(\mathrm{GL}(A_*)/\mathrm{GL}(A))^+ \xrightarrow{\theta} \mathrm{BGL}(A)^+ \to \mathrm{BGL}(A_*) \tag{1.1}$$

where $(.)^+$ is Quillen's +-construction with respect to the maximal perfect normal subgroup of the fundamental group and θ is induced by the map of simplicial sets $\mathrm{GL}(A_n) \ni \sigma \mapsto (\sigma(0)\sigma(1)^{-1},\dots,\sigma(n-1)\sigma(n)^{-1}) \in B_n\mathrm{GL}(A) = \mathrm{GL}(A)^{\times n}$. Here $\sigma(i) = (i)^*\sigma$ where $(i) : [0] \to [n]$ is the morphism in the simplicial category that sends $0 \in [0]$ to $i \in [n]$ ("the value of σ on the i^{th} vertex of the standard simplex").

For $n \geq 1$, the relative, algebraic, and topological K-groups of A are by definition (cf. [16, 18])

$$\begin{aligned} K_n^{\mathrm{rel}}(A) &:= \pi_n\big((\mathrm{GL}(A_*)/\mathrm{GL}(A))^+\big), \\ K_n(A) &:= \pi_n(\mathrm{BGL}(A)^+), \text{ and} \\ K_n^{\mathrm{top}}(A) &:= \pi_n(\mathrm{BGL}(A_*)). \end{aligned}$$

In particular, (1.1) gives a long exact sequence

$$\cdots \to K_n^{\mathrm{rel}}(A) \to K_n(A) \to K_n^{\mathrm{top}}(A) \to K_{n-1}^{\mathrm{rel}}(A) \to \ldots.$$

Remark As was mentioned by Karoubi [18, 5.7] Calvo's methods [6] show that one can impose different convergence conditions on the power series appearing in the definition of the topological K-groups. Detailed arguments in the case of overconvergent power series are provided in [25, Section 7.1].

We are particularly interested in the case where $A = K$ is a finite extension of $\mathbf{Q}_p$ with residue field k. In this situation Calvo shows that one has exact sequences [6, 3.3]

$$0 \to K_n(k) \to K_n^{\mathrm{top}}(K) \to K_{n-1}(k) \to 0.$$

By Quillen's computation [21] the algebraic K-groups $K_n(k)$ are finite for $n \geq 1$, hence the topological K-groups $K_n^{\mathrm{top}}(K)$ are finite for $n \geq 2$. In particular the canonical map $K_n^{\mathrm{rel}}(K)_{\mathbf{Q}} \to K_n(K)_{\mathbf{Q}}$ is an isomorphism for $n \geq 2$ (here we write $K_n^{\mathrm{rel}}(K)_{\mathbf{Q}}$ for $K_n^{\mathrm{rel}}(K) \otimes_{\mathbf{Z}} \mathbf{Q}$ etc.). Furthermore, if R denotes the ring of integers in K there are canonical isomorphisms $K_n^{\mathrm{top}}(R) \xrightarrow{\simeq} K_n^{\mathrm{top}}(k) \simeq K_n(k)$ [6, 2.1, 1.2]. The localization sequence in algebraic K-theory yields isomorphisms $K_n(R)_{\mathbf{Q}} \to K_n(K)_{\mathbf{Q}}$ for $n \geq 2$. Putting everything together we have canonical isomorphisms ($n \geq 2$)

$$\begin{tikzcd} K_n^{\mathrm{rel}}(R)_{\mathbf{Q}} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & K_n(R)_{\mathbf{Q}} \arrow[d, "\simeq"] \\ K_n^{\mathrm{rel}}(K)_{\mathbf{Q}} \arrow[r, "\simeq"] & K_n(K)_{\mathbf{Q}}. \end{tikzcd} \tag{1.2}$$

1.2. The regulator

The construction of the p-adic regulator is due to Karoubi. It uses the relative Chern character constructed by Karoubi [16, 17] and Connes-Karoubi [8].

Let K be a finite extension of $\mathbf{Q}_p$ with ring of integers R and uniformizing parameter π. We denote the continuous cyclic homology of K with ground ring $\mathbf{Q}$ by $HC_*^{\mathrm{top}}(K)$. Since we will not need the precise definition we do not recall it here (see e.g. [16]). Karoubi's relative Chern character then is a homomorphism $\mathrm{ch}_{2n-1}^{\mathrm{rel}} \colon K_{2n-1}^{\mathrm{rel}}(K) \to HC_{2n-2}^{\mathrm{top}}(K)$ (cf. [8, Théorème 3.6]).

Definition 1.1 The p-adic regulator r_p is defined to be the composition

$$r_p\colon K_{2n-1}^{\mathrm{rel}}(K) \xrightarrow{\mathrm{ch}_{2n-1}^{\mathrm{rel}}} HC_{2n-2}^{\mathrm{top}}(K) \xrightarrow{S} HC_0^{\mathrm{top}}(K) = K,$$

where S is the $(n-1)$-fold iterate of Connes' periodicity operator.

In order to compare the above regulator with the p-adic Borel regulator we need the explicit description of r_p given by Hamida which uses Goodwillie's relative K-theory. We recall the relevant definitions and facts from [20, Ch. 11]. For a ring A and a two-sided ideal I in A denote by $\mathcal{K}(A,I)$ the connected component of the basepoint of the homotopy fiber of $\mathrm{BGL}(A)^+ \to \mathrm{BGL}(A/I)^+$. For $n \geq 1$, $K_n(A,I)$ is defined to be $\pi_n(\mathcal{K}(A,I))$. The space $\mathcal{K}(A,I)$ has a Volodin model $X(A,I)$ constructed as follows: For any ordering γ of $\{1,\dots,n\}$ define $T_n^\gamma(A,I)$ to be the subgroup $\{1+(a_{ij}) \in \mathrm{GL}_n(A) \mid a_{ij} \in I \text{ if } i \overset{\gamma}{\not<} j\}$ of $\mathrm{GL}_n(A)$. Then $X(A,I)$ is the union of classifying spaces $\bigcup_{n,\gamma} BT_n^\gamma(A,I)$ in $\mathrm{BGL}(A)$. We consider $X(A,I)$ also as a simplicial subset of the simplicial set $\mathrm{BGL}(A)$.

Proposition 1.2 ([20, Prop. 11.3.6, Cor. 11.3.8]) *There is a natural homotopy equivalence $X(A,I)^+ \approx \mathcal{K}(A,I)$. In particular $K_n(A,I) = \pi_n(X(A,I)^+)$. Moreover the direct sum of matrices induces an H-space structure on $X(A,I)^+$ so that $K_n(A,I)_{\mathbf{Q}}$ is isomorphic to the primitive part* $\mathrm{Prim}H_n(X(A,I),\mathbf{Q})$ *of the rational homology of $X(A,I)$ via the Hurewicz homomorphism.*

Now let $A = R$ and $I = \pi R$ the maximal ideal of R. Hamida proves the following

Proposition 1.3 ([13, Thm. 1.3]) *There exists an isomorphism*

$$K_n(R,\pi R) \xrightarrow{\simeq} K_n^{\mathrm{rel}}(R).$$

It is induced by the map of simplicial sets

$$\varphi : X(R,\pi R) \to \mathrm{GL}(R_*)/\mathrm{GL}(R) \tag{1.3}$$

that sends $(g_1,\dots,g_r) \in T^\gamma(R,\pi R)^{\times r} \subset X_r(R,\pi R)$ to $\sum_{i=0}^r x_i g_{i+1}\cdots g_r \in \mathrm{GL}(R_r)$.

It is not a priori clear that $\sum_{i=0}^r x_i g_{i+1}\cdots g_r$ is invertible in $\mathrm{Mat}(R_r)$. For a proof of a similar statement see Lemma 2.4. Now the explicit description of the regulator r_p is the following

Proposition 1.4 ([13, Prop. 2.1.3]) *The composition*

$$K_{2n-1}(R,\pi R) \underset{\simeq}{\xrightarrow{\varphi}} K_{2n-1}^{\mathrm{rel}}(R) \to K_{2n-1}^{\mathrm{rel}}(K) \xrightarrow{r_p} K$$

is equal to the composition

$$K_{2n-1}(R,\pi R) \to H_{2n-1}(X(R,\pi R),\mathbf{Z}) \xrightarrow{\phi} K,$$

where the first arrow is the Hurewicz map and ϕ is given by the simplicial cocycle that sends $(g_1,\dots,g_{2n-1}) \in T^\gamma(R,\pi R)^{\times(2n-1)}$ to

$$\frac{(-1)^n(n-1)!}{(2n-1)!(2n-2)!}\operatorname{Tr}\int_{\Delta^{2n-1}}(d\nu\cdot\nu^{-1})^{2n-1} \in K \tag{1.4}$$

with $\nu = \sum_{i=0}^{2n-1} x_i g_{i+1}\cdots g_{2n-1} \in \mathrm{GL}(R_{2n-1})$.

See the Appendix for the definition of the integral.

2. An explicit description of the p-adic Borel regulator

2.1. The construction of the p-adic Borel regulator

We recall the construction of the p-adic Borel regulator and the explicit description of the Lazard isomorphism, both due to Huber and Kings [14].

As before, K denotes a finite extension of $\mathbf{Q}_p$ with ring of integers R and uniformizer π. Define $U_N(R) = 1+\pi\mathrm{Mat}_N(R) \subset \mathrm{GL}_N(R)$. Denote by $\mathfrak{gl}_N$ the K-Lie algebra of $\mathrm{GL}_N(R)$ viewed as a locally K-analytic Lie group and by $\mathcal{O}^{\mathrm{la}}(X)$ the ring of locally analytic functions on a locally K-analytic manifold X. We denote by $H^*_{\mathrm{la}}(\mathrm{GL}_N(R),K)$ the *locally analytic group cohomology* defined as the cohomology of the complex associated with the cosimplicial K-vector space $[p] \mapsto \mathcal{O}^{\mathrm{la}}(B_p\mathrm{GL}_N(R)) = \mathcal{O}^{\mathrm{la}}(\mathrm{GL}_N(R)^{\times p})$. Recall that the Lie algebra cohomology $H^*(\mathfrak{gl}_N,K)$ is the cohomology of the complex $\bigwedge^*\mathfrak{gl}_N^\vee$ with differential induced by the Lie bracket (see e.g. [26, Corollary 7.7.3]) where $\mathfrak{gl}_N^\vee$ denotes the K-dual of $\mathfrak{gl}_N$.

Huber and Kings prove the following version of Lazard's Theorem:

Theorem 2.1 (Lazard, Huber–Kings) *There are isomorphisms*

$$H^k_{\mathrm{la}}(\mathrm{GL}_N(R),K) \xrightarrow{\cong} H^k_{\mathrm{la}}(U_N(R),K) \xrightarrow{\cong} H^k(\mathfrak{gl}_N,K).$$

On the level of cochains the map to Lie algebra cohomology is induced by the map

$$\Phi : \mathcal{O}^{\mathrm{la}}(\mathrm{GL}_N(R)^{\times k}) \to \bigwedge^k \mathfrak{gl}_N^\vee,$$

which is given on topological generators by $f_1\otimes\cdots\otimes f_k \mapsto df_1(1)\wedge\cdots\wedge df_k(1)$, where $df(1)$ is the differential of f at the unit element $1 \in \mathrm{GL}_N(R)$.

This is proven in [14, Theorems 1.2.1 and 4.7.1], [15, Theorem 4.3.2].

Definition 2.2 ([14, Definitions 0.4.5 and 1.2.3]) For $n \leq N$ the (primitive) element $p_n = p_{n,N} \in H^{2n-1}(\mathfrak{gl}_N, K)$ is the class represented by the cocycle

$$X_1 \wedge \cdots \wedge X_{2n-1} \mapsto \frac{((n-1)!)^2}{(2n-1)!} \sum_{\sigma \in \mathfrak{S}_{2n-1}} \operatorname{sgn}(\sigma) \operatorname{Tr}(X_{\sigma(1)} \cdots X_{\sigma(2n-1)}).$$

Here $\mathfrak{S}_{2n-1}$ denotes the symmetric group on $2n-1$ elements. Define $b_{n,N} \in H^{2n-1}(\mathrm{GL}_N(R), K)$ to be the image of $p_{n,N}$ under the composition

$$H^{2n-1}(\mathfrak{gl}_N, K) \xleftarrow{\cong} H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K) \to H^{2n-1}(\mathrm{GL}_N(R), K),$$

where the right hand map is the canonical map from locally analytic to discrete group cohomology. Obviously, the $b_{n,N}$ are compatible for different N.

The *p-adic Borel regulator* is the composition

$$b_p \colon K_{2n-1}(R) \xrightarrow{\text{Hur.}} H_{2n-1}(\mathrm{GL}(R), \mathbf{Q}) = \varinjlim_N H_{2n-1}(\mathrm{GL}_N(R), \mathbf{Q}) \xrightarrow{\varprojlim_N b_{n,N}} K.$$

In order to compare Hamida's cocycle with the primitive element p_n above via the Lazard isomorphism we need the following general formula for the map Φ in Theorem 2.1, valid not just for functions of the form $f_1 \otimes \cdots \otimes f_k$. It is inspired by Guichardet-Wigner's formula for the van Est isomorphism [10, Appendice].

Consider $\mathrm{GL}_N(R)$ as a K-Lie group and let exp be the exponential map of $\mathrm{GL}_N(R)$ defined on a neighbourhood of zero in $\mathfrak{gl}_N$. For a locally analytic function $f \in \mathcal{O}^{\mathrm{la}}(\mathrm{GL}_N(R)^{\times k})$ we define $\Delta f \in \bigwedge^k \mathfrak{gl}_N^\vee$ by

$$\Delta f(X_1, \ldots, X_k) := \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) \frac{d^k}{dt_1 \ldots dt_k} f(\exp(t_1 X_{\sigma 1}), \ldots, \exp(t_k X_{\sigma k}))\Big|_{t_1 = \cdots = t_n = 0}.$$

Since $\exp(0) = 1$ and $d\exp(0) = \mathrm{id}_{\mathfrak{gl}_N}$ one has for f of the special form $f = f_1 \otimes \cdots \otimes f_k$

$$\frac{d}{dt_i} f(\exp(t_1 X_{\sigma 1}), \ldots, \exp(t_k X_{\sigma k}))\Big|_{t_i = 0} = \\ = f_1(\exp(t_1 X_{\sigma 1})) \cdots df_i(1)(X_{\sigma i}) \cdots f_k(\exp(t_k X_{\sigma k}))$$

and therefore

$$\begin{aligned} \Delta f(X_1, \ldots, X_k) &= \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) df_1(1)(X_{\sigma 1}) \cdots df_k(1)(X_{\sigma k}) \\ &= df_1(1) \wedge \cdots \wedge df_k(1)(X_1, \ldots, X_k) = \Phi(f)(X_1, \ldots, X_k). \end{aligned}$$

The vector space $\mathcal{O}^{\mathrm{la}}(\mathrm{GL}_N(R)^k)$ carries a natural locally convex topology [23, §12]. Using Proposition 12.4 of *loc. cit.* it is easy to see that both, Φ and Δ, are continuous for this topology. Since moreover the functions of the form $f_1 \otimes \cdots \otimes f_k$ are topological generators of $\mathcal{O}^{\mathrm{la}}(\mathrm{GL}_N(R)^{\times k})$, we get:

Corollary 2.3 *The Lazard isomorphism $H^k_{\mathrm{la}}(\mathrm{GL}_N(R), K) \xrightarrow{\simeq} H^k(\mathfrak{gl}_N, K)$ is induced by $\Delta : \mathcal{O}^{\mathrm{la}}(\mathrm{GL}_N(R)^{\times k}) \to \bigwedge^k \mathfrak{gl}_N^\vee$.*

The same description applies for $U_N(R)$ instead of $\mathrm{GL}_N(R)$.

2.2. *An explicit cocycle*

Denote by $\|.\|$ the norm on the field K, normalized such that $\|p\| = p^{-1}$. We extend the norm to $\mathrm{Mat}_N(K)$ by $\|(A_{ij})_{i,j=1}^N\| = \sup_{i,j} \|A_{ij}\|$. This norm is submultiplicative, i.e. for matrices $A, B \in \mathrm{Mat}_N(K)$ we have $\|A \cdot B\| \le \|A\| \cdot \|B\|$.

Recall the ring $R_n = R\langle x_0, \ldots, x_n\rangle / (\sum x_i - 1)$ from Section 1.1.

Lemma 2.4 *Let $g_1, \ldots, g_n$ be elements of $U_N(R)$. Then $\nu = \sum_{i=0}^n x_i g_{i+1} \cdots g_n$ is invertible, i.e. lies in $\mathrm{GL}_N(R_n)$.*

Proof: Write $g_{i+1} \cdots g_n = 1 - h_i$ with $h_i \in \pi \mathrm{Mat}_N(R)$. Then $\nu = \sum_{i=0}^n x_i - \sum_{i=0}^n x_i h_i = 1 - \sum_{i=0}^n x_i h_i =: 1 - h$. We show that $\sum_{k \in \mathbf{N}} h^k$ converges in $\mathrm{Mat}_N(R\langle x_0, \ldots, x_n\rangle)$. Its image in $\mathrm{Mat}_N(R_n)$ will be an inverse of ν. Let $p_r, r \in \mathbf{N}_0$, be the family of seminorms defining the Fréchet topology on $R\langle x_0, \ldots, x_n\rangle$ (cf. Sections 1.1 and A.1) and extend p_r to matrices by taking the supremum of the seminorms of the coefficients. It follows from Proposition A.1 that $\mathrm{Mat}_N(R\langle x_0, \ldots, x_n\rangle)$ is complete for the topology defined by this family of seminorms and hence it suffices to show that $(h^k)_{k \in \mathbf{N}}$ is a zero sequence.

Now h^k is homogeneous of degree k and for any multiindex $J = (j_0, \ldots, j_n)$ with $|J| = k$ the coefficient of x^J in h^k is a sum of products of $h_0, \ldots, h_n$ where each h_i appears exactly j_i times. From this and the submultiplicativity of $\|.\|$ it follows that

$$p_r(h^k) \le k^r \max_{|J|=k} \|h_0\|^{j_0} \cdots \|h_n\|^{j_n} \le k^r c^k$$

where $c := \max_{i=0,\ldots,n} \|h_i\| < 1$ and J runs through multiindices in $\mathbf{N}_0^{n+1}$. But $k^r c^k$ tends to zero as k tends to infinity and so $(h^k)_{k \in \mathbf{N}}$ is a zero sequence as wanted. □

Recall from Definition 2.2 that the p-adic Borel regulator is induced by the compatible system of the $b_{n,N} \in H^{2n-1}(\mathrm{GL}_N(R), K), N \ge n$, which comes from a compatible system in locally analytic group cohomology (namely from the inverse image of the $p_{n,N}, N \ge n$, under the Lazard isomorphism). By Theorem 2.1 $H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K) \simeq H^{2n-1}_{\mathrm{la}}(U_N(R), K)$ and hence the p-adic Borel regulator is determined by a compatible system of locally analytic cocycles $U_N(R)^{\times(2n-1)} \to K, N \geq n$.

Theorem 2.5 *The p-adic Borel regulator b_p is given by the compatible system of locally analytic cocycles $f_N : U_N(R)^{\times(2n-1)} \to K, N \geq n$,*

$$f_N(g_1,\dots,g_{2n-1}) = -\frac{((n-1)!)^2}{(2n-1)!}\operatorname{Tr}\int_{\Delta^{2n-1}}(d\nu\cdot\nu^{-1})^{2n-1}$$

where $\nu = \nu(g_1,\dots,g_{2n-1}) = \sum_{i=0}^{2n-1} x_i g_{i+1}\cdots g_{2n-1} \in \mathrm{GL}_N(R_{2n-1})$.

Proof: The compatibility of the f_N for different N is easy to check and we drop the index N in the following. The fact that f is locally analytic is proven in the Appendix (Proposition A.9). For the proof that f indeed defines a cocycle cf. [12, Prop. II 3.3.1].

We have to show that f is mapped to the primitive element p_n of Definition 2.2 under the Lazard isomorphism, i.e. $\Delta(f) = p_n$. To simplify notation write $\partial_i := \frac{d}{dt_i}$ and $\underline{t} := (t_1,\dots,t_{2n-1})$. We have

$$\Delta(f)(X_1,\dots,X_{2n-1}) = -\frac{((n-1)!)^2}{(2n-1)!}\sum_{\sigma\in\mathfrak{S}_{2n-1}}\operatorname{sgn}(\sigma)\partial_1\dots\partial_{2n-1}\Big|_{\underline{t}=0}$$
$$\operatorname{Tr}\int_{\Delta^{2n-1}}(d\nu\cdot\nu^{-1})^{2n-1}(\exp(t_1X_{\sigma1}),\dots,\exp(t_{2n-1}X_{\sigma(2n-1)})).$$

By Proposition A.8 we may interchange differentiation and integration. Let us first consider the $\sigma = 1$ summand. Write

$$\omega := \sum_{i=0}^{2n-1} dx_i \exp(t_{i+1}X_{i+1})\cdots\exp(t_{2n-1}X_{2n-1}),$$
$$\omega' := \sum_{i=0}^{2n-1} x_i \exp(t_{i+1}X_{i+1})\cdots\exp(t_{2n-1}X_{2n-1}),$$

and $\eta := \omega\cdot\omega'^{-1}$. Then

$$(d\nu\cdot\nu^{-1})^{2n-1}(\exp(t_1X_1),\dots,\exp(t_{2n-1}X_{2n-1})) = (\omega\cdot\omega'^{-1})^{2n-1} = \eta^{2n-1}.$$

We calculate $\partial_1\dots\partial_{2n-1}\eta^{2n-1} = \partial_1\dots\partial_{2n-2}\sum_{k=1}^{2n-1}\eta^{k-1}(\partial_{2n-1}\eta)\eta^{2n-1-k} = \dots$ using the Leibniz rule repeatedly. Since $\omega|_{\underline{t}=0} = \sum_{i=0}^{2n-1} dx_i = 0$, hence $\eta|_{\underline{t}=0} = 0$, the only terms surviving after setting $\underline{t} = 0$ are those with exactly one ∂ in front of each η:

$$\partial_1\dots\partial_{2n-1}\eta^{2n-1}\Big|_{\underline{t}=0} = \sum_{\tau\in\mathfrak{S}_{2n-1}}\partial_{\tau(1)}\eta\cdots\partial_{\tau(2n-1)}\eta\Big|_{\underline{t}=0}.$$

On the other hand, using $\omega'\big|_{\underline{t}=0} = \sum_{i=0}^{2n-1} x_i = 1$ we get

$$\partial_j \eta\big|_{\underline{t}=0} = \partial_j \left(\omega\omega'^{-1}\right)\big|_{\underline{t}=0} = (\partial_j\omega)\omega'^{-1}\big|_{\underline{t}=0} + \omega(\partial_j\omega'^{-1})\big|_{\underline{t}=0} = \\ = (\partial_j\omega)\big|_{\underline{t}=0} = \sum_{i=0}^{j-1} dx_i \cdot X_j .$$

Altogether we obtain

$$\begin{aligned}
&\partial_1 \ldots \partial_{2n-1}\eta^{2n-1}\Big|_{\underline{t}=0} = \\
&\quad = \sum_{\tau\in\mathfrak{S}_{2n-1}} \left(\sum_{i=0}^{\tau(1)-1} dx_i \cdot X_{\tau(1)}\right)\cdots\left(\sum_{i=0}^{\tau(2n-1)-1} dx_i \cdot X_{\tau(2n-1)}\right) \\
&\quad = \sum_{\tau\in\mathfrak{S}_{2n-1}} X_{\tau(1)}\cdots X_{\tau(2n-1)} \left(\sum_{i=0}^{\tau(1)-1} dx_i\right)\cdots\left(\sum_{i=0}^{\tau(2n-1)-1} dx_i\right) \\
&\quad = \sum_{\tau\in\mathfrak{S}_{2n-1}} \operatorname{sgn}(\tau) X_{\tau(1)}\cdots X_{\tau(2n-1)} dx_0 dx_1 \ldots dx_{2n-2},
\end{aligned}$$

where the last equality can be seen as follows: The sum $dx_0 + \cdots + dx_j$ appears as the $\tau^{-1}(j+1)^{\text{th}}$ factor in the product $\left(\sum_{i=0}^{\tau(1)-1} dx_i\right)\cdots\left(\sum_{i=0}^{\tau(2n-1)-1} dx_i\right)$. Hence this product is equal to $\operatorname{sgn}(\tau^{-1}) dx_0(dx_0 + dx_1)\cdots(dx_0 + \cdots + dx_{2n-2}) = \operatorname{sgn}(\tau) dx_0 dx_1 \cdots dx_{2n-2}$. It follows that

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_{2n-1}} \operatorname{sgn}(\sigma)\partial_1 \ldots \partial_{2n-1}(d\nu\cdot\nu^{-1})^{2n-1}(\exp(t_1 X_{\sigma 1}),\ldots,\exp(t_{2n-1}X_{\sigma(2n-1)}))\Big|_{\underline{t}=0} \\
&\quad = \sum_{\sigma\in\mathfrak{S}_{2n-1}} \operatorname{sgn}(\sigma) \sum_{\tau\in\mathfrak{S}_{2n-1}} \operatorname{sgn}(\tau) X_{\sigma\tau(1)}\cdots X_{\sigma\tau(2n-1)} dx_0 dx_1 \ldots dx_{2n-2} \\
&\quad = (2n-1)! \sum_{\sigma\in\mathfrak{S}_{2n-1}} \operatorname{sgn}(\sigma) X_{\sigma(1)}\cdots X_{\sigma(2n-1)} dx_0 dx_1 \ldots dx_{2n-2}.
\end{aligned}$$

Since $dx_0 \ldots dx_{2n-2} = -dx_{2n-1} dx_1 \ldots dx_{2n-2} = -dx_1 \ldots dx_{2n-1}$ and hence

$$\int_{\Delta^{2n-1}} dx_0 \ldots dx_{2n-2} = -\int_{\Delta^{2n-1}} dx_1 \ldots dx_{2n-1} = -\frac{1}{(2n-1)!}$$

(see e.g. [9, Ch. 1, Ex. 1] or [7, §6.2]) we finally obtain

$$\Delta(f)(X_1,\ldots,X_{2n-1}) = \frac{((n-1)!)^2}{(2n-1)!} \sum_{\sigma\in\mathfrak{S}_{2n-1}} \operatorname{sgn}(\sigma)\operatorname{Tr}(X_{\sigma 1}\cdots X_{\sigma(2n-1)}),$$

that is $\Delta(f) = p_n$. □

Example For $n = 1$ one can compute that $b_p\colon K_1(R) = R^\times \to K$ is just the p-adic logarithm.

3. Comparison of the two regulators

Theorem 3.1 *For $n > 1$, the diagram*

$$\begin{tikzcd} K^{\mathrm{rel}}_{2n-1}(K)_{\mathbf{Q}} \arrow[r, "\simeq"] \arrow[dr, "r_p"'] & K_{2n-1}(K)_{\mathbf{Q}} & K_{2n-1}(R)_{\mathbf{Q}} \arrow[l, "\simeq"'] \arrow[dl, "\frac{(-1)^{n-1}}{(n-1)!(2n-2)!}b_p"] \\ & K & \end{tikzcd}$$

is commutative.

Remark The p-adic Borel regulator is similar to the classical Borel regulator defined in odd degrees only. The relative Chern character that is used in the construction of r_p is also defined in positive even degrees, $\mathrm{ch}^{\mathrm{rel}}_{2n}\colon K^{\mathrm{rel}}_{2n}(K) \to HC^{\mathrm{top}}_{2n-1}(K)$, but here the target group vanishes. Since one is eventually interested in the composition $K_*(F)_{\mathbf{Q}} \to K_*(K)_{\mathbf{Q}} \to K$ where K is the completion of a number field F at a finite place, this is conform with the vanishing of the rational K-theory of number fields in positive even degrees [3].

Proof of the Theorem: We have a commutative diagram

$$\begin{tikzcd} & K^{\mathrm{rel}}_{2n-1}(K)_{\mathbf{Q}} \arrow[r, "\simeq"] \arrow[dr, phantom, "\text{cf. (1.2)}"] & K_{2n-1}(K)_{\mathbf{Q}} \\ K_{2n-1}(R,\pi R)_{\mathbf{Q}} \arrow[r, "\simeq", "\text{via } \varphi(1.3)"'] \arrow[d, "\simeq", "\text{cf. Prop. 1.2}"'] & K^{\mathrm{rel}}_{2n-1}(R)_{\mathbf{Q}} \arrow[u, "\simeq"] \arrow[r, "\simeq", "\text{via } \theta(1.1)"'] & K_{2n-1}(R)_{\mathbf{Q}} \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\ \mathrm{Prim}\, H_{2n-1}(X(R,\pi R),\mathbf{Q}) \arrow[rr, "\beta"] & & \mathrm{Prim}\, H_{2n-1}(\mathrm{BGL}(R),\mathbf{Q}) \end{tikzcd}$$

where β is induced by the composition

$$X(R,\pi R) \xrightarrow[(1.3)]{\varphi} \mathrm{GL}(R_*)/\mathrm{GL}(R) \xrightarrow[(1.1)]{\theta} \mathrm{BGL}(R)$$

which is just the natural inclusion $X(R,\pi R) \subset \mathrm{BGL}(R)$ as one easily checks. We also denote by β the induced map on homology. To prove the Theorem it suffices by Proposition 1.4 to show that

$$\begin{tikzcd} H_{2n-1}(X(R,\pi R),\mathbf{Q}) \arrow[rr, "\beta"] \arrow[dr, "\phi(1.4)"'] & & H_{2n-1}(\mathrm{BGL}(R),\mathbf{Q}) \arrow[dl, "\frac{(-1)^{n-1}}{(n-1)!(2n-2)!}b_p"] \\ & K & \end{tikzcd}$$

commutes.

It follows from the long exact sequence of Goodwillie's relative K-theory and the finiteness of $K_i(R/\pi R)$ for $i > 0$ (due to Quillen [21]) that $K_i(R,\pi R)_{\mathbf{Q}} \to K_i(R)_{\mathbf{Q}}$ is an isomorphism for all $i > 0$. Thus we know that $X(R,\pi R)^+ \to \mathrm{BGL}(R)^+$ induces an isomorphism on the subspaces of primitive elements in rational homology (both spaces are connected). Since this is obviously a map of H-spaces the Theorem of Cartier-Milnor-Moore (see e.g. [20, A.10 – A.11]) implies that it induces in fact an isomorphism $H_*(X(R,\pi R),\mathbf{Q}) \underset{\simeq}{\xrightarrow{\beta}} H_*(\mathrm{BGL}(R),\mathbf{Q})$.

Since for each N the subgroup $U_N(R)$ has finite index in $\mathrm{GL}_N(R)$ it follows that $H_{2n-1}(BU(R),\mathbf{Q}) \to H_{2n-1}(\mathrm{BGL}(R),\mathbf{Q})$ is surjective where $U(R) = \varinjlim U_N(R)$. Next $BU(R)$ is actually contained in $X(R,\pi R)$ and thus we have a commutative diagram

$$\begin{tikzcd}
H_{2n-1}(BU(R),\mathbf{Q}) \arrow[d, "\alpha"'] \arrow[dr, "\gamma"] & \\
H_{2n-1}(X(R,\pi R),\mathbf{Q}) \arrow[r, "\beta", "\simeq"'] & H_{2n-1}(\mathrm{BGL}(R),\mathbf{Q})
\end{tikzcd}$$

with γ and hence α surjective. Now ϕ is given by the cocycle (1.4) and $b_p \circ \gamma$ is given by the compatible system of cocycles f of Theorem 2.5. From the explicit formulae for ϕ and f it is clear that $\phi \circ \alpha = \frac{(-1)^{n-1}}{(n-1)!(2n-2)!} f \circ \gamma$ which proves the Theorem. □

A. Integration on the standard simplex

A.1. The ring $A\langle x_0,\ldots,x_n\rangle$

Let $(A,\|.\|)$ be an ultrametric Banach ring. For simplicity write $A\langle x\rangle$ for $A\langle x_0,\ldots,x_n\rangle$ (cf. Section 1.1). Recall the family of seminorms p_r, $r \in \mathbf{N}_0$, $p_r(\sum a_I x^I) = \sup_I \|a_I\| \cdot |I|^r$. We also write $\|.\|_r$ instead of p_r.

Proposition A.1 *$A\langle x\rangle$ is a sub-A-algebra of the algebra of formal power series with coefficients in A. Its underlying module is an ultrametric Fréchet module. Furthermore*

$$\|f \cdot g\|_r \le \sum_{s=0}^{r} \binom{r}{s} \|f\|_s \|g\|_{r-s}.$$

Proof: Let $f = \sum_I a_I x^I$ and $g = \sum b_I x^I$ be in $A\langle x\rangle$. We want to show that $f \cdot g = \sum_I c_I x^I$ with $c_I = \sum_{K+L=I} a_K b_L$ also lies in $A\langle x\rangle$. Fix a non negative integer r. For any $I \in \mathbf{N}_0^{n+1}$ we have

$$\begin{aligned} |I|^r \|c_I\| &\le \max_{K+L=I} ((|K|+|L|)^r \|a_K\| \|b_L\|) \\ &= \max_{K+L=I} \left(\sum_{s=0}^{r} \binom{r}{s} |K|^s \|a_K\| |L|^{r-s} \|b_L\| \right) \qquad (A.1) \\ &\le \sum_{s=0}^{r} \binom{r}{s} \|f\|_s \|g\|_{r-s}. \qquad (A.2) \end{aligned}$$

Set $A := \sup_K(\sum_{s=0}^r \binom{r}{s}|K|^s\|a_K\|)$, $B := \sup_L(\sum_{s=0}^r \binom{r}{s}|L|^{r-s}\|b_L\|)$. Given $\varepsilon > 0$ choose $N > 0$ such that $|K|^s\|a_K\| < \frac{\varepsilon}{B}$, $|L|^{r-s}\|b_L\| < \frac{\varepsilon}{A}$ for all $s = 0,\dots,r$ and $|K|,|L| \ge N$. Then for every $|I| \ge 2N$ with $I = K + L$ we have $|K| \ge N$ or $|L| \ge N$. In the first case

$$\sum_{s=0}^{r} \binom{r}{s} |K|^s \|a_K\| |L|^{r-s} \|b_L\| < \sum_{s=0}^{r} \binom{r}{s} \frac{\varepsilon}{B} \cdot |L|^{r-s} \|b_L\| \le \varepsilon$$

by the definition of B. The case $|L| \ge N$ is symmetric and altogether we obtain from (A.1) that $|I|^r\|c_I\| < \varepsilon$ for all $|I| \ge 2N$. Thus $f \cdot g$ in fact belongs to $A\langle x\rangle$.

The assertion on $\|f \cdot g\|_r$ follows immediately from (A.2).

It remains to show that $A\langle x\rangle$ is complete. Recall that a sequence $(f_n)_{n\in\mathbf{N}}$ in $A\langle x\rangle$ is a Cauchy sequence (resp. converges to $f \in A\langle x\rangle$) for the topology defined by the family of seminorms $\|.\|_r, r \in \mathbf{N}_0$, if it is a Cauchy sequence (resp. converges to f) with respect to each $\|.\|_r$ (cf. [22, Remark 7.1]). So let $(f_n = \sum_I a_I^{(n)} x^I)_{n\in\mathbf{N}}$ be a Cauchy sequence. In particular it is a Cauchy sequence for $\|.\|_0$ and hence is $(a_I^{(n)})_{n\in\mathbf{N}}$ for each $I \in \mathbf{N}_0^{n+1}$. By the completeness of A this last sequence converges to some $a_I \in A$ and it is not hard to check that $f := \sum_I a_I x^I$ belongs to $A\langle x\rangle$ and that $(f_n)_{n\in\mathbf{N}}$ indeed converges to f. □

Now let $\phi : [n] \to [m]$ be a monotone map (a morphism in the simplicial category). We want to define $\phi^* : A\langle x_0,\dots,x_m\rangle \to A\langle x_0,\dots,x_n\rangle$ by $x_i \mapsto \sum_{\phi(j)=i} x_j$, $i = 0,\dots,m$. We have to show that ϕ^* is well defined and continuous. Slightly more generally we have:

Lemma A.2 *Let $g = (g_0,\dots,g_m)$ be a tuple of polynomials of degree 1 in the variables $y_0,\dots,y_n$ with integral coefficients. Then for $f = \sum_I a_I x^I \in A\langle x_0,\dots,x_m\rangle$ the formal composition $f \circ g = \sum_I a_I g^I$ lies in $A\langle y_0,\dots,y_n\rangle$ and $f \mapsto f \circ g$ is continuous.*

Proof: g^I is a polynomial of degree $\le |I|$ with integral coefficients and thus $\|g^I\|_r \le \|1\| \cdot |I|^r$. Since $\|a_I\||I|^r$ tends to zero when $|I|$ tends to infinity it follows that $(\sum_{|I|\leq n} a_I g^I)_{n\in\mathbf{N}}$ is a Cauchy sequence. Since $A\langle y_0,\ldots,y_n\rangle$ is complete it converges and the limit is $f\circ g$.

Moreover $\|f\circ g\|_r = \|\sum_I a_I g^I\|_r \leq \sup_I \|a_I g^I\|_r \leq \sup_I \|a_I\|\cdot\|1\|\cdot|I|^r = \|1\|\cdot\|f\|_r$ and thus $f\mapsto f\circ g$ is continuous. □

Recall that $I_n\subset A\langle x_0,\ldots,x_n\rangle$ is the principal ideal generated by $x_0+\cdots+x_n-1$ and $A_n := A\langle x_0,\ldots,x_n\rangle/I_n$.

Lemma A.3 *The homomorphism $\eta: A\langle x_0,\ldots,x_n\rangle\to A\langle t_1,\ldots,t_n\rangle$ that sends x_i to t_i for $i>0$ and x_0 to $1-t_1-\cdots-t_n$ induces a topological isomorphism $A_n\to A\langle t_1,\ldots,t_n\rangle$.*

Proof: By the previous lemma η is well defined and continuous. We have the obvious continuous section $\iota: t_i\mapsto x_i$, $i>0$, so that η is surjective. Assume $f=\sum_I a_I x^I$ is in the kernel of η. Then

$$\begin{aligned} f &= f-\iota(\eta(f)) = \\ &= \sum_I a_I\left(x^I-(1-x_1-\cdots-x_n)^{i_0}x_1^{i_1}\cdots x_n^{i_n}\right)\\ &= \sum_I a_I\left(x_0^{i_0}-(1-x_1-\cdots-x_n)^{i_0}\right)x_1^{i_1}\cdots x_n^{i_n}\\ &= \sum_I a_I\cdot g_I\cdot(x_0+\cdots+x_n-1)\end{aligned}$$

where the g_I are polynomials with integral coefficients of total degree $\leq |I|$. In particular $\|g_I\|_r\leq\|1\|\cdot|I|^r$ and thus $\sum_I a_I g^I$ is an element of $A\langle x_0,\ldots,x_n\rangle$ which satisfies $(\sum_I a_I g^I)\cdot(x_0+\cdots+x_n-1)=f$. □

If $f\in I_m$ then clearly $\phi^*(f)\in I_n$ and thus there are induced continuous homomorphisms $\phi^*: A_m\to A_n$ for every $\phi:[n]\to[m]$ which make $[n]\mapsto A_n$ a simplicial Fréchet ring.

A.2. Integration of differential forms

Fix a non archimedean field $(K,|.|)$ of characteristic 0. We want to define the integral of an n-form with values in K over the standard simplex Δ^n as it appears in Proposition 1.4.

With the above notation set $\Omega^0(\Delta^n)=K_n$, $\Omega^1(\Delta^n)=(\bigoplus_{i=0}^n K_n dx_i)/(\sum_{i=0}^n dx_i)$ and $\Omega^r(\Delta^n)=\bigwedge_{K_n}^r\Omega^1(\Delta^n)$ with the usual differential $d:\Omega^i(\Delta^n)\to\Omega^{i+1}(\Delta^n)$.

Since $\Omega^n(\Delta^n)=K_n dx_1\ldots dx_n$ every n-form ω can be written uniquely as $\omega=f dx_1\ldots dx_n$ with $f\in K_n$. We also denote by f the image $\sum_I a_I x^I$ of f

in $K\langle x_1,\ldots,x_n\rangle$. We want to define

$$\int_{\Delta^n}\omega := \int_{\Delta^n} f dx_1\ldots dx_n := \sum_I a_I \int_{\Delta^n} x^I dx_1\ldots dx_n \in K,$$

where the integral on the right hand side is the usual integral of the n-form $x^I dx_1\ldots dx_n$ over the real standard simplex $\Delta^n = \{(x_0,\ldots,x_n) \in \mathbf{R}^{n+1} | \sum_i x_i = 1, \forall i : 0 \le x_i \le 1\} \subset \mathbf{R}^{n+1}$ with orientation given by $dx_1\ldots dx_n$, and using that this last integral is in fact a rational number.[1]

Proposition A.4 *The above integral is well defined and $\omega \mapsto \int_{\Delta^n}\omega$ gives a continuous homomorphism $\Omega^n(\Delta^n) \to K$.*

Proof: We have topological isomorphisms $\Omega^n(\Delta^n) = K_n dx_1\ldots dx_n \xrightarrow{\simeq} K_n \xrightarrow{\simeq} K\langle x_1,\ldots,x_n\rangle$ and we have to show that $K\langle x_1,\ldots,x_n\rangle \ni f \mapsto \int_{\Delta^n} f dx_1\ldots dx_n \in K$ is well defined and continuous.

The restriction of the norm $|.|$ from K to $\mathbf{Q}$ is equivalent to either the trivial norm or a p-adic norm for some prime p. In both cases there exists a constant $s \in \mathbf{N}$ such that $|\frac{1}{k}| \le k^s$ for all $k \in \mathbf{N}$. We claim that for any polynomial $f \in \mathbf{Z}[x_1,\ldots,x_n]$ of total degree $\le D$ viewed as function $\Delta^n \to \mathbf{R}$ we have

$$\left|\int_{\Delta^n} f dx_1\ldots dx_n\right| \le (n+D)^{sn}. \tag{A.3}$$

Indeed (A.3) is true for $n = 0$ since in this case the integral is just an integer, hence its norm is ≤ 1 which is the value of the right hand side. Now let $n \ge 1$ and assume (A.3) proven for $n-1$ and all $f \in \mathbf{Z}[x_1,\ldots,x_{n-1}]$. For the monomial $x_1^{i_1}\cdots x_n^{i_n}$ of total degree $D = \sum_j i_j$ we have

$$x_1^{i_1}\cdots x_n^{i_n} dx_1\ldots dx_n = \frac{(-1)^{n-1}}{i_n+1} d\left(x_1^{i_1}\cdots x_{n-1}^{i_{n-1}} x_n^{i_n+1} dx_1\ldots dx_{n-1}\right).$$

For $j = 0,\ldots,n$ set $\delta^j : \Delta^{n-1} \to \Delta^n, (x_0,\ldots,x_{n-1}) \mapsto (x_0,\ldots,x_{j-1},0,x_j,\ldots,x_{n-1})$. Then by Stokes' Theorem [9, Ch. 1, Ex. 2] we have

$$\int_{\Delta^n} x_1^{i_1}\cdots x_n^{i_n} dx_1\ldots dx_n = \\ \frac{(-1)^{n-1}}{i_n+1}\sum_{j=0}^{n}(-1)^j \int_{\Delta^{n-1}} (\delta^j)^*\left(x_1^{i_1}\cdots x_{n-1}^{i_{n-1}} x_n^{i_n+1} dx_1\ldots dx_{n-1}\right).$$

[1] This can be seen by an inductive argument as in the proof of the following proposition or else by an explicit computation (e.g. [7, §6.2]).

Now $(\delta^j)^*\left(x_1^{i_1}\cdots x_{n-1}^{i_{n-1}}x_n^{i_n+1}dx_1\ldots dx_{n-1}\right)$ can be written as $fdx_1\ldots dx_{n-1}$ where f is a polynomial in $\mathbf{Z}[x_1,\ldots,x_{n-1}]$ of total degree $\leq i_1+\cdots+i_{n-1}+i_n+1 = D+1$. By the inductive hypothesis and the strict triangle inequality we thus get

$$\left|\int_{\Delta^n} x_1^{i_1}\cdots x_n^{i_n}dx_1\ldots dx_n\right| \leq \left|\frac{(-1)^{n-1}}{i_n+1}\right|\cdot((n-1)+(D+1))^{s(n-1)}$$
$$\leq (i_n+1)^s(n+D)^{s(n-1)} \leq (n+D)^{sn}.$$

This and the strict triangle inequality again implie (A.3) for any $f \in \mathbf{Z}[x_1,\ldots,x_n]$.

For simplicity we write $\int_{\Delta^n} f$ for $\int_{\Delta^n} fdx_1\ldots dx_n$ in the following. For a monomial x^I of total degree $|I| \geq 1$ (A.3) implies $\left|\int x^I\right| \leq (n+|I|)^{sn} \leq C|I|^{sn}$ with $C = (n+1)^{sn}$.

Now, for $f = \sum_I a_I x^I \in K\langle x_1,\ldots,x_n\rangle$, $|a_I|\cdot|I|^{sn}$ tends to zero when $|I|$ tends to infinity and thus $\int_{\Delta^n} f = \sum_I a_I \int_{\Delta^n} x^I$ converges in K. Furthermore $|\int_{\Delta^n} f| \leq \sup_I |a_I \int_{\Delta^n} x^I| \leq \max\{C\cdot\sup_I|a_I|\cdot|I|^{sn}, |\int_{\Delta^n} 1|\cdot|a_0|)\} \leq (\text{const})\cdot \max\{\|f\|_{sn},\|f\|_0\}$. It follows that $f \mapsto \int_{\Delta^n} f$ is continuous. □

A.3. Dependence on parameters

The goal of this section is to prove the local analyticity of the cocycle f in Theorem 2.5 and to justify the computations carried out in the proof of that Theorem.

Fix K as before. For any K-Banach space A and $\varepsilon > 0$ we denote by $\mathcal{F}_\varepsilon(K^r, A)$ the Banach space of ε-convergent power series in r variables, i.e. power series $a = \sum_J \alpha_J y^J$, $\alpha_J \in A$, $y = (y_1,\ldots,y_r)$, $J \in \mathbf{N}_0^r$ such that $\lim_{|J|\to\infty}\|\alpha_J\|\varepsilon^{|J|} = 0$, with norm $\|f\|_\varepsilon = \sup_J \|\alpha_J\|\varepsilon^{|J|}$. If A is a Banach algebra this becomes a Banach algebra in a natural way (see e.g. [23, Proposition 5.2]).

Definition A.5 Let M be a locally analytic r-dimensional K-manifold and $f : M \to K\langle x_0,\ldots,x_n\rangle$ a function. We say that f is *locally analytic* if for every $u \in M$ there exists an $\varepsilon > 0$ and a chart $M \supset V \xrightarrow{\psi} B_\varepsilon(0) = \{|.| \leq \varepsilon\} \subset K^r$ with $\psi(u) = 0$ such that $f \circ \psi^{-1}$ is given by $\sum_I a_I x^I$ where the a_I are in $\mathcal{F}_\varepsilon(K^r, K)$ and satisfy $\|a_I\|_\varepsilon\cdot|I|^t \to 0$ as $|I| \to \infty$ for every $t \in \mathbf{N}_0$. In other words, $\sum_I a_I x^I \in \mathcal{F}_\varepsilon(K^r,K)\langle x_0,\ldots,x_n\rangle$.

Remark There is a natural continuous injection of $K\langle x_0,\ldots,x_n\rangle$ in the Banach space $\mathcal{F}_1(K^{n+1},K)$. If $f : M \to K\langle x_0,\ldots,x_n\rangle$ is locally analytic then the induced map $M \to \mathcal{F}_1(K^{n+1},K)$ is locally analytic in the ordinary sense but not vice versa.

Proposition A.6 *(i) Let $f,g : M \to K\langle x_0,\ldots,x_n\rangle$ be locally analytic. Then also $f+g$ and $f\cdot g$ are locally analytic.*

(ii) If $\varphi : M' \to M$ is locally analytic then $f \circ \varphi : M' \to K\langle x_0,\dots,x_n\rangle$ is locally analytic.

Proof: (ii) Let $u' \in M'$, $u := \varphi(u')$ and choose a chart $u \in V \xrightarrow{\psi} B_\varepsilon(0) \subset K^r$ with $\psi(u) = 0$ such that $f \circ \psi^{-1}$ is of the form $\sum_I a_I x^I$ with $\|a_I\|_\varepsilon \cdot |I|^t \to 0$ as $|I| \to \infty$ for every $t \in \mathbf{N}_0$. Choose a chart $u' \in V' \xrightarrow{\psi'} B_{\varepsilon'}(0) \subset K^{r'}$ for M' with $\psi'(u') = 0$ such that $\varphi(V') \subset V$. After possibly shrinking ε' we may assume that the induced map $\tilde\varphi \colon B_{\varepsilon'}(0) \to B_\varepsilon(0)$ is given by a ε'-convergent power series with coefficients in K^r. Since $\tilde\varphi(0) = 0$ it is of the form $\sum_{|J|>0} \alpha_J y^J, \alpha_J \in K^r$. For $\varepsilon'' \le \varepsilon'$ we then have $\|\tilde\varphi\|_{\varepsilon''} = \sup_{|J|>0} |\alpha_J|(\varepsilon')^{|J|} \cdot (\frac{\varepsilon''}{\varepsilon'})^{|J|} \le \frac{\varepsilon''}{\varepsilon'} \cdot \sup_{|J|>0} |\alpha_J|(\varepsilon')^{|J|} = \frac{\varepsilon''}{\varepsilon'} \|\tilde\varphi\|_{\varepsilon'}$. Thus we may even assume that $\|\tilde\varphi\|_{\varepsilon'} \le \varepsilon$. But then $a_I \circ \tilde\varphi$ is a well defined power series in $\mathcal{F}_{\varepsilon'}(K^{r'}, K)$ with $\|a_I \circ \tilde\varphi\|_{\varepsilon'} \le \|a_I\|_\varepsilon$ [23, Proposition 5.4]. Now $f \circ \varphi \circ \psi'^{-1} = f \circ \psi^{-1} \circ \tilde\varphi = \sum_I (a_I \circ \tilde\varphi) x^I$ and the claim follows.

Essentially the same argument shows that if the condition for the local analyticity of f is satisfied at $u \in M$ for one chart ψ with $\psi(u) = 0$ and $\psi' \colon V' \to B_{\varepsilon'}(0)$ is a second chart with $\psi'(u) = 0$ then after possibly shrinking ε' the condition is also satisfied for ψ'. Taking locally the same chart for f and g (i) then follows from Proposition A.1 with $A = \mathcal{F}_\varepsilon(K^r, K)$. □

We call $f \colon M \to K_n$ *locally analytic* if it is locally analytic in the above sense under the identification $K_n = K\langle x_1,\dots,x_n\rangle$. One can check that if $g \colon M \to K\langle x_0,\dots,x_n\rangle$ is locally analytic, so is the induced map $M \to K_n$.

Proposition A.7 *Let $f \colon M \to K_n$ be locally analytic. Then $M \ni u \mapsto \varphi(u) := \int_{\Delta^n} f(u) dx_1 \dots dx_n \in K$ is locally analytic.*

Proof: Fix a chart $M \supset V \xrightarrow{\psi} B_\varepsilon(0)$ such that $f \circ \psi^{-1} = \sum_I a_I x^I$ with $\|a_I\|_\varepsilon \cdot |I|^s \to 0$ as $|I| \to \infty$ for every $s \in \mathbf{N}_0$ as in the definition. For I fixed the function $B_\varepsilon(0) \ni v \mapsto \int_{\Delta^n} a_I(v) x^I dx_1 \dots dx_n = a_I(v) \int_{\Delta^n} x^I dx_1 \dots dx_n$ is given by the power series $a_I \cdot (\int_{\Delta^n} x^I dx_1 \dots dx_n) \in \mathcal{F}_\varepsilon(K^r, K)$. For $|I| \ge 1$ we have $\|a_I \cdot (\int_{\Delta^n} x^I dx_1 \dots dx_n)\|_\varepsilon \le \|a_I\|_\varepsilon \cdot |(\int_{\Delta^n} x^I dx_1 \dots dx_n)| \le \|a_I\|_\varepsilon \cdot C|I|^{sn}$ with C as in the proof of Proposition A.4. Since $\|a_I\|_\varepsilon |I|^{sn} \to 0$ as $|I| \to \infty$ it follows that $\sum_I a_I \cdot (\int_{\Delta^n} x^I dx_1 \dots dx_n)$ converges in $\mathcal{F}_\varepsilon(K^r, K)$. The claim follows since obviously $\varphi \circ \psi^{-1} = \sum_I a_I \cdot (\int_{\Delta^n} x^I dx_1 \dots dx_n)$. □

We will also write $\int_{\Delta^n} f dx_1 \dots dx_n$ for the function φ in the above Proposition. The next Proposition assures that we may interchange differentiation and integration. This is crucial in the proof of Theorem 2.5.

A power series $a = \sum_J \alpha_J y^J$ in $\mathcal{F}_\varepsilon(K^r, K)$ defines a locally analytic function $K^r \supset B_\varepsilon(0) \to K$ and we denote by $\partial_i a = \sum_J (j_i + 1)\alpha_{J+e_i} y^J$ with $e_i = (0,\dots,\underset{i}{1},\dots 0) \in \mathbf{N}_0^r$ its derivative with respect to the i^{th} variable y_i. Since

$|(j_i+1)\alpha_{J+e_i}|\varepsilon^{|J|} \leq \varepsilon^{-1}|\alpha_{J+e_i}|\varepsilon^{|J+e_i|}$, $\partial_i a$ is ε-convergent as well and $\|\partial_i a\|_\varepsilon \leq \varepsilon^{-1}\|a\|_\varepsilon$. In particular, $\partial_i : \mathcal{F}_\varepsilon(K^r, K) \to \mathcal{F}_\varepsilon(K^r, K)$ is continuous.

Proposition A.8 *Assume that $M = B_\varepsilon(0) \subset K^r$ and $f : M \to K_n$ is given by $\sum_I a_I x^I$ with $a_I \in \mathcal{F}_\varepsilon(K^r, K)$ and $\|a_I\|_\varepsilon |I|^t \to 0$ as $|I| \to \infty$ for every $t \in \mathbf{N}_0$.*

(i) $\partial_i f := \sum_I (\partial_i a_I) x^I$ is well defined and locally analytic.

(ii) $\int_{\Delta^n} (\partial_i f) dx_1 \ldots dx_n = \partial_i \int_{\Delta^n} f dx_1 \ldots dx_n$.

(iii) If $g : M \to K_n$ is of the same type then $\partial_i(fg) = (\partial_i f)g + f(\partial_i g)$.

Proof: By the above $\|\partial_i a_I\|_\varepsilon |I|^t \leq \varepsilon^{-1}\|a_I\|_\varepsilon |I|^t$ from which (i) follows. Then $\int_{\Delta^n} (\partial_i f) dx_1 \ldots dx_n = \sum_I (\partial_i a_I) \int_{\Delta^n} x^I dx_1 \ldots dx_n = \partial_i \left(\sum_I a_I \int_{\Delta^n} x^I dx_1 \ldots dx_n\right) = \partial_i \int_{\Delta^n} f dx_1 \ldots dx_n$ by definition of the integral and the continuity of ∂_i. The last assertion follows by a direct computation from the usual Leibniz rule for ε-convergent power series. □

More generally, if P is a free K_n-module of finite rank we say that a function $f : M \to P$ is locally analytic if all component functions with respect to a given basis of P are locally analytic. Then analogues of the above Propositions hold. In particular we are interested in the case where $P = \mathrm{Mat}_N(K_n)$ or $P = \mathrm{Mat}_N(K_n) \otimes_{K_n} \Omega^r(\Delta^n)$.

Now we can prove that the cocycle f in Theorem 2.5 is in fact locally analytic. Recall that here K is a finite extension of $\mathbf{Q}_p$ with ring of integers R and uniformizer π.

Proposition A.9 *The function $U_N(R)^{\times(2n-1)} \to K$,*

$$(g_1, \ldots, g_{2n-1}) \mapsto \mathrm{Tr} \int_{\Delta^{2n-1}} (d\nu \cdot \nu^{-1})^{2n-1}$$

where $\nu = \nu(g_1, \ldots, g_{2n-1}) = \sum_{i=0}^{2n-1} x_i g_{i+1} \cdots g_{2n-1} \in \mathrm{GL}_N(R_{2n-1})$ is locally analytic.

Proof: It suffices to show that $\nu^{-1} : (g_1, \ldots, g_{2n-1}) \mapsto \nu(g_1, \ldots, g_{2n-1})^{-1}$ and $d\nu : (g_1, \ldots, g_{2n-1}) \mapsto d(\nu(g_1, \ldots, g_{2n-1}))$ are locally analytic, where the above functions are considered as functions on $U_N(R)^{\times(2n-1)}$ with values in $N \times N$-matrices with coefficients in $\Omega^0(\Delta^{2n-1}) = K_{2n-1}$ resp. $\Omega^1(\Delta^{2n-1})$.

This is clear for $d\nu$. Set $\varepsilon := |\pi|$ and consider the global chart $\psi : U_N(R)^{\times(2n-1)} \to \pi \mathrm{Mat}_N(R) = B_\varepsilon(0) \subset K^{N \times N}$ whose inverse is given by $(M_1, \ldots, M_{2n-1}) \mapsto (1 + M_1, \ldots, 1 + M_{2n-1})$. Then $\nu^{-1} \circ \psi^{-1}$ is given by $\sum_{k=0}^\infty (\sum_{i=0}^{2n-1} x_i h_i)^k$ where $h_i : \pi \mathrm{Mat}_N(R) \to \mathrm{Mat}_N(K)$ is the function $(M_1, \ldots, M_{2n-1}) \mapsto 1 - (1 + M_{i+1}) \cdots (1 + M_{2n-1})$ (cf. the proof of Lemma 2.4). Since h_i has no constant term and only integral coefficients we have $\|h_i\|_\varepsilon \leq \varepsilon$. The coefficient of x^I in the above expansion of $\nu^{-1} \circ \psi^{-1}$ is of the form $h_0^{i_0} \cdots h_{2n-1}^{i_{2n-1}} +$ permutations and thus $\|(\text{coefficient of } x^I)\|_\varepsilon \leq \|h_0\|_\varepsilon^{i_0} \cdots \|h_{2n-1}\|_\varepsilon^{i_{2n-1}} \leq \varepsilon^{|I|}$. Since $\varepsilon < 1$ it follows that $\|(\text{coefficient of } x^I)\|_\varepsilon \cdot |I|^t$ tends to zero as $|I|$ tends to infinity for every $t \in \mathbf{N}_0$ and thus that ν^{-1} is locally analytic. □

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