Difficulties with three-dimensional weak solutions for inviscid incompressible flow

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The representation of an inviscid three-dimensional incompressible flow by vortex singularities is considered and shown to lead to dynamical inconsistencies.

I. THE TWO-DIMENSIONAL CASE

In two-dimensional rotational flow of an inviscid incompressible fluid, the representation of the flow by assemblies of line (point) vortices has proved to be a valuable concept, both qualitatively and often quantitatively, for understanding the structure and evolution of flows. One writes

$$\omega(\mathbf{r},t) = \sum_\alpha \kappa_\alpha\, \delta[\mathbf{r} - \mathbf{R}_\alpha(t)], \tag{1}$$

where δ is the two-dimensional delta function. This expression describes the velocity as a sum of delta functions of strength κ_α at points $\mathbf{R}_\alpha$. The associated velocity field is

$$\mathbf{q}(\mathbf{r},t) = \sum_\alpha \kappa_\alpha\, \mathbf{v}_\alpha + \mathbf{Q}(\mathbf{r},t), \tag{2}$$

where $\mathbf{Q}$ is an externally produced irrotational velocity field caused by the presence and movement of walls, etc., and

$$\mathbf{v}_\alpha(\mathbf{r},t) = \frac{\mathbf{k} \wedge (\mathbf{r} - \mathbf{R}_\alpha)}{2\pi|\mathbf{r} - \mathbf{R}_\alpha|^2}. \tag{3}$$

Here, $\mathbf{k}$ denotes the unit vector parallel to the vorticity and normal to the plane of the flow. The vortices move according to the equations

$$\frac{d\kappa_\alpha}{dt} = 0, \quad \frac{d\mathbf{R}_\alpha}{dt} = \sum_\beta{}' \kappa_\beta\, \mathbf{v}_\beta(\mathbf{r}_\alpha) + \mathbf{Q}(\mathbf{R}_\alpha) = \mathbf{q}_\alpha, \tag{4}$$

say, where the prime denotes that the term with $\alpha = \beta$ is omitted in the sum.

The governing equation for a piecewise smooth two-dimensional vorticity distribution is

$$\frac{\partial\omega}{\partial t} + \mathbf{q}\cdot\nabla\omega = 0. \tag{5}$$

Then the ω and $\mathbf{q}$ fields given by (1) and (2), evolving according to Eq. (4), constitute a weak solution of (5) in the sense that

$$\int f(\mathbf{r}) \left(\frac{\partial\omega}{\partial t} + \mathbf{q}\cdot\nabla\omega\right) d\mathbf{r} = 0 \tag{6}$$

for all smooth f, with the integral evaluated according to the rules for manipulating generalized functions.[1] To see this, we note that

$$\frac{\partial\omega}{\partial t} = -\sum_\alpha \kappa_\alpha \left(\frac{d\mathbf{R}_\alpha}{dt}\cdot\nabla\right)\delta_\alpha + \sum_\alpha \frac{d\kappa_\alpha}{dt}\delta_\alpha \tag{7}$$

and

$$\mathbf{q}\cdot\nabla\omega = \sum_\alpha \kappa_\alpha\,(\mathbf{q}_\alpha\cdot\nabla)\delta_\alpha + \sum_\alpha \kappa_\alpha\,\mathbf{v}_\alpha\cdot\nabla\delta_\alpha. \tag{8}$$

Here, δ_α denotes $\delta[\mathbf{r} - \mathbf{R}_\alpha(t)]$. The second term on the right-hand side of (8) vanishes; actually this is not immediately obvious and depends on the disappearance of

$$\int \delta(x_1,x_2) \left(\frac{x_1(\partial f/\partial x_2) - x_2(\partial f/\partial x_1)}{x_1^2 + x_2^2}\right) dx_1\, dx_2 \tag{9}$$

for arbitrary smooth f, which follows from a polar coordinate representation. Equations (4) then imply that (6) vanishes everywhere, including $\mathbf{r} = \mathbf{R}_\alpha$.

The fact that point vortices constitute a weak solution makes them worthy of study as a dynamical system in their own right, and not just as a particular example of the so-called vortex methods for constructing approximations to solutions of the Euler equations by replacing the vorticity field with a sum of invariant, in general overlapping, vortex patches. Physically interesting results can be obtained from studies of a few (perhaps periodic) point vortices; e.g., the Karman vortex street.

II. VORTONS

Attempts have been made to extend the weak solution approach to three-dimensional inviscid rotational incompressible flow. Saffman[2] (the idea was described in a proposal letter from Saffman to Laccabue of Control Data Corporation in January 1977) suggested a representation as a sum of vortex monopoles

$$\boldsymbol{\omega}(\mathbf{r},t) = \sum_\alpha \boldsymbol{\kappa}_\alpha(t)\delta[\mathbf{r} - \mathbf{R}_\alpha(t)] = \sum_\alpha \boldsymbol{\omega}_\alpha. \tag{10}$$

The term "vorton" was coined for these singularities. (We are grateful to Professor N. Zabusky for pointing out that he had earlier used the term vorton to describe a pair of two-dimensional vortices of finite area, see Scott and Lonngren.[3]) The velocity field is

$$\mathbf{q}(\mathbf{r},t) = \sum_\beta \frac{\boldsymbol{\kappa}_\beta \wedge (\mathbf{r} - \mathbf{R}_\beta)}{4\pi|\mathbf{r} - \mathbf{R}_\beta|^3} + \mathbf{Q}(\mathbf{r},t)$$
$$= \frac{\boldsymbol{\kappa}_\alpha \wedge (\mathbf{r} - \mathbf{R}_\alpha)}{4\pi|\mathbf{r} - \mathbf{R}_\alpha|^3} + \mathbf{q}_\alpha(\mathbf{r},t), \tag{11}$$

say, where $\mathbf{q}_\alpha$ is the contribution to the velocity field from all the vortons, except the one at $\mathbf{R}_\alpha$, plus the external irrotational velocity field. Representation (11) is what one obtains from an elementary discretization of the representation expressing the velocity as a volume integral of the vorticity field.

We now investigate whether (10) and (11) constitute a weak solution of the three-dimensional vorticity equation

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$$\frac{\partial \omega}{\partial t} + (\mathbf{q}\cdot\nabla)\omega - (\omega\cdot\nabla)\mathbf{q} = 0 \quad (12)$$

if

$$\frac{d\mathbf{R}_\alpha}{dt} = \mathbf{q}_\alpha(\mathbf{R}_\alpha), \quad \frac{d\boldsymbol{\kappa}_\alpha}{dt} = (\boldsymbol{\kappa}_\alpha\cdot\nabla)\mathbf{q}_\alpha(\mathbf{R}_\alpha). \quad (13)$$

Since $\mathbf{q}_\alpha$ is smooth in the neighborhood of $\mathbf{R}_\alpha$, it is obvious that (13) implies that

$$\frac{\partial \omega_\alpha}{\partial t} + (\mathbf{q}_\alpha\cdot\nabla)\omega_\alpha - (\omega_\alpha\cdot\nabla)\mathbf{q}_\alpha = 0 \quad (14)$$

at all points including $\mathbf{r} = \mathbf{R}_\alpha$. It therefore remains to consider whether

$$[(\mathbf{q} - \mathbf{q}_\alpha)\cdot\nabla]\omega_\alpha - (\omega_\alpha\cdot\nabla)(\mathbf{q} - \mathbf{q}_\alpha) = 0 \quad (15)$$

at $\mathbf{R}_\alpha$.

In order to investigate this question, we take local coordinates and use a suffix notation (dropping the label α). Then the condition for (15) to vanish is equivalent, after an integration by parts, to the disappearance of the following integral for arbitrary smooth f:

$$-\int \delta(\mathbf{x})\left(\epsilon_{ijk}\,\kappa_j\,\kappa_p\,f\frac{\partial}{\partial x_p}\frac{x_k}{r^3} + \epsilon_{pjk}\,\kappa_i\,\kappa_j\,\frac{\partial}{\partial x_p}\frac{f x_k}{r^3}\right)d\mathbf{x}. \quad (16)$$

Expanding f as a Taylor series, it follows from the symmetries that only even powers of the coordinates need to be considered. It is easy to see that the contribution from the constant term vanishes and only the quadratic terms in f need to be considered explicitly. (Quartic and higher order terms vanish sufficiently rapidly to cancel the delta function singularity.) Substituting $f = f_{qr}\,x_q\,x_r$ and carrying out the integrals over the angles in a spherical polar representation, and using the result that

$$\int x_i\,x_j\,x_k\,x_l\,dS \propto \delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}, \quad (17)$$

it is found that (16) is proportional to

$$f_{qr}\,\epsilon_{ijq}\,\kappa_r\,\kappa_j + f_{qr}\,\epsilon_{ijr}\,\kappa_q\,\kappa_j, \quad (18)$$

which does not vanish in general. Thus we do not have a weak solution in three dimensions.

A further difficulty is the fact that the field (10) does not satisfy the condition

$$\operatorname{div}\omega = 0. \quad (19)$$

Thus the vorticity evolution equation (12) is not equivalent to the alternative form

$$\frac{\partial \omega}{\partial t} + \operatorname{curl}(\omega \wedge \mathbf{q}) = 0. \quad (20)$$

Equivalently, expressions (10) and (11) violate the definition

$$\omega = \operatorname{curl}\mathbf{q}. \quad (21)$$

Novikov[4] replaced (10) by

$$\omega(\mathbf{r},t) = \sum_\alpha \left[\boldsymbol{\kappa}_\alpha\,\delta_\alpha + \frac{1}{4\pi}\nabla\left(\frac{\boldsymbol{\kappa}_\alpha\cdot(\mathbf{r} - \mathbf{R}_\alpha)}{|\mathbf{r} - \mathbf{R}_\alpha|^3}\right)\right]. \quad (22)$$

This ensures that (19) is satisfied and removes the problem of nonequivalence. However, the evolution equations (13) were still retained and the failure to satisfy conditions for existence of a weak solution is unaltered. In fact, the situation is now worse, as with (22) the equations of motion (14) or (20) are now not satisfied away from the singularities, i.e., they are violated for all $\mathbf{r}$, whereas with the form (10) the equations are satisfied trivially except at the singularities.

The fact that the vorton representation, either (10) or Novikov's modification (22), does not constitute a weak solution leads to violation of the invariance of vorticity and hydrodynamic impulse (momentum). In unbounded fluid, there are three linear invariants associated with conservation of vorticity and of linear and angular impulse, namely,

$$\boldsymbol{\Omega} = \int \omega\,dV, \quad (23)$$

$$\mathbf{I} = \frac{1}{2}\int \mathbf{r} \wedge \omega\,dV, \quad (24)$$

$$\mathbf{A} = -\frac{1}{2}\int r^2\,\omega\,dV. \quad (25)$$

There are also two quadratic invariants associated with conservation of kinetic energy

$$E = \frac{1}{2}\int \mathbf{q}^2\,dV = \int (\mathbf{q} \wedge \mathbf{r}\cdot\omega)dV, \quad (26)$$

and the conservation of helicity

$$J = \int \mathbf{q}\cdot\omega\,dV. \quad (27)$$

In terms of the vorton representation (10), the three linear invariants become

$$\boldsymbol{\Omega} = \sum_\alpha \boldsymbol{\kappa}_\alpha, \quad (28)$$

$$\mathbf{I} = \frac{1}{2}\sum_\alpha \mathbf{R}_\alpha \wedge \omega_\alpha, \quad (29)$$

$$\mathbf{A} = -\frac{1}{2}\sum_\alpha \mathbf{R}_\alpha^2\,\boldsymbol{\kappa}_\alpha. \quad (30)$$

With the modified representation (22), the integrals in (23)–(25) either diverge at infinity or are at best conditionally convergent.

In any case, when the vortons move according to Eq. (13), with $\boldsymbol{\Omega} = 0$ at the initial instant, it is easy to see by direct evaluation that $\boldsymbol{\Omega}$, $\mathbf{I}$, and $\mathbf{A}$ are not conserved in general. This can be demonstrated explicitly for two vortons. Here,

$$\frac{d}{dt}(\boldsymbol{\kappa}_\alpha + \boldsymbol{\kappa}_\beta) = (\boldsymbol{\kappa}_\alpha\cdot\nabla)\frac{\boldsymbol{\kappa}_\beta \wedge \mathbf{R}}{R^3} - (\boldsymbol{\kappa}_\beta\cdot\nabla)\frac{\boldsymbol{\kappa}_\alpha \wedge \mathbf{R}}{R^3}, \quad (31)$$

where $\mathbf{R} = \mathbf{R}_\alpha - \mathbf{R}_\beta$, and this is clearly nonzero in general, even if $\boldsymbol{\kappa}_\alpha + \boldsymbol{\kappa}_\beta = 0$ at $t = 0$. The noninvariance of $\mathbf{I}$ and $\mathbf{A}$ can be shown similarly, but requires more algebra.

The energy is not defined, unless an infinite self-energy is subtracted. Calculating the energy from the square of the velocity field, with the self-energy of the vortons neglected, gives

$$E = \frac{1}{8\pi} \sum_{\alpha \neq \beta} \frac{1}{|\mathbf{R}_\alpha - \mathbf{R}_\beta|} [\boldsymbol{\kappa}_\alpha \cdot \boldsymbol{\kappa}_\beta - \boldsymbol{\kappa}_\alpha \cdot (\mathbf{R}_\alpha - \mathbf{R}_\beta)\, \boldsymbol{\kappa}_\beta \cdot (\mathbf{R}_\alpha - \mathbf{R}_\beta)/|\mathbf{R}_\alpha - \mathbf{R}_\beta|^2]. \quad (32)$$

For details of the calculation, see Novikov.[4] As pointed out by Novikov, this is not a conserved quantity under the evolution equations (13). Substitution of the representation (10) into the second integral in (26) gives a different but similar expression, the difference resulting from the fact that (10) violates (19).

The self-helicity is zero, and substitution of (10) and (11) into (27) gives

$$J = \frac{1}{4\pi} \sum_{\alpha \neq \beta} \boldsymbol{\kappa}_\alpha \wedge \boldsymbol{\kappa}_\beta \cdot (\mathbf{R}_\alpha - \mathbf{R}_\beta)/|\mathbf{R}_\alpha - \mathbf{R}_\beta|^3. \quad (33)$$

This is also not conserved.

Invariance can of course be produced by imposing symmetries, e.g., the vortons could be all parallel and perpendicular to their separations (this is a quasi-two-dimensional situation). There is numerical evidence that the invariants are conserved in the limit as the number N of vortons goes to infinity,[5] and this agrees with the idea that having a large number of vortons will be equivalent to a numerical discretization of the evolution equations of a smooth vorticity distribution. A referee has pointed out that using vortons lying contiguously along a curve, which is equivalent to modeling the motion of a vortex filament by an elementary discretization of the Biot–Savart law, may not be too inaccurate for values of N of order 10. We do not discount the possibility that reasonably accurate solutions of the Euler equations may be obtained for some configurations with relatively few vortons.

But if the interest lies in the possibility of representing, with at least qualitative accuracy as can be and often is done in two dimensions, the dynamics of a general three-dimensional rotational incompressible velocity field (e.g., the motion of coherent structures) by a system with a few degrees of freedom, and this is the hope for applications of recent work in dynamical systems to turbulent flow, then it appears that the vorton representation is seriously flawed.

III. DIPOLES

Another possibility is to represent the vorticity field as a weak solution by writing the velocity as that due to a sum of dipoles of strength γ_α at points $\mathbf{R}_\alpha$. The vorticity is then written

$$\boldsymbol{\omega} = -\sum_\alpha \gamma_\alpha \wedge \nabla\delta_\alpha = \sum_\alpha \operatorname{curl}(\gamma_\alpha\, \delta_\alpha), \quad (34)$$

and the velocity field is

$$\mathbf{q} = \sum_\alpha \left\{ \gamma_\alpha\, \delta_\alpha + \nabla \left[\frac{\gamma_\alpha}{4\pi} \cdot \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{R}_\alpha|} \right) \right] \right\} + \mathbf{Q}(\mathbf{r},t). \quad (35)$$

A physical interpretation of this representation can be given in terms of the impulsive force density $\mathbf{F}$ required to generate the motion from rest. We can write

$$\mathbf{q} = \mathbf{F} + \nabla P, \quad \operatorname{curl} \mathbf{F} = \boldsymbol{\omega}, \quad (36)$$

where $\mathbf{F}$ can be chosen to be of finite extent when $\boldsymbol{\omega}$ is of finite extent. Then γ_α is the strength of the sources of impulsive force. It is readily verified that $\operatorname{div} \boldsymbol{\omega} = 0$, $\operatorname{div} \mathbf{q} = 0$, and $\boldsymbol{\omega} = \operatorname{curl} \mathbf{q}$.

The evolution equations are assumed to be

$$\frac{d\gamma_\alpha}{dt} = (\gamma_\alpha \cdot \nabla)\mathbf{q}_\alpha, \quad \frac{d\mathbf{R}_\alpha}{dt} = \mathbf{q}_\alpha. \quad (37)$$

It is easily verified that we now have a weak solution of either (12) or (20), *provided the self-interaction terms are neglected*. But this is an unreasonable restriction, because a dipole is an infinitesimal vortex ring and moves with infinite speed. The velocity of the ring can be reduced to zero by adding a vorticity field corresponding to a distribution of swirl (e.g., Moore and Saffman[6]), but we have not been able to construct a swirl velocity field for which the singularity does not have a definite structure and constitutes a weak solution in the sense of (6).

Proceeding nevertheless, we have by construction that

$$\boldsymbol{\Omega} = 0. \quad (38)$$

Further, a little algebra shows that

$$\frac{d\mathbf{I}}{dt} = 0, \quad (39)$$

i.e., linear impulse is conserved. However, angular impulse is not conserved in general. For example, for two dipoles,

$$\frac{d\mathbf{A}}{dt} = 6\mathbf{R} \wedge \gamma_\alpha(\mathbf{R}\cdot\gamma_\beta) + 6\mathbf{R} \wedge \gamma_\beta(\mathbf{R}\cdot\gamma_\alpha) \neq 0. \quad (40)$$

Thus this representation also violates a fundamental conservation requirement.

IV. CONCLUDING REMARKS

The conclusion to be drawn at present from this study is that a dynamically consistent representation of three-dimensional rotational velocity fields by a small number of vortex singularities is not at present a viable possibility. It may work in two dimensions, but unfortunately not in three dimensions. Applications[7] which use only a few vortons are therefore open to question. It may be argued that failure to conserve energy represents the formation of inviscid singularities, but the failure to conserve vorticity and momentum when only a few vortons are used and symmetries are not imposed is a serious objection.

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