Conjectured Set of Exact Bootstrap Equations*

SHU-YUAN CHU AND P. KAUS
University of California, Riverside, California 92501

AND

F. ZACHARIASEN
California Institute of Technology, Pasadena, California 91109
(Received 12 December 1969)

A set of exact bootstrop equations is conjectured, consisting of two coupled homogeneous equations for the vertex function and the propagator from which any n-legged amplitude can be constructed. One equation is analogous to the vanishing of vertex renormalization constants ($Z=0$); the second equation expresses "duality". All graphs can be reduced to the simple tree diagram. Amplitudes, if solutions exist, will be crossing symmetric and will have at least all the necessary singularities making unitarity plausible but not proved.

I. INTRODUCTION

IN the continuing search for a well-defined statement of the bootstrap hypothesis, for a set of "equations" encompassing the bootstrap theory, two primary approaches have been used. These may conveniently be called the "Regge-pole formulation"[1] and the "$Z=0$ formulation."[2] They are, presumably, equivalent ways of saying the same thing.[3]

Both of the formulations provide us with valuable insights into what we mean by the bootstrap hypothesis. The Regge-pole formulation (together, of course, with experimental input) has made us understand that we have rising trajectories, and daughter trajectories which perhaps have real particles on them, so that we must expect to deal with families containing infinite numbers of particles (albeit most are unstable) in any true bootstrap theory. It has also led us to the concept of "duality," which, simply stated, is just the idea that one must be careful in phrasing a bootstrap theory to avoid double counting.[4] Finally, we have learned from the Regge formulation how to cope with composite particles of high spin, and how to avoid the divergences which would, in field theory, be associated with such particles.

From the $Z=0$ or field-theoretic formulation, on the other hand, we have learned the value of diagrams in describing a bootstrap theory. We can phrase the bootstrap in a manifestly crossing-symmetric way, so that it is clear that the bootstrap conditions in different channels are compatible.[5] This formulation also allows us to invoke many of the well-known approximations of conventional field theory in the bootstrap. In addition, it permits us to write down, formally at least, exact equations which contain the whole of the bootstrap theory.[2]

Both the Regge bootstrap theory and the $Z=0$ bootstrap theory are, however, really unsatisfactory. The Regge approach (which, incidentally, is sometimes glorified with the title "S-matrix theory") is woefully incomplete in that its assumptions cannot be made explicit in other than rather simple cases. The Z theory is also sadly lacking because it is incapable of dealing with particles of high spin. It is, therefore, a worthwhile occupation to look for other, more satisfactory, ways of phrasing the bootstrap theory; ways which may, hopefully, permit us for the first time to write down exact, explicit, and well-defined equations encompassing the bootstrap assumption.

We should like to combine here the ideas proceeding from both of these older formulations into a new and remarkably simple way of setting up the bootstrap theory, one which consists simply of a set of coupled integral equations, which can be written down explicitly and in closed form, and a rule which permits the direct calculation of any (on- or off-shell) scattering or production amplitude in terms of these functions.

Let us begin with the idea, taken from the $Z=0$ formulation, that the bootstrap theory can be described with graphs. At this stage, we will not try to be very specific about what a graph means; we need to make explicit only the following three remarks.

(i) First, graphs are to be constructed by conventional Feynman rules out of vertex functions and propagators. These are, as usual, functions of the 4-momenta associated with each line in a graph, and they are in addition labeled by a set of indices for each line. The indices are to reflect the input from the Regge formulation, that we must expect to have to deal with infinite numbers of particles. Thus, among the indices are channel labels, that is, a total spin J, projection M, and whatever conserved internal quantum numbers may

* Work supported in part by the U. S. Atomic Energy Commission. Prepared under Contract No. AT(11-1)-68 for the San Francisco Operations Office, U. S. Atomic Energy Commission.

[1] G. F. Chew and S. Frautschi, Phys. Rev. Letters **7**, 394 (1961).

[2] A. Salam, Nuovo Cimento **25**, 224 (1962); P. Kaus and F. Zachariasen, Phys. Rev. **171**, 1597 (1968).

[3] P. Kaus and F. Zachariasen, Phys. Rev. **138**, B1304 (1965).

[4] This is not to be confused with approximate and often false statements of duality which equate s-channel resonances with a single leading t-channel Regge-pole contribution "on the average," at energies all the way down to threshold.

[5] To be specific, the number of constraints $Z=0$ exactly equals the number of coupling constants plus the number of masses, regardless of the number of channels involved (see Ref. 2).

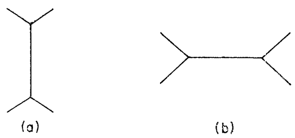

FIG. 1. Direct and exchange graphs which must not be added to avoid double counting.

exist. But we must not expect channel indices alone to suffice. If we anticipate the existence of daughter trajectories, we must allow for the possibility of many particles in each channel[6] so that the indices will, in general, include more than J, M, and internal quantum numbers. Exactly what this extra index (or set of indices) is, we do not know; for the moment, it suffices to say that it exists. Evidently, in any graph with an internal line, the indices associated with that line are to be summed over.

We thus have a set of vertex functions, and a set of propagators, labeled by 4-momenta and indices, and in terms of which we can write down any graph. Particles appear as poles (on the real axis if stable; on some sheet or other if not) in the propagator functions. The quantum numbers of the particle determines which channel, and hence which propagator, it appears in.

(ii) The next question to be answered is what is the rule by which we construct scattering amplitudes, and this leads us to our second remark.

From the Regge theory, we have the concept of duality. In terms of our graphs, this means that we cannot blindly add graphs like those of Figs. 1(a) and 1(b) without danger of double counting. But by the same token, we must be careful not to add graphs like Figs. 2(a) and 2(b) because this, as we have learned from the $Z=0$ formulation, will also involve double counting.

In contrast to a field theory with elementary particles, then, in a bootstrap theory one cannot calculate an amplitude simply by adding all graphs; rather we must select a subset of all graphs. Exactly which subset is to be included will, of course, depend on just what the vertex functions and propagators are. Whatever equations we require these to satisfy will determine which graphs comprise a physical amplitude.

The simplest and most appealing choice, to anticipate the content of the rest of this article, is to choose the vertex and propagator so that they satisfy *exactly* the two equations described graphically in Figs. 3(a) and 3(b). As will be discussed more fully later, from these equations one deduces the important result that *all graphs with the same number of external legs are equal*, irrespective of their internal structure. The consequent rule (which avoids double counting) for constructing amplitudes, is then clearly that *an n-legged amplitude is equal to any single n-legged graph*. Since all graphs are equal, any one of these suffices to give the entire amplitude. This means, in particular, that not only poles in the cross channel, but in fact all multibody singularities of the amplitude as well, will be produced at least in part by the divergence of the sums over the infinite set of indices associated with the internal lines of a given diagram. For some diagrams, indeed, *all* singularities in some channels will come about through the divergence of these sums.

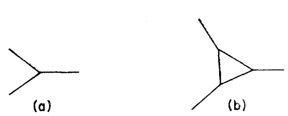

FIG. 2. Vertex graphs which must not be added to avoid double counting.

[6] We hasten to emphasize, however, that we do *not* attempt to make a one-to-one correspondence between lines in graphs and (stable or unstable) particles. One propagator (corresponding to one line) may have several poles (corresponding to several particles) or no poles (corresponding to no particles).

(iii) This brings us to our third remark. As we have said, the lines in our graphs do not strictly correspond to particles; nevertheless, it is convenient to think of them as doing so. Roughly speaking, then, our lines stand for the infinite set of composite particles contained in the bootstrap theory, and of this infinite set, almost all are unstable. In the usual field theory, on the other hand, a line in a Feynman graph stands for the stable elementary particles which are the input into the theory. In our case, furthermore, an amplitude is represented by a single graph, while in the normal language it is given as the sum of an infinite number of graphs.

Now we may think of attempting to reexpress our graphs, with the infinite number of particles in each line, in terms of a new set of graphs in which only the subset of stable particles appears in the lines. Evidently, a single graph of the first type will generate an infinite number of graphs of the second type.

Thus, we anticipate a kind of reciprocity between the usual Feynman graphs and our graphs: Our graphs amount to replacing an infinite set of graphs with a finite number of kinds of lines by a finite set of graphs with an infinite number of kinds of lines.

Having made these remarks, let us attempt to summarize what we hope to do in the body of this paper.

We wish to set up, first, equations by which vertex functions and propagators can be determined, second, rules for the calculation of graphs in terms of these functions, and third, rules for the calculation of physical amplitudes in terms of graphs.

Further, the theory so defined will be analytic, crossing symmetric, and (we hope) unitary, and it will have no double counting of either of the two types described above, so that it will truly be a bootstrap theory.

The theory we shall present is incomplete in two ways. In the first place, the "index" referred to above which distinguishes the different lines in the graphs

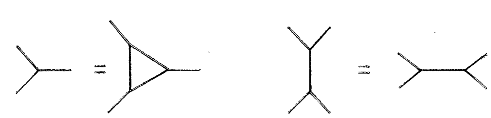

FIG. 3. Graphical representation of bootstrap equations.

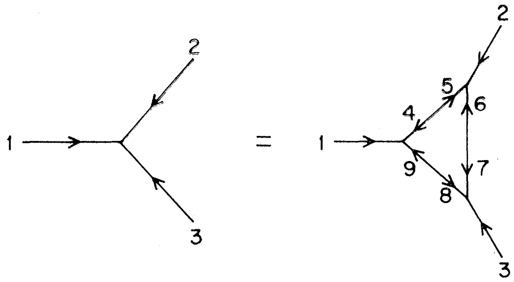

FIG. 4. "Vertex" bootstrap graph.

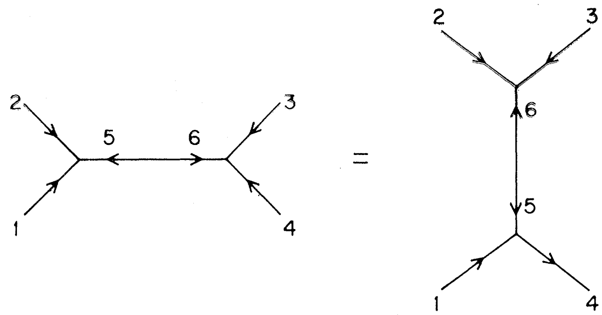

FIG. 5. "Duality" bootstrap graph.

cannot be completely defined *a priori*. What it is must be determined by the equations we write down for the vertex functions and propagators. This will be greatly elaborated on later.

In the second place, the theory is incomplete in that we have not been able to demonstrate that the amplitudes calculated by our rules are unitary. Two possibilities exist. One is that, with any solution (or solutions) of the integral equations for the vertex functions and propagators, the resulting amplitudes are unitary. The other is that these equations necessarily have many solutions, but only one, or a limited class, of them gives a unitary theory.[7] In this case unitarity would be an extra condition, in addition to the equations for the vertex and propagator, to be imposed on the theory.

In at least a trivial sense, the second of these possibilities is the one which applies. Since our equations for the vertex and propagator are homogeneous, there is evidently a freedom to change the scale in the solutions. That is, an arbitrary constant multiple of a solution for the vertex with a related constant multiple of the solution for the propagator is also a solution. We thus have at least a continuously infinite class of solutions (if we have one). However, since the unitarity relation is nonlinear, only one (at most) of this class of solutions can be unitary. Thus unitarity must be used at least to determine the scale of the theory.

Whether or not this somewhat trivial lack of uniqueness is the only one which exists we do not know; we suspect not. If it is, we would hope that the equations themselves guarantee unitarity. If it is not, we would hope that unitarity is compatible with the equations and that it can be imposed as an additional constraint on the solutions.

II. PROPOSAL

Definitions

Since much of this proposal will be based on manipulating diagrams, it is necessary to define what we mean by the various quantities appearing in them. In analogy with usual Feynman rules, we assume a graph to be constructed from "vertex functions" Γ, which we will take to join three lines at a point, and "propagators" Δ, which will be represented as lines joining one vertex with another.

First, what is meant by a line? A line symbolizes a four momentum and a set of indices which label the states (not, by any means, necessarily single particle states) in the theory. As mentioned above, the precise nature of the set of indices will have to be determined by the solution to the equations of the theory. We will come back to discuss various possibilities after the equations have been defined.

Next, we write the expressions for the basic quantities in the theory: (a) Vertex functions are written

$$\Gamma_{J_1M_1,J_2M_2,J_3M_3}(P_1,P_2,P_3;X_1,X_2,X_3) \equiv \Gamma_{i_1i_2i_3}(P_1,P_2,P_3); \quad (1)$$

(b) propagators are written

$$\Delta_{JM}(P;X_1,X_2)=\Delta_{i_1i_2}(P). \quad (2)$$

The index i_n, then, stands for the channel indices J_n, M_n, and something else which we call X_n. For convenience we will not distinguish between continuous and discrete contributions to X_n when writing summations over i_n, but we note that we have not assumed that the propagator is diagonal in all contributions to X_n, though it doubtless will contain some conserved quantities, in particular, internal conserved quantum numbers. Also for convenience we have not included the 4-momentum P_n in the index i_n.

We now write the equations which, we propose, define Γ and Δ. Graphically, as mentioned in the Introduction, they are very simple.

First, the vertex condition may be represented as in Fig. 4. Second, the expression for duality is shown graphically in Fig. 5. All internal lines are to be summed

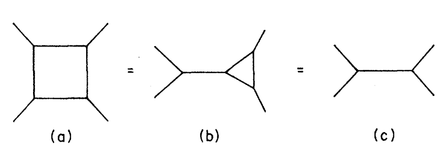

FIG. 6. Reduction of the box diagram.

[7] For the sake of completeness and honesty, we must add that a third possibility also exists, namely, that our equations are incompatible with unitarity.

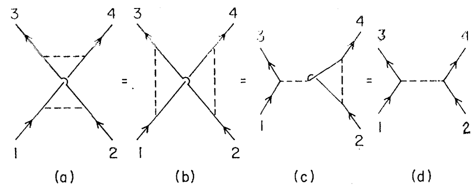

FIG. 7. Reduction of a nonplanar graph.

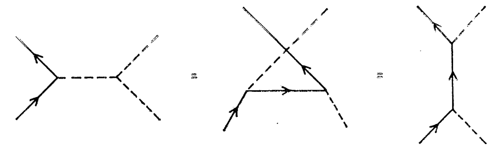

FIG. 9. Necessary kinds of duality.

or integrated over all indices consistent with conservation laws.[8]

Specifically, the graphical equations shown in Figs. 4 and 5 mean

$$\Gamma_{i_1i_2i_3}(P_1,P_2,P_3) = \sum_{i_4,i_5,i_6,i_7,i_8,i_9} \int d^4P\, \Gamma_{i_9i_1i_4}(-P_1-P,\, P_1,\, P)\Delta_{i_4i_5}(P) \times\Gamma_{i_5i_2i_6}(-P,\, P_2,\, -P_2+P)\Delta_{i_6i_7}(-P_2+P) \times\Gamma_{i_7i_3i_8}(P_2-P,\, P_3,\, P_1+P)\Delta_{i_8i_9}(P_1+P) \quad (3)$$

and

$$\sum_{i_5,i_6} \Gamma_{i_1i_2i_5}(P_1,\, P_2,\, -P_1-P_2)\Delta_{i_5i_6}(-P_1-P_2) \times\Gamma_{i_6i_3i_4}(P_1+P_2,\, P_3,\, P_4) = \sum_{i_5,i_6} \Gamma_{i_1i_4i_5}(P_1,\, P_4,\, P-{}_1-P_4)\Delta_{i_5i_6}(-P_1-P_4) \times\Gamma_{i_6i_2i_3}(P_1+P_4,\, P_2,\, P_3)\,. \quad (4)$$

An important property of our vertex functions is that the order in which the indices appear is immaterial. This is crucial in reducing nonplanar diagrams. In formulations of amplitudes based on the Veneziano representation,[9] this basic property of vertex functions is violated and the consequent classification of graphs becomes very complicated.

III. CONSEQUENCES

From Eqs. (3) and (4) we can see that all diagrams with n external legs are equivalent. Take the box diagram [Fig. 6(a)] as an example. Duality [Eq. (4)] tells us that this is equal to any planar four-legged diagram with one loop, in particular, to the one in Fig. 6(b). The vertex condition [Eq. (3)] then reduces this to the simple exchange diagram shown in Fig. 6(c).

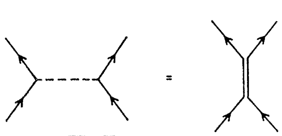

FIG. 8. Unnecessary kind of duality.

[8] It is not clear at this point exactly which conservation laws will be input and which are to be bootstrapped. Hopefully, all internal symmetries, at least, can be derived from the bootstrap assumption.

[9] K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969); K. Kikkawa, S. Klein, B. Sakita, and M. A. Virasoro, Phys. Rev. D **1**, 3258 (1970).

Nonplanar graphs can also be reduced in a similar way. Take as an example the special case where the index X includes (conserved) baryon number which can take on only two values, namely 0, indicated by a dashed line, and 1, indicated by a solid line. (The arrow here is used to distinguish baryon from antibaryon, not to indicate the sign of the momentum as in previous graphs.)

Consider now the diagram in Fig. 7(a). Applying duality to the external legs 1 and 4, we obtain Fig. 7(b). Then, applying duality to 1 and 3 gives Fig. 7(c), and finally the vertex condition applied to 2 and 4 gives Fig. 7(d). Note the important observation that at no stage was it necessary to assume the existence of a baryon-number 2 line. In other words, it is *not* implied here that duality of the type shown in Fig. 8 exists. There may or may not be such a dibaryon vertex, depending on whether or not the equations have a solution without it, but it is not needed for our reduction. We do require, on the other hand, that the types of duality expressed in Fig. 9 exist.

It is worth mentioning, incidentally, that excluding lines with exotic quantum numbers from our diagrams never forces us to say any scattering amplitude is exactly zero. There is always, for every process, at least one diagram without loops in which all lines are nonexotic.

It is fairly obvious now that all n-legged graphs are equivalent. In particular, then, the general n-legged graph may be reduced to a simple tree diagram, for example, that shown in Fig. 10.

We now make the following assumption: *The n-legged off-shell amplitude T_n is equal to any n-legged graph.* The on-shell amplitude is, of course, as usual constructed from the off-shell amplitude simply by setting the squares of the external 4-momenta equal to the squared masses of whichever particles we wish to have as the external legs, and choosing the external leg channel indices to correspond to those same particles.

Since one of the diagrams representing T_n is the tree diagram, Fig. 10, T_n can be written entirely as a product of $n-2$ vertex functions $\Gamma_{ijk}(P_iP_jP_k)$ and $n-3$ propagators $\Delta_{ij}(P)$ with the appropriate summations or integrations implied. The problem then reduces entirely

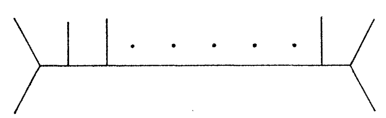

FIG. 10. Graph representing a production amplitude.

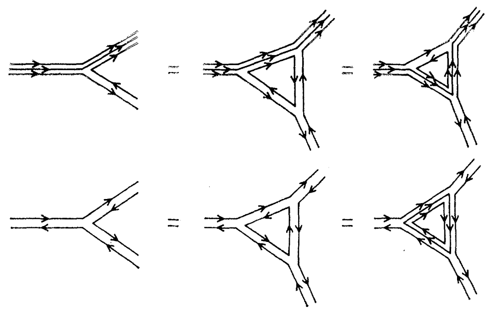

FIG. 11. Bootstrap equation in quark model.

to solving Eqs. (3) and (4). These, of course, are very complicated and we do not know at this point whether solutions exist, or whether solutions, if they exist, are unique.

That Eqs. (3) and (4) are not unique at least as to scale can be seen directly. Clearly if a solution exists for the functions Γ_{ijk} and Δ_{ij}, every Γ can be multiplied by a constant A^3, as long as every Δ is multiplied by A^{-2}. The n-legged amplitude will then be multiplied by A^n. If this is the only type of nonuniqueness, unitarity would then, presumably, determine the scale A, as mentioned in the Introduction.

We now turn to a discussion of the indices i_n. Let us convince ourselves first that these cannot be too trivial. One extreme would be that the Γ's and Δ's have no indices at all. Then to satisfy (4), $\Gamma\Delta\Gamma$ must not depend on the 4-momentum of Δ. Looking at (3) then, we see that the integrand would be independent of the loop momentum p and the integral would diverge.

The other extreme may be to think of Γ_{i_1,i_2,i_3} as $\Gamma_{J_1M_1,J_2M_2,J_3M_3}$, and independent of the 4-momenta. In that case, the couplings are $3-j$ symbols (Clebsch-Gordan coefficients), and consequently Eq. (3) reduces to the requirement that the sum $\sum_{J_4J_5J_6} W(J_1J_4J_2J_5;J_3J_6)$, where W is the Racah coefficient, must be independent of J_1, J_2, and J_3. Since the Racah coefficients do not in fact satisfy such a relation, this is not a solution either.

A combination of these two extremes are the Cutkosky relations.[10] Cutkosky showed that if a set of n vector mesons of equal mass bootstrapped themselves, then their mutual coupling constants were the structure factors of a Lie group, and this result has been genera-

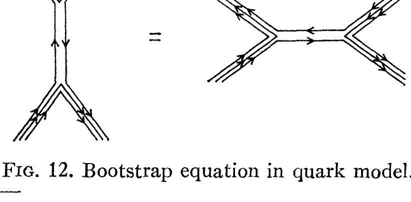

FIG. 12. Bootstrap equation in quark model.

[10] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963.)

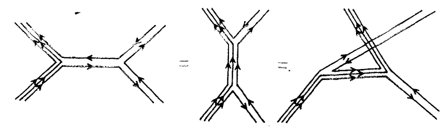

FIG. 13. Bootstrap equation in quark model.

lized to other types of particles. We would obtain Cutkosky's relations exactly if we assumed that $\Gamma_{ijk}(P_1,P_2,P_3)$ is given by $g_{ijk}I(P_1,P_2,P_3)$, where I is a universal function independent of the indices, while g_{ijk} is independent of momenta. In this case, the argument about Racah and Clebsch-Gordan coefficients no longer applies because of multiplication by the function I.[11] However, the argument that the integral equation for I must diverge is the same as in the subscriptless case discussed above. Thus this simple world cannot really satisfy our equations either. Nevertheless, the Cutkosky relations may provide a reasonably useful approximation for nearly degenerate multiplets.

The index i in $\Gamma_{i_1i_2i_3}$ and $\Delta_{i_1i_2}$ then stands for J, M, and a set of indices X as discussed before. This set X may be a discrete index, which could label, for example, all the different single-particle states in a given channel. In the case of infinitely rising trajectories, it might be identified with a daughter index. On the other hand, X may be as complicated as the field-theoretical indices which label all the states in Hilbert space. In this event, X would contain a label indicating the number of the stable particles of the field theory contained in the line, as well as momenta, spins, and whatever else is necessary to specify their configuration. A single line in our diagrams, then, will stand for varying number of lines in the conventional Feynman graphs. Our vertices with three of our lines will, for varying values of the indices, become amplitudes containing all possible numbers of field-theoretic external lines. Our Eqs. (3) and (4) will become a set of equations connecting all these field-theoretic amplitudes, and altogether, it is likely that everything is so complicated that our approach has provided little, if any, simplification.

A most attractive possibility is that the unknown part of the index will turn out to be connected to the quark model. This may go along the following lines: The index carries the quark information and a solution implies that a line must be either three quarks or quark and antiquark. A graph then implies that every quark or antiquark line coming from an external leg has to go to another external leg. We could then rewrite Eqs. (3)

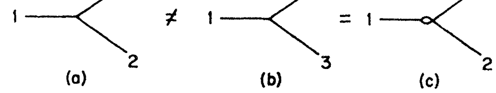

FIG. 14. Vertex graphs in FLD (Feynman-like diagram) case.

[11] C. Goebel, *Lectures in Theoretical Physics, Boulder, Colorado, 1967* (Gordon and Breach, New York, 1967), Vol. 9B.

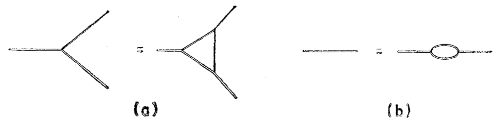

FIG. 15. $Z=0$ bootstrap graphs.

and (4) diagramatically, as shown in Figs. 11–13. In other words, X would do the job of keeping track of whether a line is a three-quark or a quark-antiquark line and stand for all that is necessary to define the line in terms of its quark content. The equations then say that the external quark lines can be connected in any manner as long as internal lines can be represented as quark-antiquark or three-quark lines, and also that internal quark loops can be added without altering the result. Note also that any selection rule satisfied by the lowest-order graphs is *a fortiori* also a selection rule of the entire amplitude; it is not necessary to argue that certain higher-order corrections are "small" in order to avoid violating such rules.[12]

To summarze, all we can say is that oversimplified assumptions for X will not be consistent with Eqs. (3) and (4); but whether there is a set X which is consistent with them and whether it is unique, we do not yet know.

The reader will by now have noticed some similarities to recent attempts[9] to construct, with so-called Feynman-like diagrams (FLD), unitary theories based on the Veneziano representation. A few remarks on the connection between these attempts and the approach under discussion here are appropriate at this point.

First, the FLD approach, just because it is based on the Veneziano representation, and because the Veneziano representation has only s-t and not s-u or t-u duality, must either accept the existence of resonances in exotic channels or else it must distinguish between planar and nonplanar diagrams. This is equivalent in our language to distinguishing between the order of indices in a vertex function: FLD must say that $\Gamma_{123}\neq\Gamma_{132}$. In terms of diagrams, FLD must say that the vertices shown in Figs. 14(a) and 14(b) are different. Evidently, such a distinction is entirely outside the spirit of conventional Feynman diagrams, and therefore, outside the spirit of what we are attempting to accomplish here.

In our approach, we must maintain the equality of the two vertices in Fig. 14; by the same token, our rules for calculation of a graph apply equally well to nonplanar as well as to planar graphs. Nevertheless, as we have already illustrated, we do not have to assume duality with exotic channels in order to reduce even nonplanar graphs to the simple lowest-order form, and the reason we can avoid this is precisely because our diagrams do *not* stand for simple Veneziano-like functions. The price we pay for this is, of course, that the FLD approach deals with explicit, known, functions as its input, while we do not know what our Γ and Δ are; indeed, we do not know if they exist.

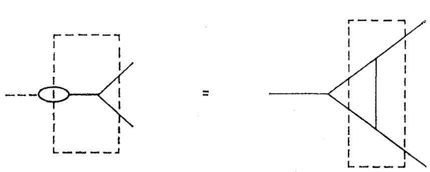

FIG. 16. Connection between duality and $Z=0$.

[12] P. G. O. Freund and R. J. Rivers, Phys. Letters **29B**, 510 (1969).

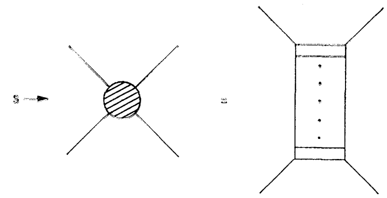

FIG. 17. Graph illustrating n-particle intermediate-state contributions to unitarity relation.

IV. DISCUSSION

Why is this theory attractive? First of all, it has a strong similarity to the field-theoretic formulation of the bootstrap hypothesis[2] based on the vanishing of well-defined renormalization constants Z. In fact, if we represent the conditions $Z_1\rightarrow 0$ and $Z_3\rightarrow 0$ pictorially by Figs. 15(a) and 15(b), then it follows that by putting 15(b) into 15(a) we obtain Fig. 16. Comparing the part of the diagrams inside the dashed rectangles, we see that duality results from the two $Z\rightarrow 0$ conditions.

This remark is not to be interpreted rigorously. Duality clearly depends on the nonconvergence of the perturbation series and therefore on the existence of an infinite series of particles of higher and higher spin. It is precisely for high-spin fields that we do not know how to define the field theory and the renormalization constants Z. We have, therefore, been unable to obtain our theory as the rigorous $Z\rightarrow 0$ limit of a Lagrangian (and therefore unitary) field theory. We can, at this stage, only point out the resemblance.

In any event, whether or not we can eventually make the connection with $Z=0$ precise, we have a theory which can be discussed on its own merits. The theory is evidently crossing symmetric, analytic, and is a bootstrap theory in the sense that there are no (obvious) free parameters. The only basic property which is not clearly present is unitarity. Intuitively, one can see that unitarity is plausible. All graphs are, after all, in a sense included; thus all the necessary multiparticle singularities in any amplitude are present. To be more specific, look at the amplitude for the two external particles to go into two other external particles. This can, as we have shown, be represented by an particular four-legged diagram; suppose we choose that shown in Fig. 17. Among the s-channel singularities of this diagram are those obtained by explicitly cutting the n

internal lines. Clearly the contribution of this particular singularity to the s-channel absorptive part is $T_{2\to n}\times T_{n\to 2}$; that is, precisely the n-body intermediate state contribution to the unitarity relation.

Thus it seems that the required singularities are present; what we cannot yet show is that *only* these are present.

Finally, if the situation regarding unitarity can be satisfactorily cleared up, we have defined a theory with all the desired properties of a true bootstrap theory. We have a well-defined set of equations, (3) and (4), for a set of vertex functions and propagators. They incorporate crossing and analyticity, and if they have solutions we have a prescription for calculating any n-legged amplitude. We then have here a set of exact bootstrap equations written in closed form, with which one can study questions of existence and uniqueness of solutions, and which present a basis for systematic approximations.

ACKNOWLEDGMENTS

We want to express our gratitude to the Aspen Center for Physics, where most of this work was done, and to our colleagues there, especially M. Gell-Mann and B. Sakita, for many interesting discussions.

PHYSICAL REVIEW D VOLUME 2, NUMBER 8 15 OCTOBER 1970

Broken Scale Invariance in Scalar Field Theory*

CURTIS G. CALLAN, JR.†
California Institute of Technology, Pasadena, California 91109
and
Institute for Advanced Study, Princeton, New Jersey 08540
(Received 4 June 1970)

We use scalar-field perturbation theory as a laboratory to study broken scale invariance. We pay particular attention to scaling laws (Ward identities for the scale current) and find that they have unusual anomalies whose presence might have been guessed from renormalization-group arguments. The scaling laws also appear to provide a relatively simple way of computing the renormalized amplitudes of the theory, which sidesteps the overlapping-divergence problem.

INTRODUCTION

THE perturbation theory of a self-interacting scalar field is about the simplest available model field theory, and a convenient laboratory for testing new ideas in strong-interaction physics. In this paper we shall be concerned with studying the concept of broken scale invariance within such a framework. We shall find that the model calls for some unexpected modifications of our ideas on broken scale invariance. At the same time, the approach suggested by broken scale invariance leads to an interesting, and simple, new approach to renormalization. We hope that this mutual illumination of two interesting questions justifies yet another paper on scalar field theory.

In Sec. I we shall review the general properties of scale invariance as a broken symmetry, leading up to the idea of a scaling law (the analog for scale invariance of PCAC low-energy theorems). In Sec. II we shall see how the general structure of renormalized perturbation theory constrains the allowable form of the scaling law and forces it to differ from naive expectations. In Sec. III we shall show how the existence of the scaling law leads to a simple prescription for computing the renormalized Green's functions of the theory. Finally, in Sec. IV, we shall demonstrate an interesting connection between the scaling law and the predictions of the renormalization group.

I. BROKEN SCALE INVARIANCE

In simple canonical field theories it is possible to introduce an acceptable energy-momentum tensor[1,2] $\Theta_{\mu\nu}$ having the following properties: (a) $\Theta=\Theta_\mu{}^\mu$ is proportional to those terms in the Lagrangian having dimensional coupling constants (such as mass terms); (b) the charge, $D=\int d^3x\, S_0$, formed from the current $S_\mu=\Theta_{\mu\nu}x^\nu$, acts as the generator of scale transformations,

$$[D(x_0),\phi(x)]=-i(d+x\cdot\partial)\phi(x)\,, \tag{1}$$

where d is the dimension of the field; (c) the current S_μ satisfies $\partial^\mu S_\mu=\Theta$ so that it is conserved when there are no dimensional coupling constants in the Lagrangian. With the help of the current S_μ and its equal-time commutation relations with fields, given above, one is able

* Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-68 and by the U. S. Air Force Office of Scientific Research under Contract No. AFOSR 70-1866.
† Alfred P. Sloan Foundation Fellow.

[1] C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N. Y.) 59, 42 (1970).
[2] M. Gell-Mann, University of Hawaii Summer School lectures, 1969 (unpublished).