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PULSE-STIMULATED RADIATION FROM A PLASMA DUE TO AN ENERGY-DEPENDENT GYROFREQUENCY*

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Abstract—A plasma excited by two short pulses at the electron gyrofrequency which have a time separation τ, is considered in the single particle approach. It is shown that the relativistic mass effect can lead to a series of radiation maxima after the second pulse. In the case of a cold plasma in an inhomogeneous magnetic field these maxima arise at multiples of the time τ; in the case of a warm plasma in a homogeneous magnetic field at multiples of $\tau/|1 \pm D|$, where D is the strength of the second pulse relative to the first one. The shape of the radiation maxima is given by the square of the Fourier transform of the distribution of the inhomogeneities or the initial energies, respectively. The two effects have the tendency to cancel each other. (i) If the plasma is excited by three pulses, the time separation of the second and third pulse being T, radiation maxima occur at times $t = K\tau + LT$, ($\pm K, L = 0, 1, 2, \ldots$ but $t > 0$) after the third pulse in the case of cold plasma with field inhomogeneities, and at $t = (K\tau + LT)/|1 \pm D \pm D_2|$ in the case of a warm plasma. (ii) If collisions are taken into account the dependence on T of the radiation maxima with $L = 0$ is determined by inelastic collisions only, while the other decay times are determined by all kinds of collisions.

1. INTRODUCTION

RECENTLY HILL and KAPLAN (1965); KAPLAN and HILL (1966) reported the observation of echoes radiated from a plasma. One type of experiment was to excite the plasma at the electron gyrofrequency by two short pulses with a time separation τ. The echo radiation was then observed at a time τ after the second pulse. This effect is related to the well-known spin echo (HAHN, 1950).

It is easy to show that any theory of echo-like phenomena has to be non-linear (GOULD, 1965, to be published; HERMANN and WHITMER, 1966). The aim of this paper is to study a special non-linearity caused by the relativistic mass effect, which can lead to such radiation maxima after the second (third . . .) pulse. There is a general relation to the spin echo (Gould, 1965, to be published), but the results obtained here show also essential differences with it.

We consider a magneto plasma, which is so dilute that the single-particle approach is valid. (In particular $\omega_c > \omega_p$ is assumed.) For simplicity the plasma dimensions are assumed to be small compared to the wavelength of the cyclotron radiation. The radiation by the plasma at the cyclotron frequency essentially depends on the relative phase of the gyrating particles. The energy radiated per second into the solid angle do is in the non-relativistic ($v \ll c$) case (LANDAU and LIFSHITZ, 1953):

$$\mathrm{d}I_c = do\,\frac{e^2\bar{\omega}_c^{\,2}}{c^3 8\pi}\,(2 - \sin^2\theta)\left|\sum_{l=1}^{N} v_l \exp\left[i(\omega_c^{\,l}t + \alpha_l)\right]\right|^2 \tag{1}$$

where N is the number of particles considered, v_l is the magnitude of the velocity of the l-th particle perpendicular to the magnetic field H, $\omega_c^{\,l}$ is its gyrofrequency and

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α_l its phase at $t = 0$. $\bar{\omega}_c$ is the average gyrofrequency and θ is the angle between the direction of observation and H. If the dimensions of the plasma are not small compared to the wavelength, retardation effects have to be taken into account.

In the case where the phases of the particles are *randomly* distributed (incoherent radiation), the last factor in (1) reduces to $\Sigma v_l^2 = N\overline{v_l^2}$. If all particles have the same phase (complete coherence) the value is $(\Sigma v_l)^2 = \bar{v}_l^2 N^2$, i.e. the radiation is increased by about a factor N compared to the incoherent case. In order to measure the degree of coherence, we introduce the function

$$\Phi(t) = \left(\sum_{l=1}^{N} v_l\right)^{-2} \left|\sum_{l=1}^{N} v_l[\exp i(\omega_c{}^l t + \alpha_l)]\right|^2 \tag{2}$$

which is unity in the case of complete coherence and $1/N$ if the phases are randomly distributed. If one neglects the statistical *fluctuations* in the phase distribution in order to substitute the sums in (2) by integrals, one obtains $\Phi = 0$ in the case of equally distributed phases.

In order to have an effect like the echo (Hill and Kaplan), the quantity Φ must depend on time and have a sharp maximum at the time the radiation peak is to occur. This means that the phase correlations between the particles have to be time dependent. In the approximation used here this requires the introduction of individual gyrofrequencies for the different particles corresponding to the procedure in the spin echo case (Hahn). One way for this to occur is through inhomogeneities of the magnetic field. An additional possibility is the relativistic mass effect or any other mechanism which causes the gyrofrequency to become energy dependent. This effect then also provides the necessary non-linearity in the equations.

The assumed validity of the single particle approach implicitly includes the assumption that the total energy contained in the radiation peaks emitted by the plasma is small compared to the total kinetic energy of the plasma.

2. THE MODEL

In the model discussed in the following, we consider the plasma as consisting of electrons with slightly different gyrofrequencies; these differences being due to inhomogeneities of the magnetic field and to the relativisitic mass effect. The latter can become important because of the long times involved, i.e. because of $\tau \gg 1/\omega_c$, although we assume throughout this paper for all velocities $v \ll c$. The relativistic mass effect causes the gyrofrequency of a given particle to change due to the interaction with the exciting pulses. On the other hand, we neglect all non-linearities during the pulses. In principle, these may also give rise to echoes. They were treated recently by HERRMANN and WHITMER and will not be considered in this paper.

At first we consider the acceleration of the electrons by the pulses. For this we treat the non-relativistic ($v \ll c$) motion of an electron in a homogeneous magnetic field under the influence of a plane electric wave. We neglect the magnetic field of the pulses as well as the spatial variation of their electric field. As we are interested in the gyration of the electrons, we assume the **E**-vector to be perpendicular to the static **H**, and we choose our co-ordinate system so that the z-axis is parallel to **H** and the x-axis parallel to **E**. The equation of motion for an electron in this approximation is

$$\dot{\mathbf{v}} = -\frac{e}{m}\left[\mathbf{E}^0 \sin \omega t + \frac{1}{c}\mathbf{v} \times \mathbf{H}\right]. \tag{3}$$

If we assume the frequency ω to be near the gyrofrequency of the considered particle $\omega = \omega \ + \Delta\omega$ and the time duration t_1 of the pulse to be short, i.e.

$$\Delta\omega t_1 \ll 1 \qquad \text{and} \qquad \frac{\Delta\omega}{\omega_c} \ll 1 \tag{4a,b}$$

the solution of (3) is

$$v_x = -pt \sin \omega_c t + v_0 \cos (\omega_c t + \alpha_0) \tag{5a}$$

$$v_y = pt \cos \omega_c t - \frac{p}{\omega_c} \sin \omega_c t + v_0 \sin (\omega_c t + \alpha_0) \tag{5b}$$

where v_0 is the component of the initial velocity of the electron under consideration perpendicular to $\mathbf{H}$, and α_0 is its phase at $t = 0$, and

$$p = \frac{e}{2m} E^0. \tag{6}$$

We now consider an ensemble of electrons, i.e. an electron plasma (without interactions) excited at the gyrofrequency by two short pulses. The quantity of interest is the radiation after the second pulse, which is characterized by $\Phi(t)$ given by (2).

In order to have differences in the gyrofrequencies after the first pulse, we consider the influence of an initial temperature and account for field inhomogeneities by attributing a different gyrofrequency to each electron. (This implies that the inhomogeneities are perpendicular to the field lines.) We assume a distribution $h(\eta)$ over the different gyrofrequencies, where $\eta = \Delta_{\text{inh}}\omega_c$ is the deviation from the average gyrofrequency due to the inhomogeneities.

At first we assume that all electrons have the same initial (transverse) energy with the corresponding (transverse) velocity v_0, but different, equally distributed phases. Then we have, in the two-dimensional v-space, the distribution given in Fig. 1(a). As it follows from (2) that only phase *differences* are essential, it is convenient to

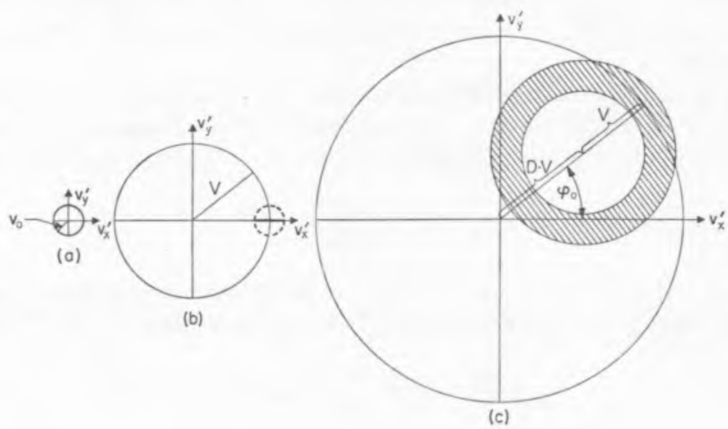

FIG. 1.—Particle distribution in v-space, (a) before the first pulse, (b) at the end of the first pulse, (c) at the end of the second pulse.

consider this diagram in a velocity space system, rotating with an average gyrofrequency $\bar{\omega}_c$ [defined by equations (12) and (18)]. Then φ is the phase *difference* with respect to a specified particle.

The velocities v_x' and v_y' in this rotating system are obtained from the velocities in the non-rotating system by the transformation

$$v_x' = -v_x \sin \bar{\omega}_c t + v_y \cos \bar{\omega}_c t \tag{7a}$$

$$v_y' = -v_x \cos \bar{\omega}_c t - v_y \sin \bar{\omega}_c t. \tag{7b}$$

If we apply this transformation to equation (5) we obtain:

$$v_x' = pt + v_0 \sin \alpha_0 - \frac{p}{\omega_c} \sin \omega_c t \cos \omega_c t \tag{8a}$$

$$v_y' = -v_0 \cos \alpha_0 + \frac{p}{\omega_c} \sin^2 \omega_c t. \tag{8b}$$

If we assume v_0, $pt \gg p/\omega_c$ *or* $\sin \bar{\omega}_c t_1 = 0$ (i.e. the pulse consisting of an integer number of cycles), we can neglect in (8) the terms with p/ω_c. We then have at the end of the first pulse the distribution given in Fig. 1(b) with

$$V = pt_1 = \frac{e}{2m} E^0 t_1. \tag{9}$$

The electrons are now equally distributed on the small dashed circle. The velocities of the electrons are now between $V - v_0$ and $V + v_0$, while the phase differences are smaller or equal to 2 arctan v_0/V. For the treatment which follows we further assume that

$$v_0 \ll V \tag{10}$$

i.e. that the energy the electrons gained during the first pulse is large compared to the initial (thermal) energy. Then all particles in Fig. 1(b) have almost the same phase and we approximate the distribution on the dashed circle by a uniform distribution on its solidly drawn diameter or, as we later allow for different initial energies, by a distribution $g(v)$ on this line. This means, the only effect of the initial energy we keep is that the particles have different energies after the first pulse according to their phase at the onset of the pulse.

As time proceeds, phase differences arise between the particles according to the differences in their gyrofrequencies. In our model the gyrofrequency depends on the energy and on the local magnetic field, and is given by (Landau and Lifshitz)

$$\omega_c^l = \frac{ecH}{E^l(v_l)} \tag{11}$$

where E^l is the relativistic energy. As we are interested in the phase *differences*, we ask for the *differences* in the gyrofrequency. With the assumption $v_l^2 \ll c^2$ we have, to first order,

$$\Delta\omega_c^l = A(1 - v_l^2/V^2) + \eta_l \tag{12}$$

with

$$A = \bar{\omega}_c V^2/2c^2 \tag{13}$$

when we attribute $\Delta\omega_c^{\,l} = 0$ to particles with $v_l = V$. Correspondingly the relative phase at time τ after the first pulse is

$$\varphi_l(\tau) = A\tau(1 - v_l^2/V^2) + \eta_l\tau. \tag{14}$$

Figure 1(c) gives the distribution in the v'-space at the end of the second pulse, which follows the first pulse after a time τ. The quantities v_l^* and $\varphi_l^*(t)$ of a particle after the second pulse are determined by the corresponding quantities v_l and $\varphi_l(\tau)$ at the onset of the second pulse and we have

$$v_l^{*2}/V^2 = v_l^2/V^2 + D^2 + 2(v_l/V)\,D\cos[\varphi(\tau) - \varphi_0] \tag{15}$$

$$\varphi_l^*(t = 0) = \varphi_0 + \beta_l \tag{16}$$

$$\sin\beta_l = (v_l/v_l^*)\sin[\varphi(\tau) - \varphi_0] \tag{17a}$$

$$\cos\beta_l = \frac{v_l^{*2}/V^2 + D^2 - v_l^2/V^2}{2Dv_l^*/V} \tag{17b}$$

$$\Delta^*\omega_c^{\,l} = A(B - v_l^{*2}/V^2) + \eta_l \tag{18}$$

$$\varphi_l^*(t) = \varphi_l^*(t = 0) + \Delta^*\omega_c^{\,l}t. \tag{19}$$

φ_0 gives the phase of the electric field of the second pulse relative to a particle with $\varphi(\tau) = 0$. In any actual experiment this is a statistical quantity. B is an arbitrary constant which defines the particle with respect to which $\Delta^*\omega_c$ is measured; D gives the strength of the second pulse relative to the first one ($p_2t_2 = DV$), and t is now the time measured from the end of the second pulse.

Having determined the velocity and the relative phase of the particles for any time after the second pulse, we now calculate $\Phi(t)$ according to (2) and study its time dependence. For this we introduce distribution functions $h(\eta)$ and $g(v)$ giving the distribution of the particles over the inhomogeneities of the magnetic field and over the velocities after the first pulse, and substitute the sums in (2) by integrals over η and v. We then find

$$\Phi(t) = \left(\iint \mathrm{d}v\,\mathrm{d}\eta\, g(v)h(\eta)v^*\right)^{-2} \times \left| V\iint \mathrm{d}v\,\mathrm{d}\eta\, g(v)h(\eta)\{D\exp(if) + (v/V)\exp(if_1)\}\right|^2 \tag{20}$$

with

$$f = \varphi_0 + \Delta^*\omega_c t \tag{21}$$

$$f_1 = \varphi(\tau) - \varphi_0 + f. \tag{22}$$

3. RESULTS

We now calculate $\Phi(t)$ from (20) using several approximations. The normalizing factor in (20) can be given approximately by

$$\left(\iint \mathrm{d}v\,\mathrm{d}\eta\, g(v)g(\eta)v^*\right)^2 \approx V^2(1 + D^2). \tag{23}$$

From equations (15), (18) and (21) we find

$$f = \varphi_0 + At\{B - v^2/V^2 - D^2 - 2(v/V)\,D\cos[\varphi(\tau) - \varphi_0]\} + \eta t. \tag{24}$$

By virtue of the relation

$$\exp(-iz\cos\theta) = \sum_{l=-\infty}^{+\infty} J_l(z)\exp[il(\theta - \pi/2)] \tag{25}$$

J_l being the Bessel function of order l, we find

$$\exp(if) = \sum_{l=-\infty}^{+\infty} J_l(2AtDv/V)\exp\{i[(t + l\tau)[\eta + A(1 - v^2/V^2)] - (l-1)\varphi_0 - l\pi/2 + At(B - D^2 - 1)]\}. \tag{26}$$

Before discussing the general case, we consider two special cases:

3.1 *An initially cold plasma in an inhomogeneous magnetic field*

In this limit we ignore the influence of the initial temperature, i.e. we assume

$$g(v) = \delta(v - V). \tag{27}$$

Then we have only to perform the integration over η. We see from (26) that at a time

$$t_l = -l\tau \tag{28}$$

the l-th term in the sum (26) becomes independent of η. If we integrate for a time t_l the first term in (20), the l-th of (26) gives the essential contribution. Or more precisely: If we perform the integral and consider it as a function of t, then the l-th term of (26) gives a contribution which is the Fourier transform of $h(\eta)$ with its maximum at t_l. If the width of this maximum is small compared to the separation from the next maximum which is due to the next term in (26), then the maximum at t_l is essentially determined only by the l-th term. In this case $\Phi(t)$ can easily be calculated. It shows maxima at times given by (28) (with the condition $t > 0$), i.e. we have a series of radiation maxima at times after the second pulse that are multiples of τ (KEGEL and GOULD, 1965).

The second term in (20) has essentially the same structure as the first. As we have, according to (22)

$$f_1 = \eta\tau + A\tau(1 - v^2/V^2) - \varphi_0 + f \tag{29}$$

we see that at a time t where the l-th term of (26) gives the main contribution to the integral, the $(l-1)$-th term of the corresponding expansion of $\exp(if_1)$ contributes. These two contributions have a phase difference of $\pi/2$. So we find for the maxima

$$\Phi(t_l) = (1 + D^2)^{-1}\{|DJ_l(2ADt_l)|^2 + |J_{l-1}(2ADt_l)|^2\}. \tag{30}$$

If the argument of the Bessel functions is small, we may use the approximation (WATSON, 1958)

$$J_l(z) = \frac{z^l}{2^l l!}; \quad l \geq 0 \text{ integer} \tag{31}$$

and we obtain

$$\Phi(t_l) = \frac{1}{1 + D^2}\left[\frac{D^2(ADt_l)^{2|l|}}{(|l|!)^2} + \frac{(ADt_l)^{2|1-l|}}{(|l-1|!)^2}\right]. \tag{32}$$

In this approximation the amplitude of the radiation maxima grows with the pulse strength and grows with increasing τ.

If, on the other hand, the argument of the Bessel function is large, we may use the approximation (Watson)

$$J_l(z) = \sqrt{\frac{2}{\pi z}} \cos\left[z - (l/2 + 1/4)\pi\right]. \tag{33}$$

Then we obtain

$$\Phi(t_l) = \frac{1}{\pi A t_l} \cdot \frac{1 + (D^2 - 1)\cos^2\left[2ADt_l - (l/2 + 1/4)\pi\right]}{1 + D^2}. \tag{34}$$

Figure 2 gives $\Phi(t_1)$ for the first radiation maximum ($l = -1$) as a function of τ,

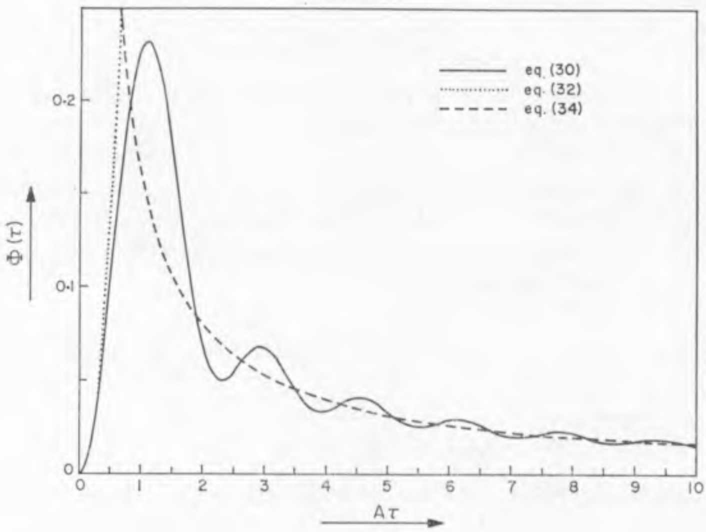

FIG. 2.—Amplitude of the first radiation maximum as a function of τ in different approximations as given by equations (30), (32) and (34) with $D = 1$.

calculated from (30), (32) and (34) with $D = 1$. If one takes into account collisions, the decay for large τ becomes nearly exponential. The maximum is then determined by the pulse strength and by the collision frequency.

We observe that the normalizing factor (23) is proportional to V^2 and in the approximation (34) we have $\Phi(t_l) \sim 1/A \sim 1/V^2$. This shows that in the region of validity of (34) the *absolute* intensity of the radiation maxima does not depend on A, i.e. on the strength of the pulse, in the case $D = 1$.

The shape of the radiation peaks is the square of the Fourier transform of $h(\eta)$. If $h(\eta)$ is a Gaussian with a width η_0, then the shape of the radiation peaks is also a Gaussian with a width $\Delta_0 t = 4/\eta_0$. In making our approximations we assumed that the width of the radiation peaks is small compared to the time separation of the different peaks. This assumption is equivalent to the assumption

$$\eta_0 \tau \gg 1 \tag{35}$$

where η_0 is a characteristic spread in the gyrofrequencies due to the inhomogeneities. If this condition is not fulfilled, or more precisely, if $\eta_0\tau$ is of the order π or less, the wings of the different peaks overlap and this means that one has really to employ the entire sum (26) in order to determine $\Phi(t)$. One sees that in this case the actual value of $\Phi(t)$ depends now in an essential way on φ_0 which determines the relative phase of the different terms. As in any actual experiment φ_0 is a statistical quantity, one should find under this condition that the amplitudes of the radiation peaks are different, each time one performs the experiment without changing any of the other parameters.

For $\eta_0 \ll 1$ there arise no maxima in $\Phi(t)$.

3.2 *A warm plasma in a homogeneous magnetic field*

In this limit we neglect the influence of the field inhomogeneities and ask only for the effects due to the initial temperature; i.e. we make the assumption of a strictly homogeneous magnetic field

$$h(\eta) = \delta(\eta). \tag{36}$$

In this limit we deal only with the dependence of (26) on v. In this case we have the integration variable v not only in the exponential function, but also in the argument of the Bessel functions. In order to simplify (26) so that the v-integration in (20) can be performed explicitly, we assume the argument of the Bessel functions in (26) to be large, so that the approximation (33) is valid. If we write the cosine in (33) as the sum of two exponential functions, we find for (26) (having performed in (20) the integration over η, i.e. dropped the η dependence):

$$\exp(if) = \left(\frac{1}{4\pi A D t v/V}\right)^{1/2} \left\{\sum_{l=-\infty}^{+\infty} \exp(iF_l) + \sum_{m=-\infty}^{+\infty} \exp(iF_m')\right\} \tag{37}$$

with

$$F_l = -At(D + v/V)^2 + lA\tau(1 - v^2/V^2) - (l-1)\varphi_0 + \pi/4 + ABt \tag{38a}$$

$$F_m' = -At(D - v/V)^2 + mA\tau(1 - v^2/V^2) - (m-1)\varphi_0 - m\pi - \pi/4 + ABt. \tag{38b}$$

We now write $v/V = 1 + u$ and make use of the assumption (10) by neglecting in (38) all terms which are quadratic in u. We then have

$$F_l = -2Au[t(D+1) + l\tau] - (l-1)\varphi_0 + \pi/4 + At[B - (1+D)^2] \tag{39a}$$

$$F_m' = 2Au[t(D-1) - m\tau] - (m-1)\varphi_0 - m\pi - \pi/4 + At[B - (1-D)^2]. \tag{39b}$$

Using the same arguments as in the cold plasma case, we conclude that we now have two series of radiation maxima at times

$$t_l = -l\frac{\tau}{D+1}; \qquad t_m = m\frac{\tau}{D-1}. \tag{40a,b}$$

The essential difference from the cold plasma case is that the time for the occurrence of the radiation peaks depends on the relative pulse strength D (KEGEL, 1965).

The term $\exp(if_1)$ in (20) can be treated in the same way. We find again that the

$(l-1)$-th term of the expansion contributes to the integral at a time t where the l-th term of (37) contributes. These two terms now have either the same phase, if they are determined by (39a), or have opposite sign, if they are determined by (39b). So we find for the maxima of $\Phi(t)$

$$\Phi(t_l) = \frac{1}{4\pi ADt_l}\frac{(1+D)^2}{1+D^2} \tag{41a}$$

$$\Phi(t_m) = \frac{1}{4\pi ADt_m}\frac{(1-D)^2}{1+D^2}. \tag{41b}$$

In the case that

$$-l(D-1) = m(D+1) \tag{42}$$

the contributions from both series add to one radiation peak. The relative phase between these two contributions depends on the last term in (39a) and (39b), i.e. depends on A and D.

The shape of the radiation peaks is now essentially the square of the Fourier transform of $g(v)$. If we assume $g(v)$ to be a Maxwellian with its maximum at $v = V$, then the shape of the radiation peaks is also a Gaussian $\sim \exp[-(t-t_l)^2 v(\Delta_0 t)^2]$, where

$$\Delta_0 t = \frac{1}{2A(D \pm 1)v_0/V} \tag{43}$$

v_0 being the velocity of a particle with the energy kT_e, i.e. $v_0/V = (kT_e/E_1)^{1/2}$ where T_e is the initial temperature and E_1 is the energy an electron gains by the first pulse if it is initially at rest. If we introduce the dimensionless quantity

$$\chi = A\tau v_0/V \tag{44}$$

we have

$$\Delta_0 t = \frac{\tau}{2\chi(D \pm 1)}. \tag{43a}$$

According to (43a) the initial temperature of the plasma can, in principle, be determined by measuring the width of the radiation maxima.

The total energy in one of the radiation peaks is proportional to $\int\Phi(t)\,\mathrm{d}t$. If the shape of the peak is a Gaussian with a width given by (43), we have

$$\int \Phi(t)\,\mathrm{d}t = \Phi(t_l)\Delta_0 t\pi^{1/2}. \tag{45}$$

The assumption that the width of the peaks is small compared to the separation between the different peaks is now equivalent to

$$\chi \gg 1 \tag{46}$$

and corresponds to the condition (35) in the cold plasma case. For $\chi \ll 1$ no radiation maxima arise.

Figure 3 gives a numerical example for the case of a warm plasma in a homogeneous magnetic field, computed from equation (20) without further approximations. The initial energy distribution was assumed to be Maxwellian. The parameters for this example were chosen to $A\tau = 50$, $v_0/V = 0{\cdot}1$ $\varphi_0 = \pi$, and $D = 1$. Figure 3

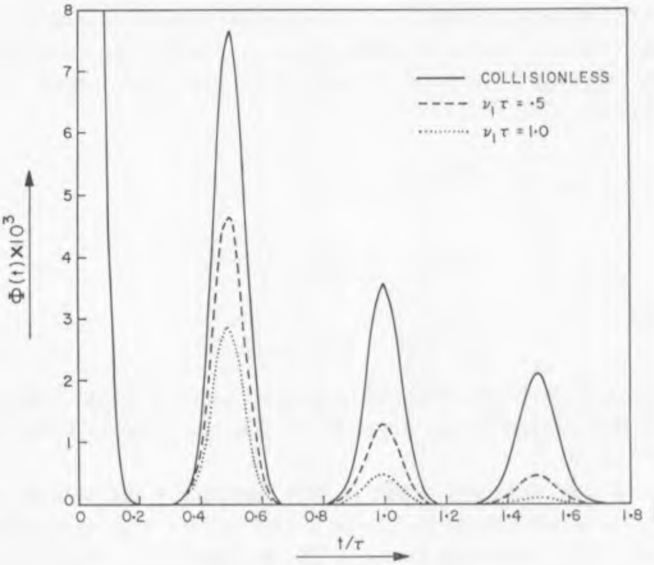

FIG. 3.—Radiation after the second pulse in the case of a warm plasma in a homogeneous magnetic field and $D = 1$.

shows also how the results obtained from (20) are modified by collisions according to equation (71). A $\Phi(t)$ of 10^{-3} means that the radiation is $N \,.\, 10^{-3}$ times the intensity which the plasma would radiate if the electrons had the same energy distribution but *randomly* distributed phases, i.e. for 10^8 electrons this would be a factor of 10^5.

We now consider the general case, i.e. the simultaneous influence of the field inhomogeneities and of the initial temperature. If the condition (35) is fulfilled, we retain for a time t_l given by (28) just the l-th term of (26) after integration over η. If we further assume the argument of the Bessel functions to be large, i.e. the approximation (33) to be valid, and the distribution over the initial velocities to be Maxwellian, the integration over v gives us the result (34) multiplied by a correction factor

$$\exp(-2l^2D^2\chi^2). \tag{47}$$

If we, on the other hand, assume (46) to be valid and integrate at first over v, we find correspondingly for times given by (40) the result (41) multiplied by a factor

$$\exp\left[-\frac{1}{2}\left(\frac{\eta_0 l D}{D \pm 1}\right)^2\right] \tag{48}$$

if $h(\eta)$ is assumed to be Gaussian.

This result shows that the effect of the initial temperature and that of the inhomogeneities have the tendency to cancel each other. If both conditions (35) and (46) are fulfilled simultaneously, essentially no radiation maxima arise.

If gradients of the magnetic field along the lines of force are taken into account, there arises an additional damping (Gould, to be published).

4. EXCITATION BY THREE OR MORE PULSES

Having considered in some detail the case of two exciting pulses, it is in principle not difficult to treat the more general case of several exciting pulses. However, the mathematical complication increases with the number of pulses.

Let us now consider a plasma exicted by three short pulses. Let D_2 be the strength of the third pulse relative to the first one, and φ_1 the phase of the electric field of the third pulse in the rotating co-ordinate system and v_2, $\varphi_2(t)$, β_2, $\Delta_2\omega$, B_2 be quantities after the third pulse. Furthermore, let T be the time between the second and third pulse. Then we have, corresponding to equations (15)–(19):

$$v_2^2/V^2 = v^{*2}/V^2 + D_2^2 + 2D_2(v^*/V)\cos[\varphi^*(T) - \varphi_1] \tag{49}$$

$$\varphi_2(t = 0) = \varphi_1 + \beta_2 \tag{50}$$

$$\sin\beta_2 = \frac{v^*}{v_2}\sin[\varphi^*(T) - \varphi_1] \tag{51a}$$

$$\cos\beta_2 = \frac{v_2^2/V^2 + D_2^2 - v^{*2}/V^2}{2D_2v_2/V} \tag{51b}$$

$$\Delta_2\omega_c = A(B_2 - v_2^2/V^2) + \eta \tag{52}$$

$$\varphi_2(t) = \varphi_2(t = 0) + \Delta_2\omega_c t \tag{53}$$

where t now is measured from the end of the third pulse. By analogy with (20) we find

$$\Phi(t) = \left(\iint \mathrm{d}v\,\mathrm{d}\eta g(v)h(\eta)v_2\right)^{-2}\Bigg|V\iint \mathrm{d}v\,\mathrm{d}\eta g(v)h(\eta) \times \{D_2\exp(if_2) + D\exp(if_3) + (v/V)\exp(if_4)\}\Bigg|^2 \tag{54}$$

with

$$f_2 = \varphi_1 + \Delta_2\omega_c t \tag{55a}$$

$$f_3 = \varphi_0 - \varphi_1 + \Delta^*\omega_c T + f_2 \tag{55b}$$

$$f_4 = -\varphi_0 + \Delta\omega_c\tau + f_3. \tag{55c}$$

More explicitly, we have

$$f_2 = \varphi_1 + t\eta - At\{-B_2 + D_2^2 + D^2 + v^2/V^2 + 2(v/V)D\cos[\varphi(\tau) - \varphi_0] + 2D_2D\cos[\varphi_0 - \varphi_1 + \Delta^*\omega_c T] + 2D_2(v/V)\cos[\varphi(\tau) - \varphi_1 + \Delta^*\omega_c T]\} \tag{56}$$

where $\Delta^*\omega_c T$ is to be calculated from (18).

We now specialize again to the case of an initially cold plasma in a slightly inhomogeneous magnetic field, i.e. we make the assumption (27). Performing the v-integration in (54) means essentially dropping the v-dependence. As (56) contains three cosines, the expansion into Bessel functions now gives a three-fold product of sums of the type (25). As $\Delta^*\omega_c T$ still contains a cosine, we apply (35) once more and obtain finally:

$$\exp(if_2) = \sum_{k,l,m,n=-\infty}^{+\infty} J_k(2ADt)J_l(2ADD_2t)J_m(2AD_2t)J_n[2ADT(m + l)] \times \exp\{i[\eta(t + (k + m + n)\tau + (l + m)T) - (k + l + m + n)\pi/2 - (l + m - 1)\varphi_1 + (l - k - n)\varphi_0 + (l + m)(B - 1 - D^2)AT - At(D_2^2 + D^2 + 1 - B_2)]\}. \tag{57}$$

With the assumption (35) we conclude from (57) that in the case of three exciting pulses we have radiation maxima at times

$$t_{KL} = -K\tau - LT \tag{58}$$

K and L being integers with the restriction $t_{KL} > 0$. The assumption (35) implies that at a time t_{KL} only those terms of (57) give an essential contribution, which are independent of η. For these terms we have $k + m + n = K$ and $l + m = L$. With these two conditions the four-fold sum reduces to a double sum over k and m. We perform the sum over k by means of the addition theorem (Watson)

$$J_\nu(x + y) = \sum_{\mu=-\infty}^{+\infty} J_\mu(x) J_{\nu-\mu}(y) \tag{59}$$

and obtain†

$$G_2 \sim \sum_m J_{K-m}(x_1) J_{L-m}(x_2) J_m(x_3) \exp(im\,\pi/2) \tag{60}$$

$$= \sum_m J_{K-m}(x_1) J_{L-m}(x_2) J_{-m}(x_3) \exp(-im\pi/2) \tag{60a}$$

with

$$x_1 = -2ADK\tau;\ x_2 = 2ADD_2 t_{KL};\ x_3 = 2AD_2 t_{KL} \tag{61a,b,c}$$

$$G_2 = \int d\eta h(\eta) \exp(if_2). \tag{61d}$$

It seems that the expression (60) can only be simplified by making further restrictive assumptions. If we assume $D = 1$, i.e. $x_2 = x_3$, we can use the formula (Watson)

$$J_\mu(z) J_\nu(z) = \frac{2(-1)^\nu}{\pi} \int_0^{\pi/2} J_{\mu-\nu}(2z\cos\psi) \cos(\mu + \nu)\psi\, d\psi. \tag{62}$$

We apply (62) to the last two Bessel functions in (60a) so that $J_{K-m}(x_1)$ remains the only Bessel function depending on m. We now can perform the sum over m by virtue of (25). Thus we find

$$G_2 \sim \frac{2}{\pi} \int_0^{\pi/2} J_L(2x_2 \cos\psi) \exp\{ix_1 \cos 2\psi - iK\pi/2\} \cos(2K - L)\psi\, d\psi. \tag{63}$$

Only in special cases have we found an analytic expression for the integral (63). If $K = 0$, from which it follows that $x_1 = 0$, we can apply (62) and find

$$G_2 \sim J_L(-2AD_2LT) J_0(-2AD_2LT). \tag{64}$$

In the case that $x_1 \ll 1$ *and* $x_1 \ll x_2$, one can neglect the exponential function in (63) and obtain

$$G_2 \sim J_{-K}(2AD_2 t_{KL}) J_{L-K}(2AD_2 t_{KL}) \exp(iK\pi/2). \tag{65}$$

This approximation is *not* valid for $L = 0$ as we then have $x_1 \approx x_2$ and the variation of the exponential function in (63) with ψ must not be neglected in comparison with the variation of the Bessel function.

† In (60) there is only a phase-factor omitted, which is common to all terms in the sum.

In the same way one finds for the other two terms in (54):

$$G_3 \sim \sum_m J_{K-m}(x_1)J_{L-1-m}(x_2)J_m(x_3)\exp(im\pi/2) \tag{66}$$

$$G_4 \sim \sum_m J_{K-1-m}(x_1)J_{L-1-m}(x_2)J_m(x_3)\exp(im\pi/2) \tag{67}$$

where G_3 and G_4 are defined as similar to G_2 (61d). These sums can be treated as (60).

If we consider the case of a warm plasma in a homogeneous magnetic field excited by three pulses, we obtain an expression similar to (57) where the arguments of the Bessel functions, except the second, depend on v. Making the large argument approximation (33) leads us to expect radiation maxima at times

$$t_{KL} = \frac{-K\tau - LT}{1 \pm D \pm D_2} \tag{68}$$

where K and L are again integers with the restriction $t_{KL} > 0$ and the signs of D and D_2 may occur in each combination.

The results obtained for the case of three exciting pulses may be generalized in a straightforward manner to the case of $n + 1$ pulses. The relations (49)–(53) become recurrence formulae by substituting the index 2 by n, the index 1 and the asterisk by the index $(n - 1)$, and T by T_n. For t being the time elapsed after the last pulse, one has

$$\Phi(t) \sim \left| \iint \mathrm{d}v\, \mathrm{d}\eta\, g(v)h(\eta)\{D_n \exp[i(\varphi_{n-1} + \Delta_n\omega_c t)] + v_{n-1}\exp[i(\varphi_{n-1}(T_n) + \Delta_n\omega_c t)]\} \right|^2 \tag{69}$$

In the case of an initially cold plasma in an inhomogeneous magnetic field one concludes that radiation maxima arise at times

$$t = \sum_{i=1}^{n} L_i T_i \tag{70}$$

where L_i are integers with the requirement that $t > 0$.

5. THE INFLUENCE OF COLLISIONS

In the previous sections we did not account for collisions at all. However, as collisions destroy phase correlations, it is obvious that they give rise to a much faster decay of the radiation maxima than that given by the previous formulae. It can be shown that the influence of collisions in the two-pulse case is essentially different from that in the three-pulse case. We shall limit our discussion to this effect. We make the assumption that the probability to undergo a collision is the same for all particles. By this we neglect the velocity dependence of the collision cross section, which by itself can give rise to echoes. This latter effect was studied in detail by Kaplan and Hill.

It was observed in the experiment of Hill and Kaplan that the dependence of the radiation maxima on τ was determined by all phase-destroying collisions, while in the three-pulse case the dependence on T was determined by the inelastic collisions only. This is due to the fact that after the second pulse there is information stored not only in the phases but also in the energy distribution.

Let us consider at first the two-pulse case. If we assume that the phase of a particle after a collision is not related to its phase before the collision, it follows that only particles can contribute to the radiation maxima which did not undergo any collision. With the assumption that the probability of undergoing a collision is the same for all particles, we have

$$\Phi_c(t) = \Phi(t) \exp\left[-2(t+\tau)\nu_1\right] \tag{71}$$

where $\Phi(t)$ is the value obtained for a collisionless plasma and ν_1 is the collision frequency accounting for all kinds of collisions.

Under conditions as in the experiment of HILL and KAPLAN most of the collisions the electrons undergo are with neutrals. As the mass of an electron is very small compared to that of an atom, most collisions only change the phases of the electrons but not their energies. We call these collisions elastic.

We now consider the three-pulse case and study at first the influence of the elastic collisions during the time between the second and third pulse in the limit $T\nu_{\text{el}} \gg 1$. In this case the phases have been randomized at the onset of the third pulse and $\alpha = \Delta^* \omega_c T$ is now a statistical quantity which is to be integrated over, instead of being given by (18). Furthermore, the problem now has become three dimensional, as the electrons can, through collisions, acquire a velocity component parallel to the magnetic field. So we have

$$v_\perp^*(t=T) = v^*(t=0)\sin\theta; \quad v_\parallel^*(t=T) = v^*(t=0)\cos\theta \tag{72}$$

where θ is the angle between the electron velocity and the z-axis. We also have to integrate our final result over θ. Consequently we have to substitute in (49) and (51) v^* by $v_\perp^*(t=T)$ and (52) by

$$\Delta_2\omega_c = A[B_2 - v_2^2/V^2 - (v^{*2}(t=0)/V^2)\cos^2\theta] + \eta \tag{73}$$

v_2 now being again only the transverse part of the velocity. With these modifications we now find instead of (56):

$$\begin{aligned} f_2 = \varphi_1 + t\eta - At\{&-B_2 + D_2^2 + D^2 + v^2/V^2 + 2(v/V)D\cos[\varphi(\tau) - \varphi_0] \\ &+ 2DD_2\sin\theta\cos[\varphi_0 - \varphi_1 + \alpha] + 2(v/V)D_2\sin\theta\cos[\varphi(\tau) - \varphi_1 + \alpha]\}. \end{aligned} \tag{74}$$

If we now make the expansion into Bessel functions, we see that only terms with $L = 0$ give a contribution after integrating over α. So there are, in spite of $\nu_1 T \gg 1$, still radiation maxima after the third pulse, but their number is decreased by the limitation to $L = 0$.†

In the case of a cold plasma in a homogeneous magnetic field we derive from (74) again equation (60) with $L = 0$ and x_2 and x_3 to be substituted by $x_2 \sin\theta$ and $x_3 \sin\theta$. Assuming again $D = 1$ we can perform the θ-integration by means of (62). Thus we obtain

$$G_2' \sim \int_0^{\pi/2} J_{1/2}^2(x_2\cos\psi)\exp\left\{ix_1\cos 2\psi + iK\frac{\pi}{2}\right\}\cos 2K\psi\, d\psi. \tag{75}$$

In order to account for the inelastic collisions during the time between the second and third pulse and all kinds of collisions during the time before the second and after the

† A corresponding result was obtained in the spin echo case (Hahn).

third pulse, the value (75) has to be multiplied by the factor

$$\exp[-(\tau + t)\nu_1 - T\nu_2] \tag{76}$$

where ν_2 is an effective collision frequency for inelastic collisions.

6. DISCUSSION

In the previous sections it has been shown that the relativistic mass effect can give rise to radiation maxima in a plasma excited by a sequence of short pulses at the electron gyrofrequency. The essential point in the treatment was that the influence of the relativistic mass effect was *neglected during* the exciting pulses, while it was taken into account *between* the pulses. This approximation can only be made if (4) is fulfilled, i.e. if the pulses are short and if $\tau \gg t_1$ and further if $v \ll c$.

The results obtained here are, of course, to some degree similar to those in the spin echo case (Hahn). There are, however, essential differences due to the fact that the physical mechanism which gives rise to the radiation maxima is very different. Some of these differences are (a) the fact that the amplitude of the first radiation maximum goes to zero as τ goes to zero (Fig. 2); (b) the occurrence of a series of radiation maxima with a relatively slow decay; and (c) the fact that in the case of a warm plasma in a homogeneous magnetic field the time at which the radiation peaks occur depends on the relative pulse strength D.

A not very essential assumption in our treatment was that the dimensions of the plasma are small compared to the wavelength of the radiation at the gyrofrequency. If this is no longer true, one sees that the **k**-vector of the exciting pulses has to be perpendicular to the static magnetic field. If **k** and **H** are parallel, particles excited at different phases can interchange their places by moving along the lines of force, giving rise thereby to statistical phase differences and spoiling the correlations which have been generated. A further consequence is that the radiated energy in this case is essentially radiated into the same direction as the exciting pulse.

It has been shown by Gould (1965; to be published) in a general discussion that there are other non-linearities besides the relativistic mass effect which can give rise to pulse stimulated radiation from a plasma. The relative importance of the different effects depends on the details of the experimental conditions. It should be noted, however, that the validity of the results obtained in this paper is not restricted to the case of the relativistic mass effect, but the results are in first order correct for all non-linearities which lead to an energy-dependent gyrofrequency. The only change that must be made is that equation (13) is to be substituted by a new definition of the quantity A which determines the amount by which the gyrofrequency is changed when the velocity is changed. Hirshfield and Wachtel (1966a), for example, have shown explicitly that the influence of spatial gradients of the static magnetic or a static electric field lead to a velocity-dependent gyrofrequency.

Finally, it should be mentioned that the effect of an initial temperature combined with a velocity dependence of the gyrofrequency can give rise to radiation maxima even if the plasma is excited by one pulse only, as was shown by Hirshfield and Wachtel (1966a,b). This effect does not show up in our approximation because of the assumption $v_0 \ll V$ in which case all particles have essentially the same phase at the end of the first pulse. On the other hand Hirshfield and Wachtel assumed $v_0 \gg V$.

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