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Duality of the fermionic 2d black hole and $\mathcal{N} = 2$ Liouville theory as mirror symmetry

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Abstract: We prove the equivalence of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki model, which is a fermionic generalization of the 2d Black Hole, and $\mathcal{N} = 2$ Liouville theory. We show that this duality is an example of mirror symmetry. The essential part of the derivation is to realize the fermionic 2d Black Hole as the low energy limit of a gauged linear sigma-model. Liouville theory is obtained by dualizing the charged scalar fields and taking into account the vortex-instanton effects, as proposed recently in non-dilatonic models. The gauged linear sigma-model we study has many useful generalizations which we briefly discuss. In particular, we show how to construct a variety of dilatonic superstring backgrounds which generalize the fermionic 2d Black Hole and admit a mirror description in terms of Toda-like theories.

Keywords: Field Theories in Lower Dimensions, Supersymmetry and Duality, Conformal Field Models in String Theory.

Contents

1. Introduction

The mirror dual of an $\mathcal{N} = 2$ supersymmetric non-linear sigma-model on a toric variety has been derived in [1] by realizing the model as the low energy limit of a gauged linear sigma-model [2], and dualizing the phases of charged scalar fields. This can be viewed as T-duality applied to the fibers of a torus fibration. When a circle fiber shrinks to zero size at some locus of the base, one could naïvely expect that the dual circle blows up at the same locus. What really happens is the following. To each such degenerating fiber there corresponds a superpotential term, generated by the vortex-instanton of the gauge system (analogously to [3]), that diverges toward the degeneration locus. The superpotential also breaks the rotational symmetry of the dual theory, accounting for the loss of winding number in the original system due to the degeneration of the circle. This is the story for $(2,2)$ supersymmetric non-dilatonic sigma-models on toric manifolds, but it would be interesting to see how universal this phenomenon is.

Some time ago, Fateev, Zamolodchikov and Zamolodchikov (FZZ) [4] conjectured a duality between the conformal field theory of a two-dimensional euclidean black hole [5] and a Landau-Ginzburg theory, called *sine-Liouville theory*. The 2d Black Hole is defined as the level k $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset model and has the following target-space metric and dilaton for large k

$$\begin{aligned} \mathrm{d}s^2 &= k[\mathrm{d}\rho^2 + \tanh^2\!\rho\, \mathrm{d}\varphi^2]\,, \\ \Phi &= \Phi_0 - 2\log\cosh\rho\,. \end{aligned} \tag{1.1}$$

Here φ is a periodic variable of period 2π. The coset theory is well-defined for all $k > 2$. On the other hand, the sine-Liouville theory is a theory of scalar fields $-\infty < \varrho < \infty$ and $\vartheta \equiv \vartheta + 2\pi$ with the following action

$$\widetilde{S} = \frac{1}{4\pi}\int \left[\frac{1}{k-2}(\mathrm{d}\varrho)^2 + \frac{1}{k}(\mathrm{d}\vartheta)^2 - \frac{1}{k-2}R_h\varrho + \mu^2\,\mathrm{e}^{-\varrho}\cos\vartheta\right]\sqrt{h}\mathrm{d}^2x\,, \tag{1.2}$$

where h is the world-sheet metric (with Ricci scalar R_h) and μ is some mass scale. We refer the reader to [6] for a review of this conjectural duality. The duality was used in [6] as the starting point for the Matrix Model formulation of string theory in the black hole background.

The 2d Black Hole has an asymptotic region, $\rho \to +\infty$, where the geometry is that of a cylinder of radius $\sqrt{k}$ and the dilaton is linear, $\Phi \sim -2\rho$. At $\rho = 0$ the circle shrinks to zero size, and therefore the overall geometry is that of a semi-infinite cigar. Sine-Liouville theory also has an asymptotic region, $\varrho \to \infty$, where the potential is exponentially small and the theory is the sigma-model on a cylinder of radius $1/\sqrt{k}$ with a linear dilaton $\widehat{\Phi} \sim -\varrho/(k-2)$. Note that the sine-Liouville potential is unbounded from below, and therefore for small k, where the radius of the cylinder is large and semiclassical reasoning is valid, we expect the model to be unstable. This corresponds to the fact that the coset model is well-defined only for $k > 2$.

If we compare the radii of the two asymptotic regions, we notice that the two theories may be related by T-duality. The shrinking of the circle as one goes towards $\rho = 0$ on the 2d Black Hole side corresponds to the exponentially growing potential which breaks rotational symmetry on the sine-Liouville side. Thus FZZ duality is strongly reminiscent of mirror duality between $(2,2)$ sigma-models and $(2,2)$ Landau-Ginzburg models mentioned above.

In this paper, we prove the supersymmetric version of FZZ duality using the method of [1]. Instead of a 2d Black Hole we consider a fermionic 2d Black Hole, defined as the level k $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ Kazama-Suzuki supercoset model [7], and instead of sine-Liouville theory we consider $\mathcal{N} = 2$ supersymmetric Liouville theory [8]. This duality was conjectured in [9] from the space-time point of view; closely related ideas were discussed earlier in [10, 11, 12], and the duality was studied more recently in [13]. The supercoset model can be viewed as an $\mathcal{N} = 1$ supersymmetric sigma-model with target-space metric (1.1). The action for $\mathcal{N} = 2$ Liouville theory on a flat world-sheet is given by

$$\widetilde{S} = \frac{1}{2\pi}\int \mathrm{d}^2x \left[\int \mathrm{d}^4\theta \frac{1}{2k}|Y|^2 + \frac{1}{2}\left(\int \mathrm{d}^2\theta\, \mu\, \mathrm{e}^{-Y} + h.c.\right)\right], \tag{1.3}$$

where Y is a chiral superfield with period $2\pi i$ and μ is a mass scale. (A linear dilaton is hidden in this action.) As in the bosonic case, the two theories have asymptotic regions that are related by T-duality, and the shrinking of the circle on one side corresponds to growing superpotential breaking rotational symmetry on the other side. Unlike in the bosonic case, the supercoset theory is well-defined for all $k > 0$. This corresponds to the fact that $\mathcal{N} = 2$ Liouville theory makes sense for all $k > 0$.

The crucial part of our proof is showing that the $(2,2)$ superconformal field theory of the fermionic 2d Black Hole arises as the infrared limit of a certain super-renormalizable gauge theory. The candidate system is the U(1) gauge theory with two chiral superfields Φ and P on which the gauge transformation acts as $\Phi \to \mathrm{e}^{i\alpha}\Phi$ and $P \to P + i\alpha$. The action is

$$S = \frac{1}{2\pi}\int \mathrm{d}^2x\, \mathrm{d}^4\theta \left[\overline{\Phi}\, \mathrm{e}^V \Phi + \frac{k}{4}(P + \overline{P} + V)^2 - \frac{1}{2e^2}|\Sigma|^2\right]. \tag{1.4}$$

We will first give some numerical evidence. We will show that the sigma-model that arises after integrating out the gauge multiplet flows under one-loop renormalization group flow to the supersymmetric sigma-model with target-space metric (1.1). We will explicitly see how the linear dilaton in the asymptotic region is generated. The one-loop approximation is valid for large k. To go beyond this approximation, we compute the infrared central charge of the above gauged linear sigma-model (GLSM).

Following [14, 15], we identify the right-moving $\mathcal{N} = 2$ superconformal algebra in the ring of left-chiral operators. The classical gauge theory (1.4) has both vector and axial R-symmetries, but on the quantum level the axial R-symmetry is anomalously broken. However, one can modify the current using the field P to make it conserved. This allows us to identify the right-moving R-current, and then the full $\mathcal{N} = 2$ superconformal algebra. The correction terms in the superconformal currents are linear in P and generate linear dilaton in the asymptotic region. (Alternatively, one can obtain the whole current superfield by cancelling the Konishi anomaly [16] associated to the axial anomaly.) We find that the central charge is

$$c = 3\left(1 + \frac{2}{k}\right), \tag{1.5}$$

which coincides with the central charge of the level k $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ Kazama-Suzuki model. The asymptotic behavior of the target-space metric also agrees in the two theories. Then we argue the uniqueness of the SCFT with this value of the central charge, asymptotic behaviour, and symmetries. This establishes that our gauge theory (1.4) flows to the fermionic 2d Black Hole for all $k > 0$.

Since the UV central charge of the GLSM is 9, and the IR central charge (1.5) becomes arbitrarily large as $k \to 0$, one may wonder how these results are consistent with Zamolodchikov's c-theorem [17]. The resolution of this apparent paradox is well known [18, 19]. Technically, the c-theorem is not applicable here because the IR conformal field theory violates one of the assumptions made in [17], namely the assumption that there exists a normalizable $\mathrm{SL}(2,\mathbb{C})$-invariant vacuum state. A more satisfactory explanation is that in general the central charge is not a good measure of the number of degrees of freedom. For example, if one does not assume normalizability of vacuum, Cardy's formula [20] says that the growth of the density of states is determined not by c, but by $c_{eff} = c - 24h_{\min}$, where $h_{\min}$ is the lower boundary of the spectrum of L_0.[1] If a unitary CFT has a normalizable vacuum, then $h_{\min} = 0$, but in general the effective number of degrees of freedom is different from c. For the supercoset model $h_{\min} = \frac{1}{4k}$ (this can be derived either by using the fact that the supercoset is asymptotic to a linear dilaton theory with background charge $Q = 1/\sqrt{k}$ and applying the Seiberg bound [18], or by the direct analysis of the operator spectrum), and therefore $c_{eff} = 3$. Thus the effective number of degrees of freedom decreases as one flows towards the infrared, in agreement with expectations.

Once the flow to the fermionic 2d Black Hole is established, the rest is a straightforward generalization of [1]. Dualizing the phase of Φ and the imaginary part of P, we obtain twisted chiral superfields Y and Y_P of period $2\pi i$. The superpotential of the dual system is

$$\widetilde{W} = \Sigma(Y + Y_P) + \mu\, \mathrm{e}^{-Y}\,, \tag{1.6}$$

[1]We assume that world-sheet parity is a symmetry of the theory. Otherwise $h_{\min}$ is defined as the smaller of the lower boundaries of the spectra of L_0 and $\tilde{L}_0$.

where the term linear in Σ is present already at the classical level, and the exponential term is generated by the vortex of Φ. Note that the P-vortex is absent, and therefore no nonperturbative superpotential is generated for Y_P. The Kähler potential is

$$K = -\frac{1}{2e^2}|\Sigma|^2 - \frac{1}{2k}|Y_P|^2 + \cdots , \tag{1.7}$$

where dots denote a possible correction term that vanishes in the asymptotic region $\mathrm{Re}\, Y \to \infty$. In the infrared limit $e \to \infty$ it is appropriate to integrate out Σ, and this gives a constraint $Y + Y_P = 0$. Thus, we obtain a theory of a single periodic chiral superfield Y with the superpotential e^{-Y}. Using the uniqueness of the supersymmetric coset, one can show that the corrections to the Kähler potential indicated by dots in (1.7) are in fact absent. Note that in general the methods of [1] do not allow to control the Kähler potential. What makes the present case different is that one can continuously deform the gauge theory (1.4) to the $\mathcal{N} = 2$ Liouville theory without breaking any symmetries. Since the supersymmetric coset is rigid, this implies that the infrared limit of the theory (1.4) is equivalent to $\mathcal{N} = 2$ Liouville theory. This alternative way of deriving the mirror dual is less general than that used in [1], but provides more information about the dual theory.

We also describe some obvious generalizations of the model (1.4), compute their infrared central charge and find mirror duals. Some of these models flow to non-trivial $(2, 2)$ superconformal field theories and can be used to construct a variety of higher-dimensional superstring backgrounds with a non-constant dilaton and fermionic symmetries. Others are massive field theories which upon integrating out the gauge fields reduce to sigma-models on "squashed" toric varieties. Mirror symmetry relates these sigma-models to Landau-Ginzburg models; for example, the sigma-model on a "squashed" $\mathbb{CP}^1$ (the supersymmetric "sausage model") is mirror to the $\mathcal{N} = 2$ sine-Gordon model with a finite Kähler potential. In fact, in this particular case both theories are integrable, and their equivalence has been conjectured by Fendley and Intriligator [21]. (The squashed toric sigma models and the mirrors are also introduced and studied from a different but related point of view in [22].)

2. The gauged linear sigma-model

The field content of the gauged linear sigma-model will be the following: two chiral superfields Φ and P and a vector superfield V. Our superfield conventions are collected in appendix A. The gauge transformations laws are defined to be

$$\begin{aligned} \Phi &\to \mathrm{e}^{i\Lambda}\Phi, \\ P &\to P + i\Lambda, \\ V &\to V - i\Lambda + i\overline{\Lambda}\,, \end{aligned} \tag{2.1}$$

where Λ is a chiral superfield, $\overline{D}_+\Lambda = \overline{D}_-\Lambda = 0$. We take the gauge group to be U(1), and $\mathrm{Im}\, P$ is periodically identified with period 2π.

The action of the system is

$$S = \frac{1}{2\pi}\int \mathrm{d}^2x\, \mathrm{d}^4\theta \left[\overline{\Phi}\, \mathrm{e}^V \Phi + \frac{k}{4}(P+\overline{P}+V)^2 - \frac{1}{2e^2}|\Sigma|^2\right]. \tag{2.2}$$

Here $\Sigma = \overline{D}_+ D_- V$ is a twisted chiral superfield, $\overline{D}_+\Sigma = D_-\Sigma = 0$. We did not include the Fayet-Iliopoulos term as it can be absorbed into P. Neither did we include its superpartner, the theta-angle, since it breaks world-sheet parity, while we want the theory to flow to a parity-invariant supercoset model (see appendix D for details about the definition of world-sheet parity for the coset models).

The chiral superfield P can be gauged away completely, after which one is left with Φ and a massive vector superfield described by V. Thus the action (2.2) describes massive $\mathcal{N}=2$ QED. Alternatively, one can choose the Wess-Zumino gauge for V and retain P. Then the action in terms of component fields reads

$$\begin{aligned}
\frac{1}{2\pi}\int \mathrm{d}^2x\, \Big[& -\mathcal{D}^\mu\overline{\phi}\mathcal{D}_\mu\phi + i\overline{\psi}_-(\mathcal{D}_0+\mathcal{D}_1)\psi_- + i\overline{\psi}_+(\mathcal{D}_0-\mathcal{D}_1)\psi_+ + D|\phi|^2 + |F|^2 + \\
& - |\sigma|^2|\phi|^2 - \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_- - i\overline{\phi}\lambda_-\psi_+ + i\overline{\phi}\lambda_+\psi_- + i\overline{\psi}_+\overline{\lambda}_-\phi - \\
& - i\overline{\psi}_-\overline{\lambda}_+\phi + \frac{k}{2}\Big(-\mathcal{D}^\mu\overline{p}\mathcal{D}_\mu p + i\overline{\chi}_-(\partial_0+\partial_1)\chi_- + i\overline{\chi}_+(\partial_0-\partial_1)\chi_+ + \\
& \qquad + D(p+\overline{p}) + |F_P|^2 - |\sigma|^2 + i\chi_+\lambda_- - i\chi_-\lambda_+ + \\
& \qquad + i\overline{\chi}_+\overline{\lambda}_- - i\overline{\chi}_-\overline{\lambda}_+\Big) + \qquad\qquad (2.3)\\
& + \frac{1}{2e^2}\Big(-\partial^\mu\overline{\sigma}\partial_\mu\sigma + i\overline{\lambda}_-(\partial_0+\partial_1)\lambda_- + i\overline{\lambda}_+(\partial_0-\partial_1)\lambda_+ + v_{01}^2 + D^2\Big)\Big].
\end{aligned}$$

Here ϕ and p are the lowest components of Φ and P, respectively, ψ and χ are their superpartners, and v_μ, λ, and D are components of a vector multiplet in the Wess-Zumino gauge. $\mathcal{D}_\mu\phi$ and $\mathcal{D}_\mu\psi_\pm$ are the standard covariant derivatives, while $\mathcal{D}_\mu p := \partial_\mu p + iv_\mu$. After one gauges away the imaginary part of p, one can see that the gauge field and its superpartners have mass $e\sqrt{k}$.

This field theory is free in the UV and super-renormalizable. We are interested in its infrared limit. At energies much lower than $e\sqrt{k}$ one can integrate out Σ and set the D-term potential to zero. The D-term is given by

$$D(\phi,p) = |\phi|^2 + k\,\mathrm{Re}\, p\,.$$

To obtain the low-energy effective action for Φ we set $\mathrm{Im}\, p = 0$ (this is a gauge choice), express $\mathrm{Re}\, p$ in terms of ϕ by means of $D(\phi,p)=0$, and integrate out V omitting the last term in the action (because the infrared limit is equivalent to taking $e\to\infty$). Equivalently, we can take the flat space parametrized by ϕ and p with Kähler potential

$$K(\phi,p) = |\phi|^2 + \frac{k}{2}|p|^2\,,$$

and compute its Kähler quotient with respect to the action of U(1) given by

$$\phi \to \phi e^{i\lambda}, \quad p \to p + i\lambda\,.$$

Either way, one concludes that the low-energy theory is described by a supersymmetric non-linear sigma-model with the following target space metric:

$$ds^2 = \left(1 + \frac{r^2}{k}\right) dr^2 + \frac{r^2}{1 + r^2/k} d\theta^2\,. \tag{2.4}$$

Here $r = \sqrt{2}|\phi| \in [0, +\infty)$, $\theta = \arg\phi \in \mathbb{R}/(2\pi\mathbb{Z})$. This metric is smooth near the origin $r = 0$, while for $r \to \infty$ it approaches a flat metric on a cylinder of circumference $2\pi\sqrt{k}$. Thus it describes a cigar, i.e. a 2d Riemannian manifold diffeomorphic to $\mathbb{R}^2$ with a metric which has a U(1) isometry and asymptotes to a flat metric on a cylinder.

The metric (2.4) is different from the usual 2d Black Hole metric [5]. If one sets $r = \sqrt{k}\sinh\rho$, the metric (2.4) takes the form

$$ds^2 = k\left(\cosh^4\rho\ d\rho^2 + \tanh^2\rho\ d\theta^2\right), \tag{2.5}$$

while the 2d Black Hole metric is

$$ds^2 = k\left(d\rho^2 + \tanh^2\rho\ d\theta^2\right). \tag{2.6}$$

Qualitatively, the difference between the two metrics is the following. Let us define a natural "radial" variable $v \in [0, +\infty)$ by

$$v(\rho) = \int_0^\rho \sqrt{g_{\rho\rho}(\rho)}d\rho\,. \tag{2.7}$$

In terms of v, θ any cigar-like metric has the form

$$ds^2 = dv^2 + F^2(v)\ d\theta^2$$

for some function $F(v)$ which approaches a constant for $v \to \infty$. For our metric the difference $F(v) - \sqrt{k}$ is of order $1/v$ for large v, while for the 2d Black Hole it is exponentially small.

The metric (2.5) does not define a conformal field theory, and flows in a non-trivial way under the renormalization group. We will show that the end-point of the flow is the fermionic 2d Black Hole.

Let us conclude this section by listing the symmetries of the action (2.2). Classically, we have $(2,2)$ supersymmetry, axial and vector R-symmetries (such that the lowest components of Φ and P have zero R-charges), world-sheet parity, and a global non-R symmetry which which shifts $\mathrm{Im}\, p$ by a constant and leaves all other fields invariant. The generator of the latter symmetry will be called momentum, since the corresponding symmetry in the low energy nonlinear sigma-model shifts θ and leaves ρ and the fermions invariant. Quantum-mechanically, the naïve axial R-current is anomalous, but one can nevertheless define a conserved gauge-invariant axial R-current. This is discussed in detail in Section 4. In the infrared the R-symmetry gets promoted to a pair of affine U(1) current algebras (one left-moving and one right-moving). In contrast, the left and right components of the momentum current are not conserved separately even in the infrared. Nevertheless, this symmetry will play an important role in our analysis.

3. Flow to 2d black hole I: one-loop approximation

For $r \to \infty$ the metric (2.5) is flat, and therefore is unchanged by the RG flow. In other words, the RG flow deforms the cigar metric without modifying its asymptotic behavior. We would like to show that in the infrared the supersymmetric sigma-model with the metric (2.5) flows to the fermionic 2d Black Hole (2.6) with the same value of the asymptotic radius. In this section we limit ourselves to the one-loop approximation, which is valid for large enough k.

Consider the one-loop beta-function for the sigma-model metric:

$$\beta_{ij} = -\frac{1}{2\pi} R_{ij} \,. \tag{3.1}$$

Its only zero is a flat metric, and since any cigar has a nonzero curvature near the tip, naïvely it appears that a cigar-like metric cannot be a fixed point of the RG flow. The resolution of this puzzle is well-known (see e.g. [23]) and is related to the possibility of having a dilaton gradient. In the usual formulation, the dilaton affects the coupling of the sigma-model to a curved world-sheet metric. Alternatively, if one prefers to stay on a flat world-sheet, one may say that a non-trivial dilaton gradient in space-time is equivalent to assigning a non-trivial Weyl transformation law to target-space coordinates.

Once the possibility of a non-trivial Weyl transformation law for X^i is recognized, it is easy to see in what sense a cigar can be invariant under RG flow. Let us fix a conformally-flat gauge for the space-time metric G_{ij}, so that it has the form

$$ds^2 = e^{\Psi(u)} \left(du^2 + d\theta^2 \right) . \tag{3.2}$$

The function $\Psi(u)$ does not depend on θ because we are only interested in the sigma-models which have a U(1) isometry. The tip of the cigar corresponds to $u \to -\infty$, while the cylindrical asymptotics is reached for $u \to +\infty$. From the known behavior at the tip and at infinity we infer that

$$\begin{aligned} &\Psi(u) \sim 2u + \cdots \quad \text{for } u \to -\infty \,, \\ &\Psi(u) \sim \log k + \cdots \quad \text{for } u \to +\infty \,. \end{aligned} \tag{3.3}$$

The functions $\Psi(u)$ and $F(v)$ are related as follows:

$$F(v) = e^{\Psi(u)/2}, \qquad v = \int_{-\infty}^{u} e^{\Psi(u)/2} du\,. \tag{3.4}$$

Note that both (3.2) and (3.3) are left invariant by reparametrizations $u \to u+c, \theta \to \theta+c'$, where c, c' are constants. This is what remains of reparametrization invariance after we fix the gauge (3.2,3.3). Hence the most general transformation law for u and θ under the Weyl rescaling of the world-sheet metric by t^2 is

$$u \to u + at\,, \qquad \theta \to \theta + a't\,,$$

where a, a' are real constants. Saying that the metric approaches a fixed limit under such a modified Weyl transformation is equivalent to saying that for $\mu \to \infty$ the function $\Psi(u,t)$ depends only on the difference $u - at$:

$$\Psi(u,t) \to \Psi_{IR}(u - at)\,.$$

Since Ψ does not depend on θ, by a t-dependent reparametrization of θ one can make $a' = 0$.

The one-loop RG equation for Ψ is

$$\frac{\partial \Psi(u,t)}{\partial t} = \frac{1}{4\pi} e^{-\Psi(u,t)}\, \frac{\partial^2 \Psi(u,t)}{\partial u^2}\,. \tag{3.5}$$

Letting $\Psi(u,t) = \Psi_{IR}(u - at)$, we obtain an equation for $\Psi_{IR}(u)$:

$$\frac{1}{4\pi} e^{-\Psi_{IR}(u)}\; \Psi''_{IR}(u) + a\Psi'_{IR}(u) = 0\,.$$

The general solution of this equation is

$$e^{\Psi_{IR}(u)} = \frac{1}{e^{-\lambda(u-b)} + 4\pi a/\lambda}\,,$$

where λ, b are constants. Imposing the conditions (3.3), we obtain

$$\lambda = 2, \quad a = \frac{1}{2\pi k}, \quad e^{\Psi_{IR}(u)} = \frac{1}{e^{-2(u-b)} + 1/k}\,.$$

Thus $\Psi_{IR}(u)$ is completely fixed up to residual reparametrizations of u (shifts by a constant). In addition, the constant a in the modified Weyl transformation law is determined by the asymptotic radius of the cigar. By a change of variables

$$\sqrt{k}\tanh\rho = e^{\Psi_{IR}(u)/2}$$

the metric

$$ds^2 = e^{\Psi_{IR}(u)}(du^2 + d\theta^2)$$

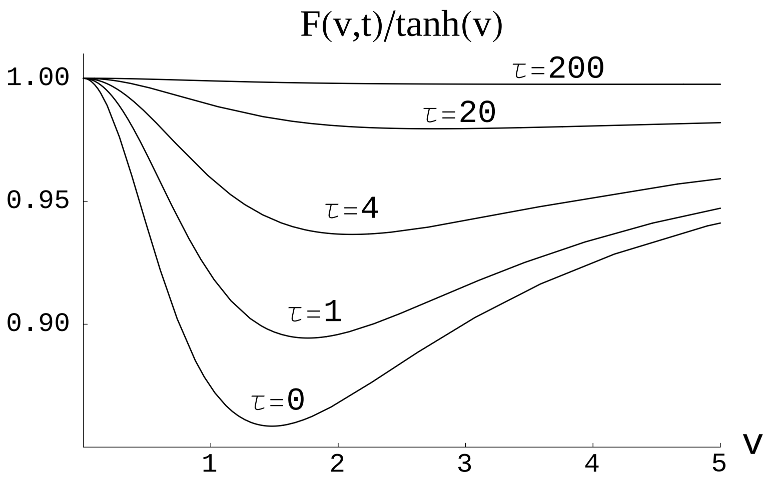

Figure 1: RG evolution of the cigar metric. We plotted $F(v,t)/\tanh v$ as a function of v for several values of the rescaled RG time $\tau = t/(4\pi)$.

is transformed to the form eq. (2.6). This proves that the only cigar-like fixed point of the one-loop RG equations is the 2d Black Hole.

We now would like to show that our metric (2.5) indeed flows to this infrared fixed point. We set

$$\Psi(u,t) = f(u - t/(2\pi k), t)\,,$$

and solve numerically the RG equation for $f(u,t)$. The initial condition is implicitly given by the metric (2.5). Explicitly, $\Psi(u,0) = \Psi_0(u)$ can be written in a parametric form

$$e^{\Psi_0(u(r))} = \frac{kr^2}{k+r^2}\,, \qquad u(r) = \log r + \frac{r^2}{2k}\,.$$

It is useful to note that the equation (3.5) is invariant with respect to the transformation

$$\Psi(u,t) \to \Psi(u,t) + \log q\,, \qquad t \to qt\,.$$

This means that we can absorb k into the defintion of the RG time t. Therefore in the remainder of this section we set $k = 1$.

For numerical integration we used an implicit scheme, which requires solving a sparse (tri-diagonal) system of linear equations at each step (see e.g. [24]). It is also convenient to reparametrize the variable u so that it runs over a finite rather than an infinite interval.

The results of the numerical integration of the RG equation are presented in figure 1. We chose to plot the ratio $F(v,t)/\tanh v$ where $F(v,t)$ is related to $\Psi(u,t)$ by (3.4). For the 2d Black Hole this ratio is equal to 1. From Figure 1 it is evident that $F(v,t)/\tanh v$ approaches 1 as $t \to +\infty$. Hence at one-loop level the sigma-model with target-space metric (2.5) flows to the 2d Black Hole (2.6).

The discussion in this section clarifies how a linear dilaton is generated by the RG flow. The point is simply that as the RG time increases, the cigar tries to shrink, so that its tip moves towards positive u. In order to “keep up” with the tip, one has to make a t-dependent reparametrization of the u-coordinate, which is equivalent to redefining the Weyl transformation law for u.

4. An exact computation of the central charge

In the previous section we have analyzed the renormalization group flow in the one-loop approximation which is valid for large k. In this section, using the method of [14, 15], we show that the central charge of the IR superconformal fixed point has to be exactly $c = 3 + 6/k$. This computation is used in the next section to prove that the GLSM (2.2) flows to the fermionic 2d Black Hole for all $k > 0$.

4.1 $\overline{Q}_+$-cohomology

One of the distinguishing properties of $(2,2)$ and $(0,2)$ theories is the existence of topological sectors that are protected from renormalization. The topological sector relevant in the present context is the chiral ring, or the right-moving chiral algebra to be more precise. Let us choose one of the four supersymmetry generators, say $\overline{Q}_+$. It is a nilpotent operator whose anti-commutator with its conjugate Q_+ is the left-moving translation operator:

$$(\overline{Q}_+)^2 = 0\,, \qquad \{\overline{Q}_+, Q_+\} = H + P\,. \tag{4.1}$$

By the nilpotency, one can consider $\overline{Q}_+$ cohomology of operators. By the second property, the left translation operator acts trivially on the cohomology group; if $[\overline{Q}_+, \mathcal{O}] = 0$ then $[H + P, \mathcal{O}] = \{\overline{Q}_+, [Q_+, \mathcal{O}]\} \simeq 0$. Thus correlation functions of $\overline{Q}_+$-closed operators are independent of $x^+ = x^0 + x^1$, that is, they depend only on the $x^- = x^0 - x^1$ coordinates of the insertion points. (In the Euclidean theory they are holomorphic functions.) In particular they form a right-moving operator product algebra (i.e. a chiral algebra).

Suppose a $(2,2)$ field theory flows to a $(2,2)$ superconformal field theory. Then $(2,2)$ supersymmetry is enhanced in the IR limit to left-moving and right-moving $\mathcal{N} = 2$ super-Virasoro algebras whose generators (anti-)commute with each other. In particular, the right-moving super-Virasoro is contained in the chiral algebra of $\overline{Q}_+$-cohomology classes. By the standard argument, this $\mathcal{N} = 2$ superconformal algebra should be observable even at finite energy (except in the rare case where the IR SCFT has another copy of currents with the same right-moving quantum numbers but with the left-moving R-charge equal to ± 1, in which case the super-Virasoro currents can pair up with them and disappear from the $\overline{Q}_+$-cohomology at finite energy). Therefore, if one can uniquely identify such a chiral algebra at finite energy, one can learn about the right-moving superconformal algebra in the IR limit, and in particular compute its central charge.

So let us look for such a superconformal algebra in the $\overline{Q}_+$ cohomology of the gauge theory in question. A right-moving $\mathcal{N}=2$ superconformal algebra consists of four currents that constitute a $(2,0)$ superfield. Its lowest component is the right-moving R-current. What we will look for is a $(2,2)$ superfield $\mathcal{J}$ that obeys

$$\overline{D}_+\mathcal{J} = 0\,. \tag{4.2}$$

Then the lowest term in the $\theta^+,\overline{\theta}^+$ expansion of $\mathcal{J}$ obeys the right-chiral condition

$$\{\overline{Q}_+, \mathcal{J}|_{\theta^+=\overline{\theta}^+=0}\} = 0\,, \tag{4.3}$$

because $\overline{D}_+ = \overline{Q}_+ + 2i\theta^+\partial_+$. Hence it is a $(2,0)$ superfield that represents a $\overline{Q}_+$ cohomology class. Its lowest component will flow to the right-moving R-current of the IR theory (modulo $\overline{Q}_+$-exact terms). Thus, if we can identify the right-moving R-symmetry in the high energy theory, we have a candidate for $\mathcal{J}$.

4.2 The current and its anomaly

The classical system has both vector and axial U(1) R-symmetries, under which the superfields Φ, P and Σ have charges $(q_V, q_A) = (0,0), (0,0)$ and $(0,2)$, respectively.[2] The corresponding currents are

$$\begin{aligned}
j_V^{\pm} &= \overline{\psi}_{\mp}\psi_{\mp} + \frac{k}{2}\overline{\chi}_{\mp}\chi_{\mp} - \frac{1}{2e^2}\overline{\lambda}_{\mp}\lambda_{\mp}\,,\\
j_A^{\pm} &= \pm\psi_{\mp}\overline{\psi}_{\mp} \pm \frac{k}{2}\chi_{\mp}\overline{\chi}_{\mp} \pm \frac{1}{2e^2}\lambda_{\mp}\overline{\lambda}_{\mp} + \frac{i}{e^2}(\partial_{\mp}\overline{\sigma}\sigma - \overline{\sigma}\partial_{\mp}\sigma)\,.
\end{aligned} \tag{4.4}$$

The right-moving R-current $j_R^{\pm} = \frac{1}{2}(j_A^{\pm} - j_V^{\pm})$ is therefore expressed as

$$\begin{aligned}
j_R^{+} &= \psi_-\overline{\psi}_- + \frac{k}{2}\chi_-\overline{\chi}_- + \frac{i}{2e^2}\left(\partial_-\overline{\sigma}\sigma - \overline{\sigma}\partial_-\sigma\right),\\
j_R^{-} &= \frac{1}{2e^2}\overline{\lambda}_+\lambda_+ + \frac{i}{2e^2}\left(\partial_+\overline{\sigma}\sigma - \overline{\sigma}\partial_+\sigma\right).
\end{aligned} \tag{4.5}$$

In the limit $e^2 \to \infty$ where the Σ multiplet becomes very massive, j_R^- vanishes and j_R^+ obeys the right-moving condition $\partial_+ j_R^+ = 0$ classically. Let us consider a superfield

$$\begin{aligned}
\mathcal{J}^{\circ} &= D_-(\Phi\,\mathrm{e}^V)\,\mathrm{e}^{-V}\overline{D}_-(\,\mathrm{e}^V\overline{\Phi}) + \frac{k}{2}D_-(P+\overline{P}+V)\overline{D}_-(P+\overline{P}+V)\\
&\quad + \frac{i}{2e^2}\Sigma(\partial_0 - \partial_1)\overline{\Sigma}\,.
\end{aligned} \tag{4.6}$$

[2]As usual, there is a room to modify the R-currents by other global symmetries of the system. We will discuss this ambiguity in section 4.4.

It is invariant under the gauge transformation (2.1), and its lowest component $\psi_-\overline{\psi}_- + \frac{k}{2}\chi_-\overline{\chi}_- + \frac{i}{e^2}\sigma\partial\overline{\sigma}$ is equal to j_R^+ up to $1/e^2$ terms. Using the equations of motion

$$\overline{D}_+\overline{D}_-(\mathrm{e}^V\overline{\Phi}) = 0\,, \tag{4.7}$$

$$\overline{D}_+\overline{D}_-(P+\overline{P}+V) = 0\,, \tag{4.8}$$

$$\Phi\,\mathrm{e}^V\overline{\Phi} + \frac{k}{2}(P+\overline{P}+V) + \frac{1}{2e^2}(D_+\overline{D}_-\Sigma + D_-\overline{D}_+\overline{\Sigma}) = 0\,, \tag{4.9}$$

it is easy to check that this superfield obeys the right-chiral condition $\overline{D}_+\mathcal{J}^\circ = 0$ on the classical level.

However on the quantum level this condition is violated:

$$\overline{D}_+\mathcal{J}^\circ = \frac{1}{2}\overline{D}_-\Sigma\,. \tag{4.10}$$

This is a supersymmetric extension of the chiral anomaly equation

$$\partial_\mu j_A^\mu = 2F_{+-}\,. \tag{4.11}$$

The factor 2 in front of F_{+-} can be understood by noting that there are n zero modes for *both* ψ_- and $\overline{\psi}_+$ for a generic gauge field with first Chern class $n = -\frac{1}{2\pi}\int F$. The equation (4.10) is a $(1+1)$-dimensional version of the Konishi anomaly [16], and its detailed derivation is given in appendix C.

Usually, the anomalous current cannot be modified in a gauge-invariant way so that it is conserved. The situation is different in the present theory where we have a field $\varphi_P := \mathrm{Im}\,p$ that shifts under the gauge transformation. Then, the curvature F_{+-} can be expressed as a differential of a gauge invariant quantity

$$A_\mu = \partial_\mu\varphi_P + v_\mu\,, \tag{4.12}$$

namely $F_{+-} = \partial_+v_- - \partial_-v_+ = \partial_+A_- - \partial_-A_+$. Then the modified axial current

$$\widetilde{j}_A^{\,+} = j_A^+ - 2A_-\,, \quad \widetilde{j}_A^{\,-} = j_A^- + 2A_+\,,$$

is gauge-invariant and conserved. This story has a supersymmetric generalization. Letting

$$\delta\mathcal{J} = \frac{1}{2}(\overline{D}_-D_- - D_-\overline{D}_-)(P+\overline{P}+V)\,, \tag{4.13}$$

we can derive from (4.8) that $\overline{D}_+\delta\mathcal{J} = -(1/2)\overline{D}_-\Sigma$. This is correct quantum mechanically, since the equation of motion (4.8) is used linearly. Thus the modified current

$$\mathcal{J} := \mathcal{J}^\circ + \delta\mathcal{J} \tag{4.14}$$

satisfies the right-chiral condition on the quantum level:

$$\overline{D}_+\mathcal{J} = 0\,. \tag{4.15}$$

For instance, let us look at the lowest component

$$\mathcal{J}|_{\theta^\pm=\overline{\theta}^\pm=0} = \psi_-\overline{\psi}_- + \frac{k}{2}\chi_-\overline{\chi}_- + \frac{i}{e^2}\sigma\partial_-\overline{\sigma} - 2A_- \,. \tag{4.16}$$

From the chiral anomaly (4.11) and the conservation law $\partial_\mu j_V^\mu = 0$, it follows that $\partial_\mu j_R^\mu = F_{+-}$, or equivalently

$$\partial_+\left(\psi_-\overline{\psi}_- + \frac{k}{2}\chi_-\overline{\chi}_- + \frac{i}{e^2}\sigma\partial_-\overline{\sigma}\right) + \partial_-\left(\frac{1}{2e^2}\overline{\lambda}_+\lambda_+ - \frac{i}{e^2}\overline{\sigma}\partial_+\sigma\right) = F_{+-} = 2\partial_+A_- \,, \tag{4.17}$$

where we have used the φ_P equation of motion $\partial^\mu A_\mu = 0$ in the last step. We note that $\overline{\lambda}_+\lambda_+ - 2i\overline{\sigma}\partial_+\sigma = \{\overline{Q}_+, \overline{\sigma}\overline{\lambda}_+\}$. Thus we find

$$\partial_+\left(\psi_-\overline{\psi}_- + \frac{k}{2}\chi_-\overline{\chi}_- + \frac{i}{e^2}\sigma\partial_-\overline{\sigma} - 2A_-\right) = 0 \ \text{ modulo } \{\overline{Q}_+, \ldots\}\,, \tag{4.18}$$

as expected from (4.15).

4.3 The superconformal algebra

We define the currents $j_-, G_-, \overline{G}_-, T_-$ as the lowest components of the right-chiral superfields $\mathcal{J}, D_-\mathcal{J}, \overline{D}_-\mathcal{J}, \frac{1}{4}[\overline{D}_-, D_-]\mathcal{J}$. They have the following expressions in terms of component fields:

$$\begin{aligned}
j_- &= \psi_-\overline{\psi}_- + \frac{k}{2}\chi_-\overline{\chi}_- + \frac{i}{e^2}\sigma\partial_-\overline{\sigma} + i(\mathcal{D}_-p - \mathcal{D}_-\overline{p})\,,\\
G_- &= -2i\psi_-\mathcal{D}_-\overline{\phi} - ki\chi_-\mathcal{D}_-\overline{p} + \frac{1}{e^2}\sigma\partial_-\overline{\lambda}_- + i\partial_-\chi_-\,,\\
\overline{G}_- &= 2i\mathcal{D}_-\phi\overline{\psi}_- + ki\mathcal{D}_-p\overline{\chi}_- - \frac{1}{e^2}\lambda_-\partial_-\overline{\sigma} - i\partial_-\overline{\chi}_-\,,\\
T_- &= 2\mathcal{D}_-\phi\mathcal{D}_-\overline{\phi} + k\mathcal{D}_-p\mathcal{D}_-\overline{p} + \frac{1}{2e^2}(\partial_-\sigma\partial_-\overline{\sigma} - \sigma\partial_-^2\overline{\sigma}) +\\
&\quad + \frac{i}{2}(\psi_-\mathcal{D}_-\overline{\psi}_- - \mathcal{D}_-\psi_-\overline{\psi}_-) + \frac{ik}{4}(\chi_-\partial_-\overline{\chi}_- - \partial_-\chi_-\overline{\chi}_-) + \frac{i}{2e^2}\lambda_-\partial_-\overline{\lambda}_- -\\
&\quad - \frac{1}{2}\partial_-(\mathcal{D}_-p + \mathcal{D}_-\overline{p})\,.
\end{aligned} \tag{4.19}$$

(j_- is of course identical to (4.16), as $-2A_- = i(\mathcal{D}_-p - \mathcal{D}_-\overline{p})$.) The quadratic terms in the currents come from $\mathcal{J}^\circ$, and the linear terms are from the "quantum correction" $\delta\mathcal{J}$. Since they are the lowest components of right-chiral superfields, they represent right-moving $\overline{Q}_+$-cohomology classes.

Now let us compute the OPE of these currents. We start with $j_-(x)j_-(0)$:

$$\begin{aligned}
j_-(x)j_-(0) &\sim \psi_-\overline{\psi}_-(x)\psi_-\overline{\psi}_-(0) + \frac{k^2}{4}\chi_-\overline{\chi}_-(x)\chi_-\overline{\chi}_-(0) - \frac{1}{e^4}\sigma\partial_-\overline{\sigma}(x)\sigma\partial_-\overline{\sigma}(0) +\\
&\quad +4A_-(x)A_-(0)\\
&\sim \frac{(-i)^2}{(x^-)^2} + \frac{k^2}{4}\frac{(-2i/k)^2}{(x^-)^2} - \frac{1}{e^4}\frac{e^2(-e^2)}{(x^-)^2} + 4\frac{-1/2k}{(x^-)^2} = -\frac{1+2/k}{(x^-)^2}\,.
\end{aligned} \tag{4.20}$$

Similarly, we can show that the rest of the OPE has the form

$$\begin{aligned}
j_-(x)G_-(0) &\sim \frac{-i}{x^-}G_-(0)\,, \qquad j_-(x)\overline{G}_-(0) \sim \frac{i}{x^-}\overline{G}_-(0)\,,\\
T_-(x)j_-(0) &\sim \frac{-1}{(x^-)^2}j_-(0) + \frac{-1}{x^-}\partial_- j_-(0)\,,\\
T_-(x)G_-(0) &\sim \frac{-3/2}{(x^-)^2}G_-(0) + \frac{-1}{x^-}\partial_- G_-(0)\,,\\
T_-(x)\overline{G}_-(0) &\sim \frac{-3/2}{(x^-)^2}\overline{G}_-(0) + \frac{-1}{x^-}\partial_- \overline{G}_-(0)\,,\\
T_-(x)T_-(0) &\sim \frac{3(1+2/k)}{2(x^-)^4} + \frac{-2}{(x^-)^2}T_-(0) + \frac{-1}{x^-}\partial_- T_-(0)\,,\\
G_-(x)\overline{G}_-(0) &\sim \frac{2i(1+2/k)}{(x^-)^3} - \frac{2}{(x^-)^2}j_-(0) + \frac{-2i}{x^-}\left(T_-(0) - \frac{i}{2}\partial_- j_-(0)\right). \qquad (4.21)
\end{aligned}$$

This is an $\mathcal{N}=2$ superconformal algebra with central charge

$$c = 3\left(1+\frac{2}{k}\right). \qquad (4.22)$$

4.4 Ambiguity and its resolution

In general, the R-current is not unique: it can be modified by other global symmetry currents. This leaves an ambiguity in the definition of the R-current and therefore in the value of the central charge. In the present system, there is one other continuous global symmetry, namely the shift of the imaginary part of p:

$$p \to p + i\alpha_2\,. \qquad (4.23)$$

The phase rotation of Φ is another symmetry, but that is gauge equivalent to (4.23).

The right-chiral current associated with (4.23) is given by

$$\mathcal{J}_2 = D_-\overline{D}_-(P+\overline{P}+V)\,, \qquad (4.24)$$

which indeed obeys $\overline{D}_+\mathcal{J}_2 = 0$ by virtue of the equations of motion (4.8).[3] This current is free of Konishi anomaly or $\overline{Q}_+$-anomaly in the sense of [15], because the conservation equation $\overline{D}_+\mathcal{J}_2 = 0$ is derived by using the equation of motion linearly. Thus it appears that one can modify the current $\mathcal{J}$ by an arbitrary multiple of $\mathcal{J}_2$

$$\mathcal{J}' = \mathcal{J} + a\mathcal{J}_2\,. \qquad (4.25)$$

It is easy to see that the four currents $j'_-, G'_-, \overline{G}'_-, T'_-$ defined as above form an $\mathcal{N}=2$ superconformal algebra with a central charge $c' = 3 + 6(1-a)/k$. Which of these $\mathcal{J}'$

[3] The chiral current for the phase rotation of Φ is $D_-\overline{D}_-(\Phi\,\mathrm{e}^V\overline{\Phi})$. This is equal to $-\frac{k}{2}D_-\overline{D}_-(P+\overline{P}+V) + \overline{D}_+(i\partial_- D_-\overline{\Sigma}/e^2)$ by the equation of motion (4.9). Therefore this current is proportional to $\mathcal{J}_2$ modulo $\overline{D}_+$ exact terms.

yields the superconformal algebra in the infrared limit? Since the central charge has to be real, we know that a is real, but we still have an ambiguity.

One can fix this ambiguity using a mild assumption about the low energy limit of the theory. Let us look at the expression for j'_-:

$$\begin{aligned} j'_- &= \psi_-\overline{\psi}_- + \frac{k}{2}\chi_-\overline{\chi}_- + \frac{i}{e^2}\sigma\partial_-\overline{\sigma} + i(\mathcal{D}_- p - \mathcal{D}_-\overline{p}) + ia\mathcal{D}_-\overline{p}, \\ &= \mathrm{Re}\, j'_- + i\left(\frac{1}{2e^2}\partial_-|\sigma|^2 + a\partial_-\mathrm{Re}\, p\right). \end{aligned} \tag{4.26}$$

The current in the infrared limit has to be real and therefore the imaginary part in (4.26) has to vanish up to $\overline{Q}_+$-exact terms. The mild assumption is the existence of the asymptotic region at $\mathrm{Re}\, p \to -\infty$ where the theory flows to the sigma-model on a flat cylinder, possibly with a linear dilaton of *some* slope. The term $\frac{1}{2e^2}\partial_-|\sigma|^2$ is negligible in that region, because σ has a large mass due to large values of $|\phi|^2 \sim -\mathrm{Re}\, p/2$. On the other hand, the field $\partial_-\mathrm{Re}\, p$ survives in the IR limit as a free field (possibly with a background charge), and is not $\overline{Q}_+$-exact. Thus for the current to be real up to $\overline{Q}_+$-exact terms, we have to set

$$a = 0\,. \tag{4.27}$$

It follows that $j_-, G_-, \overline{G}_-, T_-$ are the unique currents with the right properties, and the central charge of the IR fixed point is exactly $c = 3 + 6/k$. Note that the slope of the linear dilaton is uniquely fixed by the chiral anomaly.

5. Flow to 2d black hole II: exact treatment

In the previous sections, we have seen that the gauged linear sigma model (2.2) flows to a $(2,2)$ superconformal field theory with the same central charge, symmetries, and asymptotic behavior as the fermionic 2d Black Hole. However, there remains a possibility that it flows not to the supercoset itself, but to some other nearby fixed point with the same properties. The goal of this section is to argue that this does not happen.

5.1 General remarks

The fermionic 2d Black Hole is defined as the supersymmetric $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset at level k. The central charge is

$$c = 3\left(1 + \frac{2}{k}\right). \tag{5.1}$$

Unlike in the bosonic case, here the expansion of the central charge in powers of $1/k$ terminates at one-loop order. For large k this CFT is weakly coupled and is equivalent to the sigma-model with target (2.6). Note that the central charge of the fermionic 2d Black Hole at level k is exactly the same as the IR central charge of the GLSM (2.2) computed in section 4. In the asymptotic region of the target space both models become equivalent to the theory of a free chiral superfield with radius $\sqrt{k}$ and a background charge. The $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ supercoset is an example of a Kazama-Suzuki model and has $(2,2)$ supersymmetry. The world-sheet parity is also a symmetry of the model (see appendix D). There is also a global non-R symmetry, the momentum symmetry (this is clear from the fact that the sigma-model metric (1.1) describing the supercoset has a U(1) isometry which shifts φ). Thus the supercoset has the same symmetries as the IR fixed point of the GLSM.

The analysis of section 3 shows that for $k \to \infty$ the GLSM (2.2) flows to the fermionic 2d Black Hole at level k. For finite k we only know that the GLSM flows to a $(2,2)$ superconformal field theory with the same central charge, symmetries, and asymptotic behavior as the fermionic 2d Black Hole at level k. It could be that for finite k the GLSM flows not to the supercoset, but to some other fixed point nearby. But if this is the case, then the supercoset theory must admit a marginal operator which deforms it to the IR fixed point to which the GLSM flows to. This operator must preserve all the symmetries of the 2d Black Hole and leave its asymptotic behavior unchanged. If we can show that such marginal operators are absent, then the GLSM (2.2) has no choice but to flow to the fermionic 2d Black Hole for all $k > 0$.

5.2 Marginal deformations of the bosonic coset

As a warm-up, let us discuss marginal deformations of the bosonic $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset. This problem has been previously addressed in [25, 26]. We will focus on marginal deformations which preserve all the obvious symmetries of the coset, i.e. momentum and world-sheet parity. In addition we require the deformation to decay or stay constant towards $\rho \to \infty$, so that the asymptotic behavior of the model is not drastically altered.

First, let us consider marginal operators in the coset which correspond to normalizable states in the parent WZW theory. The quantization of the $\mathrm{SL}(2,\mathbb{R})$ WZW has been a subject of interest for many years, but the precise spectrum of the theory was determined only recently [27]. According to [27], one should include the following representations of $\mathrm{SL}(2,\mathbb{R})$ as the Kac-Moody primaries:

(i) $\mathcal{D}_j^+$: principal discrete representation with lowest weight of spin j, $1/2 < j < \frac{k-1}{2}$.

(ii) $\mathcal{D}_j^-$: principal discrete representation with highest weight of spin $-j$, $1/2 < j < \frac{k-1}{2}$.

(iii) $\mathcal{C}_j^\alpha$: principal continuous representations with $j = 1/2 + is$, $s \in \mathbb{R}$ and $0 \le \alpha < 1$.

We will work on the universal cover of $\mathrm{SL}(2,\mathbb{R})$, in which case Re j is not quantized. The primaries transforming in the principal discrete representations are normalizable, while the primaries in the principal continuous representations are delta-function normalizable.

As usual, positive-energy representations of the $\mathrm{SL}(2,\mathbb{R})$ current algebra are obtained by declaring that $J_n^{\pm}, J_n^3$ annihilate the primaries for all $n>0$. We denote these representations by $\widehat{\mathcal{D}}_j^{\pm}, \widehat{\mathcal{C}}_j^{\alpha}$. However, one should also include other representations labeled by an integer w [27]. These are obtained by declaring that the primary is annihilated by J_{n+w}^{+}, J_n^3, and J_{n-w}^{-} for $n>0$. One says that these new representations are obtained from the usual positive-energy representations by the spectral flow. They are denoted by $\widehat{\mathcal{D}}_j^{\pm,w}$ and $\widehat{\mathcal{C}}_j^{\alpha,w}$. Under the spectral flow by w units, the L_0 and J_0^3 eigenvalues of a state change as $(h,m)\mapsto(h+wm-kw^2/4, m-kw/2)$. In general spectral flow takes a positive-energy representation of $\widehat{\mathrm{SL}(2,\mathbb{R})}$ to a representation with energy unbounded from below. The exceptions to this rule are $\widehat{\mathcal{D}}_j^{+,w=-1}$ and $\widehat{\mathcal{D}}_j^{-,w=1}$. They are equivalent to $\widehat{\mathcal{D}}_{\frac{k}{2}-j}^{-}$ and $\widehat{\mathcal{D}}_{\frac{k}{2}-j}^{+}$, respectively. More generally, we have

$$\widehat{\mathcal{D}}_j^{-,w} \simeq \widehat{\mathcal{D}}_{\frac{k}{2}-j}^{+,w-1}\,. \tag{5.2}$$

Hence, to avoid double-counting, we should include in the spectrum $\widehat{\mathcal{D}}_j^{+,w}$ and $\widehat{\mathcal{C}}^{\alpha,w}$ for all $w\in\mathbb{Z}$, but exclude $\widehat{\mathcal{D}}_j^{-,w}$.

The amount of spectral flow in the left-moving and right-moving sectors must be the same [27]. Thus the space of states of the $\mathrm{SL}(2,\mathbb{R})$ WZW model at level k is the sum of $\widehat{\mathcal{D}}_j^{+,w}\times\widehat{\mathcal{D}}_j^{+,w}$ ($1<j<\frac{k-1}{2}$, $w\in\mathbb{Z}$) and $\widehat{\mathcal{C}}_j^{\alpha,w}\times\widehat{\mathcal{C}}_j^{\alpha,w}$ ($j\in\frac{1}{2}+i\mathbb{R}$, $w\in\mathbb{Z}$). Before the spectral flow the spin-j primary state with $J_0^3=m$, $\tilde{J}_0^3=\tilde{m}$ has conformal weights

$$L_0=\tilde{L}_0=-\frac{j(j-1)}{k-2}\,. \tag{5.3}$$

After the spectral flow by w its quantum numbers become

$$J_0^3=m-\frac{kw}{2},\quad \tilde{J}_0^3=\tilde{m}-\frac{kw}{2}\,, \tag{5.4}$$

$$L_0=-\frac{j(j-1)}{k-2}+wm-\frac{kw^2}{4}\,,\qquad \tilde{L}_0=-\frac{j(j-1)}{k-2}+w\tilde{m}-\frac{kw^2}{4}\,. \tag{5.5}$$

States of the coset theory are represented by states of the parent WZW theory obeying

$$J_0^3+\tilde{J}_0^3=0\,, \tag{5.6}$$

$$J_n^3=\tilde{J}_n^3=0\,,\qquad n\geq 1\,. \tag{5.7}$$

The momentum in the coset theory is given by

$$J_0^3-\tilde{J}_0^3\,.$$

The Virasoro generators are represented by $L_n-L_n^{\mathrm{U}(1)}$, $\tilde{L}_n-\tilde{L}_n^{\mathrm{U}(1)}$ where $L_n^{\mathrm{U}(1)}$ and $\tilde{L}_n^{\mathrm{U}(1)}$ are the Sugawara operators of the U(1) subalgebra at level k.

We are interested in Virasoro primaries in the coset theory which have dimension $(1,1)$ and zero momentum. This means that we are looking for Virasoro primaries of the parent WZW theory satisfying (5.7) together with

$$J_0^3 = \tilde{J}_0^3 = 0 \quad \text{and} \quad L_0 = \tilde{L}_0 = 1\,. \tag{5.8}$$

A little high-school algebra shows that in the discrete representations there are two such states for $k > 3$:

$$J_{-1}^- \tilde{J}_{-1}^- |j = 1\rangle^+\,, \tag{5.9}$$

$$\left[J_0^+ \tilde{J}_0^+ |j = \tfrac{k}{2} - 1\rangle^+\right]^{w=-1}\,. \tag{5.10}$$

Here $|j\rangle^\pm$ is the lowest/highest weight primary state of $\widehat{\mathcal{D}}_j^\pm \times \widehat{\mathcal{D}}_j^\pm$, and $[-]^w$ is the spectral flow of $[-]$ by w units. These two states are related by world-sheet parity. This becomes clearer if we use the isomorphism of $\widehat{\mathcal{D}}_{\frac{k}{2}-1}^{+,w=-1}$ and $\widehat{\mathcal{D}}_1^-$ and write the second state as[4]

$$J_{-1}^+ \tilde{J}_{-1}^+ |j = 1\rangle^-\,. \tag{5.11}$$

Since world-sheet parity exchanges $J^\pm$ and $\tilde{J}^\mp$ and $\widehat{\mathcal{D}}_j^+ \times \widehat{\mathcal{D}}_j^+$ and $\widehat{\mathcal{D}}_j^- \times \widehat{\mathcal{D}}_j^-$ (see appendix D), the statement becomes obvious.

The above two states are in the spectrum if $1 < (k-1)/2$, i.e. for $k > 3$. For $k = 3$ the states become delta-function normalizable and appear in the continuous representations with $j = 1/2, \alpha = 1/2, w = \pm 1$ (see below). For $2 < k < 3$ the states are not normalizable.

Thus for $k > 3$ there are two marginal operators in the $\mathrm{SL}(2,\mathbb{R})$ WZW theory which come from discrete representations and could give rise to marginal momentum-conserving deformations of the coset. It is easy to write down their explicit form. Following [27] we use the coordinates (ρ, t, φ) on $\mathrm{SL}(2,\mathbb{R})$ defined by

$$g = \mathrm{e}^{i\sigma_2(t+\varphi)/2}\, \mathrm{e}^{\sigma_3\rho}\, \mathrm{e}^{i\sigma_2(t-\varphi)/2} \tag{5.12}$$

(ϕ of [27] is replaced here with φ to avoid confusion with the scalar component of Φ). The vertex operators corresponding to the two states in (5.9), (5.11) are given by

$$\left(\frac{\partial_+\rho}{\cosh\rho} \mp i\sinh\rho\; \partial_+(-t-\varphi)\right) \cdot \left(\frac{\partial_-\rho}{\cosh\rho} \mp i\sinh\rho\; \partial_-(-t+\varphi)\right). \tag{5.13}$$

They are complex-conjugates of each other and are exchanged by world-sheet parity $\partial_+ \leftrightarrow \partial_-$, $t \leftrightarrow -t$ (appendix D).

[4]This should not be confused with the "field identification" in coset models [28] which would happen only if the gauge group had a non-trivial fundamental group [29]. We are considering the universal cover of $\mathrm{SL}(2,\mathbb{R})$ modded out by the gauge group $\mathbb{R}$. Since $\pi_1(\mathbb{R}) = \{1\}$, there is no non-trivial field identification.

The coset can be realized as a gauged WZW model [5, 25]. The gauging is with respect to the translation symmetry $t \to t - \alpha$, and the gauged action is obtained from the ordinary WZW action by replacing $\partial_\mu t$ with the gauge invariant expression $\partial_\mu t - A_\mu$:

$$\begin{aligned} S &= k S_{\mathrm{WZW}}(A, g) \qquad (5.14)\\ &= \frac{k}{4\pi}\int \mathrm{d}^2x \Big[-\eta^{\mu\nu}\Big(\partial_\mu\rho\partial_\nu\rho + \sinh^2\!\rho\,\partial_\mu\varphi\partial_\nu\varphi - \cosh^2\!\rho\,\partial_\mu t\partial_\nu t\Big) - \\ &\qquad - 4\sinh^2\!\rho\,(\partial_- t\partial_+\varphi - \partial_+ t\partial_-\varphi) - 4\cosh^2\!\rho\,A_+A_- + \\ &\qquad + 4(\cosh^2\!\rho\,\partial_- t - \sinh^2\!\rho\,\partial_-\varphi)A_+ + 4(\cosh^2\!\rho\,\partial_+ t + \sinh^2\!\rho\,\partial_+\varphi)A_-\Big]. \end{aligned}$$

The vertex operators in the coset model corresponding to the states (5.9), (5.11) are obtained from (5.13) by replacing $\partial_\mu t \to \partial_\mu t - A_\mu$. The equations of motion for A_μ imply

$$A_- - \partial_- t = -\tanh^2\!\rho\,\partial_-\varphi\,, \qquad A_+ - \partial_+ t = \tanh^2\!\rho\,\partial_+\varphi\,. \tag{5.15}$$

Substututing these expressions into (5.14), we obtain the world-sheet Lagrangian for the cigar (1.1). (The variable t completely disappears from the action and can be ignored.) Substuting the same expressions into the gauged versions of (5.13), we see that the vertex operators reduce to

$$\frac{1}{\cosh^2\!\rho}\Big(\partial_+\rho\partial_-\rho + \tanh^2\!\rho\,\partial_+\varphi\partial_-\varphi\Big) \pm i\frac{\tanh\rho}{\cosh^2\!\rho}(\partial_-\rho\partial_+\varphi - \partial_+\rho\partial_-\varphi)\,. \tag{5.16}$$

The real part is a metric deformation, and at first sight it seems non-trivial, but in fact it is a total derivative on the world-sheet. To see this, note that an infinitesimal reparametrization of the ρ coordinate, $\rho' = \rho + \epsilon\tanh\rho$, changes the metric of the cigar by

$$\frac{2\epsilon}{\cosh^2\rho}\left(\mathrm{d}\rho^2 + \tanh^2\rho\;\mathrm{d}\varphi^2\right).$$

Thus deformation by the real part of (5.16) is equivalent to a reparametrization of ρ. This implies in turn that this deformation is a total derivative on the world-sheet. Using equations of motion, one can check that (5.16) is proportional to

$$\partial_+\partial_- \log\cosh^2\rho\,. \tag{5.17}$$

The imaginary part of (5.16) is a B-field term which is parity-odd. This, of course, corresponds to the fact that the two states in (5.9), (5.11) are exchanged by world-sheet parity.

The conclusion is that for $k > 3$ the discrete series give rise to two momentum-conserving marginal deformations in the coset theory (while for $k \leq 3$ they give none). One is a parity-odd B-field, and the other is a total derivative on the world-sheet. If we restrict ourselves only to parity-even deformations, then we are left with the total derivative operator. Can one simply discard this operator as trivial? If the world-sheet is compact without a boundary, then one is certainly justified in doing so, but if the world-sheet has a boundary, or is noncompact, like $\mathbb{R}^2$, then the answer depends on boundary conditions. Since we are studying a conformal field theory, it is natural to impose boundary conditions which preserve Weyl invariance. In this case, the total derivative operator is trivial. Indeed, recall that the variables u and ρ are related by $u = \log\sinh\rho$, and therefore a change of variables $\rho \to \rho + \epsilon\tanh\rho$ is equivalent to $u \to u + \epsilon$. But the latter change of variables is also effected by the Weyl transformation. Hence with Weyl-invariant boundary conditions adding the operator (5.17) has no effect on the theory.

As for the principal continuous series, for general k the only states that give rise to marginal operators in the coset theory are the Kac-Moody primaries with

$$j = \frac{1}{2} \pm i\sqrt{k - \frac{9}{4}}\,, \qquad m = \overline{m} = 0\,. \tag{5.18}$$

Such operators decay as $\exp(-2j\rho)$, and since for $k > 9/4$ j has a nonzero imaginary part, they exhibit oscillatory behavior. Note that for $k = \frac{9}{4}$ there appears a non-oscillatory vertex operator decaying as $e^{-\rho}$. This is related to the fact that for $k = 9/4$ the central charge of the coset is 26, and the "tachyon" in the corresponding critical string theory is massless. The above vertex operator then describes the emission of the zero mode of the tachyon [25].

In addition, for $k = 3$ there appear two additional $(1,1)$ states in $\widehat{\mathcal{C}}_{\frac{1}{2}}^{\alpha=\frac{1}{2},w=\pm 1}$:

$$\begin{aligned} &\left[J_0^+\tilde{J}_0^+|j = \tfrac{1}{2}, \alpha = \tfrac{1}{2}\rangle^+\right]^{w=1}\,, \\ &\left[J_0^-\tilde{J}_0^-|j = \tfrac{1}{2}, \alpha = \tfrac{1}{2}\rangle^-\right]^{w=-1}\,. \end{aligned} \tag{5.19}$$

Note that for $\alpha = 1/2$ the continuous representation becomes reducible and decomposes into a direct sum of a highest weight representation with highest spin $-1/2$ and a lowest weight representation with lowest spin $1/2$. This explains superscripts $\pm$ in the above formula. The states (5.19) can be regarded as the $k \to 3$ limit of the discrete states (5.9), (5.11). To see this, one should use the isomorphism (5.2).

Now let us turn to operators corresponding to non-normalizable states. Recall that the zero-mode wave-functions of primary states with spin j decay as $\exp(-2j\rho)$ [27]. Since the volume element of $\mathrm{SL}(2,\mathbb{R})$ is proportional to

$$\sinh 2\rho\; dt\; d\varphi\,,$$

the zero-mode wave-function is normalizable for $\mathrm{Re}\, j > 1/2$. This was the origin of the restriction $j > 1/2$ for the discrete series. If we do not require normalizability, but would like the wave-function to decay towards $\rho \to \infty$ or at least not to grow, we can relax this constraint to $j \geq 0$. Consistency with the spectral flow then requires $0 \leq j \leq k/2$. This is to be compared with the normalizability condition $\frac{1}{2} < j < \frac{k-1}{2}$. It is easy to check that relaxing the conditions on j gives just one extra $(1,1)$ state, namely the one with $j = 0, m = \overline{m} = 0, w = 0$ and vertex operator

$$J^3(x^-)\tilde{J}^3(x^+)\,.$$

But this operator becomes zero after passing to the coset theory.

In addition, relaxing the constraint on j has the following effect. Recall that for $2 < k < 3$ the states with $j = 1$, and in particular the $(1,1)$ states (5.9), (5.11), are not in the spectrum. With the relaxed constraint $0 < j < k/2$ these two operators are allowed all the way down to $k = 2$. As explained above, the parity-even combination of the two operators is trivial if Weyl-invariant boundary conditions are used on the world-sheet, while the parity-odd one is a B-field.

To summarize, the only non-trivial marginal deformation of the bosonic coset which preserves momentum and world-sheet parity is a tachyon potential corresponding to the parity-even combination of the states (5.18). This operator exists for all $k \geq 9/4$ and is delta-function normalizable. For $k = 9/4$ it becomes the usual Liouville potential deformation. This result is somewhat puzzling from the perspective of the FZZ duality conjecture. According to [6], the sine-Liouville theory admits at least two marginal deformations: the Liouville potential and the radius-changing operator. In the coset theory we see the former, but no trace of the latter. However, it seems plausible that the deformation of the supercoset which changes the asymptotic radius of the cigar leads to a conical singularity at $\rho = 0$. The above analysis assumes from the beginning that the deformation is everywhere smooth and therefore cannot detect the radius-changing operator.[5] Since the bosonic FZZ duality is not the subject of this paper, we will not dwell any further on this issue. The situation in the supersymmetric case is somewhat different, as discussed below.

5.3 Marginal deformations of the supersymmetric coset

We now analyze marginal deformations of the Kazama-Suzuki supersymmetric coset model which preserve $(2,2)$ supersymmetry, R-symmetry and world-sheet parity. The model is defined as the $\mathrm{SL}(2,\mathbb{R})$ WZW model at level $(k+2)$ plus a Dirac fermion, modded out by a U(1) acting on the $\mathrm{SL}(2,\mathbb{R})$ part as before and axially on the fermion [7]. Thus the analysis is different from the bosonic case by a shift of the level $k \to k+2$ and by the addition of the fermionic sector. Fermionic oscillators $\psi_r, \overline{\psi}_r$ (right-moving) and $\tilde{\psi}_r, \overline{\tilde{\psi}}_r$ (left-moving) have the following commutation relation with J_0^3 and $\tilde{J}_0^3$:

$$\begin{aligned} &[J_0^3, \psi_r] = -\psi_r\,, \qquad &&[J_0^3, \overline{\psi}_r] = \overline{\psi}_r\,, \\ &[\tilde{J}_0^3, \tilde{\psi}_r] = \tilde{\psi}_r\,, \qquad &&[\tilde{J}_0^3, \overline{\tilde{\psi}}_r] = -\overline{\tilde{\psi}}_r\,. \end{aligned} \tag{5.20}$$

[5] We are grateful to Steve Shenker and Juan Maldacena for emphasizing this point to us.

For our purpose, we can work in the NS-NS sector, $r \in \frac{1}{2} + \mathbb{Z}$, which has a vacuum $|0\rangle$ that is annihilated by oscillators of positive frequency modes (and therefore is also annihilated by $J_0^3, \tilde{J}_0^3$).

The state space of the theory before the gauging of U(1) is a tensor product of the state space of the $\mathrm{SL}(2,\mathbb{R})_{k+2}$ WZW model and the Fock space $\mathcal{F}$ of the Dirac fermion. The former is the same as in the bosonic case with a shift of the level $k \to k+2$. (The representation $\widehat{\mathcal{D}}_j^{\pm}$ is now isomorphic to $\widehat{\mathcal{D}}_{\frac{k}{2}+1-j}^{\mp, w=\pm 1}$.) The spectral flow acts on the fermions as well as the bosons and sends the Fock space $\mathcal{F}$ to itself. In particular,

$$
|0\rangle \underset{w}{\overset{-w}{\rightleftarrows}} \begin{cases} \overline{\psi}_{-w+\frac{1}{2}} \cdots \overline{\psi}_{-\frac{1}{2}} \tilde{\psi}_{-w+\frac{1}{2}} \cdots \tilde{\psi}_{-\frac{1}{2}} |0\rangle, & w \geq 1\,, \\ \psi_{-|w|+\frac{1}{2}} \cdots \psi_{-\frac{1}{2}} \overline{\tilde{\psi}}_{-|w|+\frac{1}{2}} \cdots \overline{\tilde{\psi}}_{-\frac{1}{2}} |0\rangle, & w \leq -1\,. \end{cases} \tag{5.21}
$$

We can regard the total state space as the tensor sum of

$$
(\widehat{\mathcal{D}}_j^{+} \times \widehat{\mathcal{D}}_j^{+}) \otimes \mathcal{F} \quad (\tfrac{1}{2} < j < \tfrac{k+1}{2})\,, \quad \text{and} \tag{5.22}
$$

$$
(\,\widehat{\mathcal{C}}_j^{\alpha} \times \widehat{\mathcal{C}}_j^{\alpha}\,) \otimes \mathcal{F} \quad (j \in \tfrac{1}{2} + i\mathbb{R},\ 0 \leq \alpha < 1)\,, \tag{5.23}
$$

and their spectral flows. Before the spectral flow, the spin j primary state with $J_0^3 = m, \tilde{J}_0^3 = \tilde{m}$ has conformal weights

$$
L_0 = \tilde{L}_0 = -\frac{j(j-1)}{k}\,. \tag{5.24}
$$

After the spectral flow by w units, it becomes a state with

$$
J_0^3 = m - \frac{kw}{2}\,, \qquad \tilde{J}_0^3 = \tilde{m} - \frac{kw}{2}\,, \tag{5.25}
$$

$$
L_0 = -\frac{j(j-1)}{k} + wm - \frac{k}{4}w^2\,, \qquad \tilde{L}_0 = -\frac{j(j-1)}{k} + w\tilde{m} - \frac{k}{4}w^2\,. \tag{5.26}
$$

Despite the level shift $k \to k+2$, the coefficient of w in (5.25) and w^2 in (5.26) is proportional to k, as in (5.4) and (5.5), because the fermionic sector contributes -2.

States of the coset model must obey $J_0^3 + \tilde{J}_0^3 = 0$ and $J_n^3 = \tilde{J}_n^3 = 0$ for $n \geq 1$. The momentum generator is given by $J_0^{3(b)} - \tilde{J}_0^{3(b)}$, where $J_0^{3(b)}$ is the bosonic part of the $\mathrm{SL}(2,\mathbb{R})$ generator J_0^3. The Virasoro generators are defined as usual. There are also $(2,2)$ superconformal generators defined as follows [7]:

$$
\begin{aligned}
G_r &\propto \textstyle\sum_{n\in\mathbb{Z}} \psi_{r+n} J_{-n}^{+}\,, & \tilde{G}_r &\propto \textstyle\sum_{n\in\mathbb{Z}} \tilde{\psi}_{r+n} \tilde{J}_{-n}^{-}\,, \\
\overline{G}_r &\propto \textstyle\sum_{n\in\mathbb{Z}} \overline{\psi}_{r+n} J_{-n}^{-}\,, & \overline{\tilde{G}}_r &\propto \textstyle\sum_{n\in\mathbb{Z}} \overline{\tilde{\psi}}_{r+n} \tilde{J}_{-n}^{+}\,, \\
J_n &\propto J_n^{3(f)} + \tfrac{2}{k+2} J_n^{3(b)}\,, & \tilde{J}_n &\propto \tilde{J}_n^{3(f)} + \tfrac{2}{k+2} \tilde{J}_n^{3(b)}\,,
\end{aligned} \tag{5.27}
$$

where $J_n^{3(f)}$ and $J_n^{3(b)}$ are fermionic and bosonic parts of $J_n^3 = J_n^{3(f)} + J_n^{3(b)}$.

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We would like to find all even states in the coset model with zero axial and vector charges which preserve momentum and whose integral over the world-sheet is a $(2,2)$ superconformal invariant. The last requirement means that they must be Virasoro primaries of weight $(1,1)$ and be annihilated by $G_{-1/2}, \tilde{G}_{-1/2}$ up to total derivatives. Momentum conservation requires $J_0^{3(b)} - \tilde{J}_0^{3(b)} = 0$, and together with R-invariance this implies that the fermionic and bosonic parts of $J_0^3, \tilde{J}_0^3$ have to vanish independently.

We start with the discrete representations and their spectral flows. A little more high-school algebra reveals that the only normalizable states satisfying the above requirements are

$$
\begin{aligned}
&J_{-1}^{-}\tilde{J}_{-1}^{-}|j=1\rangle^{+} \otimes |0\rangle\,, &(5.28)\\
&J_{-1}^{+}\tilde{J}_{-1}^{+}|j=1\rangle^{-} \otimes |0\rangle\,. &(5.29)
\end{aligned}
$$

They can be equivalently written as

$$
\begin{aligned}
&\left[J_0^{+}\tilde{J}_0^{+}|j=\tfrac{k}{2}\rangle^{+} \otimes \psi_{-\frac{1}{2}}\overline{\tilde{\psi}}_{-\frac{1}{2}}|0\rangle\right]^{w=-1}\,,\\
&\left[J_0^{-}\tilde{J}_0^{-}|j=\tfrac{k}{2}\rangle^{-} \otimes \overline{\psi}_{-\frac{1}{2}}\tilde{\psi}_{-\frac{1}{2}}|0\rangle\right]^{w=1}\,. &(5.30)
\end{aligned}
$$

These states are in the spectrum for $k>1$. They are supersymmetry-descendants of $(\frac{1}{2},\frac{1}{2})$ primary states. Namely (5.28) and (5.29) can be expressed respectively as $\overline{G}_{-\frac{1}{2}}\tilde{G}_{-\frac{1}{2}}$ and $G_{-\frac{1}{2}}\overline{\tilde{G}}_{-\frac{1}{2}}$ applied to the $(\frac{1}{2},\frac{1}{2})$ states

$$
|j=1\rangle^{+} \otimes \psi_{-\frac{1}{2}}\overline{\tilde{\psi}}_{-\frac{1}{2}}|0\rangle\,, \quad \text{and} \quad |j=1\rangle^{-} \otimes \overline{\psi}_{-\frac{1}{2}}\tilde{\psi}_{-\frac{1}{2}}|0\rangle\,. \qquad (5.31)
$$

Furthermore, these two $(\frac{1}{2},\frac{1}{2})$ states are primaries of the $(2,2)$ superconformal algebra annihilated by $G_{-\frac{1}{2}}, \overline{\tilde{G}}_{-\frac{1}{2}}$ and $\overline{G}_{-\frac{1}{2}}, \tilde{G}_{-\frac{1}{2}}$, respectively, and therefore are twisted (anti-)chiral primaries. Thus, the integrals of operators corresponding to (5.28) and (5.29) are twisted F-terms. Since they have vanishing R-charges, they are in fact exactly marginal deformations of the supercoset theory. However, as in bosonic case, the parity-even combination of these operators is essentially trivial. We now explain this.

A Kazama-Suzuki supercoset can be realized as a supersymmetric gauged WZW model [30]. The Dirac fermion transforms under the U(1) gauge group as $\psi_{\mp} \to \mathrm{e}^{\mp i\alpha}\psi_{\mp}$ and $\overline{\psi}_{\mp} \to \mathrm{e}^{\pm i\alpha}\overline{\psi}_{\mp}$. This is equivalent to (5.20), if ψ_r and $\tilde{\psi}_r$ are the modes of ψ_- and ψ_+, respectively. The action is given by

$$
\begin{aligned}
S =\ & (k+2)S_{\mathrm{WZW}}(A,g) +\\
& + \frac{1}{2\pi}\int \mathrm{d}^2x \left[2i\overline{\psi}_-(\partial_+ - iA_+)\psi_- + 2i\overline{\psi}_+(\partial_- + iA_-)\psi_+\right]. &(5.32)
\end{aligned}
$$

The states (5.28) and (5.29) are identical to (5.9), (5.11) up to tensor product with the vacuum vector of the fermionic Fock space. Thus, the vertex operators for the former states are still given by (5.13). The difference between the bosonic and supersymmetric cases arises only after gauging. The equation of motion for A_μ is solved by

$$A_\mp - \partial_\mp t = \mp \tanh^2\!\rho\, \partial_\mp \varphi \pm \frac{1}{(k+2)\cosh^2\!\rho}\overline{\psi}_\mp \psi_\mp \,. \tag{5.33}$$

Substituting this into the action (5.32), we obtain the lagrangian

$$\begin{aligned}
&-\frac{k+2}{2}\eta^{\mu\nu}\Big(\partial_\mu\rho\partial_\nu\rho + \tanh^2\!\rho\,\partial_\mu\varphi\partial_\nu\varphi\Big) + \\
&+2i\overline{\psi}_-(\partial_+ - i\partial_+ t - i\tanh^2\!\rho\,\partial_+\varphi)\psi_- + 2i\overline{\psi}_+(\partial_- + i\partial_- t - i\tanh^2\!\rho\,\partial_-\varphi)\psi_+ - \\
&-\frac{2}{(k+2)\cosh^2\!\rho}\psi_+\psi_-\overline{\psi}_-\overline{\psi}_+ \,,
\end{aligned} \tag{5.34}$$

which describes the supersymmetric cigar.[6] Substituting (5.33) into the gauged version of (5.13), we obtain explicit expressions for vertex operators in the supercoset:

$$\frac{1}{\cosh^2\!\rho}\Big(\partial_+\rho\partial_-\rho + \tanh^2\!\rho\,\widetilde{\partial_+\varphi}\widetilde{\partial_-\varphi}\Big) \pm i\frac{\tanh\rho}{\cosh^2\!\rho}(\partial_-\rho\widetilde{\partial_+\varphi} - \partial_+\rho\widetilde{\partial_-\varphi}) \,, \tag{5.35}$$

where we denoted

$$\widetilde{\partial_\mp\varphi} = \partial_\mp\varphi + \frac{1}{k+2}\overline{\psi}_\mp\psi_\mp \,. \tag{5.36}$$

The real part is proportional to the variation of the action under the change of variables $\delta\rho = \epsilon\tanh\rho$. As in the bosonic case, this means that this deformation is trivial if the world-sheet is compact, or if Weyl-invariant boundary conditions are imposed on the world-sheet boundary. The imaginary part corresponds to switching on the B-field. It is parity-odd, in agreement with the fact that the two states (5.28) and (5.29) are exchanged by world-sheet parity.

One can in fact identify both of the above deformations in the gauged linear sigma-model: they are the Fayet-Iliopoulos term and the theta-angle:

$$\mathrm{Re}\int \mathrm{d}^2\widetilde{\theta}\,(r - i\theta)\Sigma \,. \tag{5.37}$$

The Fayet-Iliopoulos deformation is trivial as it can be absorbed into the real part of P, while the theta-angle breaks world-sheet parity. One can easily check that in the

[6] It can also be written as $\frac{k+2}{2}\int \mathrm{d}^4\theta\, K(Z,\bar{Z})$ where Z is a chiral superfield with components

$$z = \log\sinh\rho + i\varphi, \quad \chi_\pm = \sqrt{\frac{2}{k+2}}\coth\rho\,\mathrm{e}^{-i\varphi\pm it}\psi_\pm \,,$$

and the Kähler potential is such that $K_{z\bar{z}} = 1/(1+|\,\mathrm{e}^{-z}|^2)$. This shows that (5.27) is in the standard convention with respect to chiral versus twisted chiral. z is a good variable away from the tip of the cigar. A good coordinate near the tip is $w = \mathrm{e}^z$, with $K_{w\bar{w}} = 1/(1+|w|^2)$.

presence of the theta-angle integrating out the gauge field yields a supersymmetric sigma-model with target metric (2.4) and a B-field

$$B_{\rho\varphi} \sim \frac{\tanh \rho}{\cosh^2 \rho}\,. \tag{5.38}$$

Topological terms in the action, like a B-field with vanishing $H = dB$, are not subject to RG flow. Therefore it is gratifying that the expression (5.38) obtained by a classical computation agrees with the imaginary part of (5.35).

Next we consider continuous representations and their spectral flow. It is easy to see that the only states obeying $J_0^{3(b,f)} = \tilde{J}_0^{3(b,f)} = 0$ and $L_0 = \tilde{L}_0 = 1$ are the primary states $|j\rangle^{\alpha=0} \otimes |0\rangle$ with spin $j = 1/2 \pm i\sqrt{k - 1/4}$. These states exist for $k \geq 1/4$. In particular, for $k = 1/4$ this is simply the cosmological constant term. But these states are not annihilated by any of the four supercharges $G_{-1/2}, \overline{\tilde{G}}_{-1/2}, \overline{G}_{-1/2}, \tilde{G}_{-1/2}$. Hence these deformations break supersymmetry and do not concern us.

Finally, as in the bosonic case, we should allow deformations which correspond to non-normalizable states, if their vertex operators do not grow towards $\rho \to \infty$. This means that we should relax the constraint on j for the discrete series to $0 \leq j \leq k/2 + 1$. It is easy to check that this does not yield any new deformations in the supercoset which would preserve all the symmetries. The only effect of allowing such non-normalizable states is to extend the range of k for which the operators (5.28) and (5.29) exist: if we do not impose normalizability, then they exist for all $k > 0$.

To summarize, in the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ Kazama-Suzuki model there are two $(1,1)$ operators that could lead to supersymmetric marginal deformations preserving momentum. They combine into a twisted superpotential term and correspond to the FI-Theta deformation (5.37) of the GLSM. However, the real part is trivial and can be absorbed into a field redefinition, while the imaginary part is parity-odd and will not be generated if the high-energy theory is parity even (i.e. has $\theta = 0$). We conclude that the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ Kazama-Suzuki model is rigid, and does not admit non-trivial deformations preserving all the symmetries. This in turn implies that the GLSM (2.2) flows to this superconformal theory for all $k > 0$.

Note that unlike in bosonic case, there is no puzzle associated with the absence of a radius-changing operator, because the dual $\mathcal{N} = 2$ Liouville theory does not have it either. While in the sine-Liouville theory the asymptotic radius and the central charge can be varied independently, in the supersymmetric case the radius is quantized in units of $1/\sqrt{k}$ if the central charge is $3+6/k$. The easiest way to see this is to notice that given the action (1.3), one still has the freedom to choose the period of $\mathrm{Im}\, Y$. The form of the superpotential constrains the period to be $2\pi n, n \in \mathbb{N}$. We will see in the next section that the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ supercoset is dual to $\mathcal{N} = 2$ Liouville with the smallest possible radius corresponding to $n = 1$.

6. Liouville theory as the mirror

In this section, we will find a dual description of our gauge theory, using the method of [1]. We will see that the dual theory flows in the IR limit to $\mathcal{N}=2$ supersymmetric Liouville theory. Thus we may conclude that the fermionic 2d Black Hole is mirror to $\mathcal{N}=2$ Liouville theory.

6.1 The dual theory

We start with the classical dualization of the system. We T-dualize the phase of Φ as well as the imaginary part of P. The dual of a charged chiral superfield is a neutral twisted chiral superfield which is coupled to the field strength superfield Σ via a twisted superpotential. In other words, the lowest component of the dual superfield is a dynamical theta-angle. Using the method of [31, 1], we find that the dual action is

$$\begin{aligned}\widetilde{S} = \frac{1}{2\pi}\int \mathrm{d}^2x \Bigg\{ \int \mathrm{d}^4\theta \left[-\frac{1}{2e^2}|\Sigma|^2 - \frac{1}{2}\left(Y+\overline{Y}\right)\log\left(Y+\overline{Y}\right) - \frac{1}{2k}|Y_P|^2 \right] + \\ + \frac{1}{2}\left(\int \mathrm{d}^2\widetilde{\theta}\, \Sigma(Y+Y_P) + h.c. \right) \Bigg\} , \qquad (6.1)\end{aligned}$$

where Y and Y_P are the duals of Φ and P respectively. The lowest components of both Y and Y_P are periodically identified with period $2\pi i$. Gauge-invariant composites of the original fields are expressed in terms of the dual fields as

$$\begin{aligned}\overline{\Phi}\,\mathrm{e}^V\Phi &= \frac{1}{2}\left(Y+\overline{Y}\right), \\ P+\overline{P}+V &= \frac{1}{k}\left(Y_P+\overline{Y}_P\right). \qquad (6.2)\end{aligned}$$

Let us now include perturbative quantum corrections to this dualization procedure. The one-loop divergence in the $|\phi|^2$ one-point function requires an additive renormalization of P:

$$P(\Lambda_{\mathrm{UV}}) = P(\mu) - \frac{1}{k}\log\left(\frac{\Lambda_{\mathrm{UV}}}{\mu}\right). \qquad (6.3)$$

Here Λ_{UV} is the UV cut-off. This induces a similar renormalization of Y_P and hence of Y so that the twisted F-term in (6.1) is finite:

$$Y(\Lambda_{\mathrm{UV}}) = Y(\mu) + \log\left(\frac{\Lambda_{\mathrm{UV}}}{\mu}\right), \qquad Y_P(\Lambda_{\mathrm{UV}}) = Y_P(\mu) - \log\left(\frac{\Lambda_{\mathrm{UV}}}{\mu}\right).$$

This renormalization does not affect the twisted F-term but changes the Y-part of the D-term in (6.1). In particular, the Kähler metric for $y=y(\mu)$ and $y_P=y_P(\mu)$ is given by

$$\mathrm{d}s^2 = \frac{|\mathrm{d}y|^2}{2\log(\Lambda_{UV}) + 2\mathrm{Re}\, y} + \frac{1}{k}|\mathrm{d}y_P|^2. \qquad (6.4)$$

In the continuum limit $\Lambda_{\mathrm{UV}} \to \infty$, the metric for y degenerates to zero, but that for y_P remains finite.

The axial R-rotation shifts the imaginary part of y [1] as

$$\mathrm{Im}\, Y \to \mathrm{Im}\, Y - 2\beta\,. \tag{6.5}$$

If not for Y_P, this anomalous transformation law would induce a change in the theta-angle. This is a reflection of the fact that in the theory of Φ and V the axial R-symmetry is anomalous. In the presence of Y_P the anomaly can be cancelled by assigning an anomalous transformation law to Y_P:

$$\mathrm{Im}\, Y_P \to \mathrm{Im}\, Y_P + 2\beta\,. \tag{6.6}$$

Thus the dual system has an axial R-symmetry such that e^{-Y} and e^{-Y_P} have axial R-charges 2 and -2, respectively. The vector R-charge is zero in both cases. The anomalous transformation law for Y_P corresponds to the modification $j_A^{\pm} \to j_A^{\pm} \mp 2A_{\mp}$ of the axial R-current in the original system.

Finally, let us include non-perturbative effects. The vortex-instanton of the original gauge system can generate a twisted superpotential in the dual theory. To find the precise form of the superpotential, it is best to extend the gauge symmetry to $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$, where $\mathrm{U}(1)_1$ acts as the phase rotation of Φ while $\mathrm{U}(1)_2$ shifts the imaginary part of P. The first system is the $\mathcal{N} = 2$ QED with one massless flavor, which has been studied in detail in [1]. The twisted superpotential of its dual theory is

$$\widetilde{W}_1 = \Sigma_1\, Y + \mathrm{e}^{-Y}\,. \tag{6.7}$$

The correction term e^{-Y} is generated by the vortex-instantons of the (Φ, V_1) system. On the other hand, the system of P and V_2 is equivalent to a free theory of a massive vector multiplet. Hence the classical dualization is exact, and the twisted superpotential is

$$\widetilde{W}_2 = \Sigma_2\, Y_P\,. \tag{6.8}$$

The absence of the vortex-instanton correction can also be understood by noting that the (P, V_2) system has no vortex solutions because the target space for P is $\mathbb{R} \times \mathbb{S}^1$.

To get back to the original GLSM (2.2) we only have to freeze $\Sigma_1 - \Sigma_2$ by tuning the D-term couplings [1]. Since changing the D-terms cannot affect the twisted F-terms, the twisted superpotential of the dual theory is exactly given by

$$\widetilde{W} = \Sigma(Y + Y_P) + \mathrm{e}^{-Y}\,. \tag{6.9}$$

An alert reader should have noticed that a similar argument can be used to "prove" that the $\overline{Q}_+$-cohomology and hence the IR central charge of a gauged linear sigma-model is independent of the D-terms. On the other hand, we have seen in section 4 that for the GLSM (2.2) the $\overline{Q}_+$ cohomology and the central charge do depend on k in a non-trivial way. In fact, this is crucial for the whole approach described here. The loophole in the formal argument is that it requires integration

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by parts on the target space of the low-energy sigma-model. This can be easily seen in the path-integral formulation. Thus if the target space is noncompact, and the D-term deformation does not decay fast enough at infinity, then the formal argument may fail. Varying k changes the asymptotic behavior of the target-space metric, and therefore it is not surprising that the $\overline{Q}_+$-cohomology depends on k. On the other hand, modifying the gauge couplings has a vanishingly small effect at infinity, because the gauge fields are massive there.

We have no control over the Kähler potential of the dual theory, as it can get both perturbative and nonperturbative corrections. The only statement that we can make is that the corrections to the semi-classical expression (6.4) are small for $\mathrm{Re}\, Y \to +\infty$ and $-\mathrm{Re}\, Y_P \to +\infty$, because the gauge fields are very massive in this region, and the interactions are negligible.

6.2 Liouville theory as the IR limit of the dual theory

At low energies the vector multiplet V, which has mass of order $e\sqrt{k}$, can be integrated out. In the dual theory this gives a constraint

$$Y + Y_P = 0\,. \tag{6.10}$$

Thus we are left with a single twisted chiral superfield Y with the twisted superpotential

$$\widetilde{W} = \mathrm{e}^{-Y}\,. \tag{6.11}$$

The above arguments tell us that the Kähler potential has the form

$$K(Y,\overline{Y}) = -\frac{1}{2k}|Y|^2 + \cdots\,, \tag{6.12}$$

where the terms denoted by dots go to zero for $\mathrm{Re}\, Y \to +\infty$. Otherwise the Kähler potential is undetermined.

The superpotential (6.11) is the Liouville potential. It is known that the theory with this superpotential and a flat Kähler potential $K_\gamma = -\frac{1}{2\gamma^2}|Y|^2$ is a $(2,2)$ superconformal field theory with central charge

$$c = 3\left(1 + \frac{2}{\gamma^2}\right). \tag{6.13}$$

In fact the current superfield

$$\widetilde{\mathcal{J}} = \frac{1}{2\gamma^2}\overline{D}_- Y D_- \overline{Y} + \frac{1}{\gamma^2}(\partial_0 - \partial_1)\mathrm{Im}\, Y \tag{6.14}$$

obeys $\overline{D}_+ \widetilde{\mathcal{J}} = 0$, and the lowest components of $\widetilde{\mathcal{J}}, D_-\widetilde{\mathcal{J}}, \overline{D}_-\widetilde{\mathcal{J}}, \frac{1}{4}[\overline{D}_-, D_-]\widetilde{\mathcal{J}}$ generate $\mathcal{N} = 2$ superconformal algebra with central charge $c = 3 + 6/\gamma^2$. The linear term in (6.14) shows that there is a linear dilaton with the slope proportional to $1/\gamma^2$.

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We are now very close to proving that the IR limit of the dual theory is the $\mathcal{N}=2$ Liouville theory with $\gamma^2=k$. Indeed, we already know that the twisted superpotential, the central charge, and the asymptotic behavior of the Kähler potential are the same in the two theories, if we set $\gamma^2=k$. But there still remains a remote possibility that there is another twisted Landau-Ginzburg model with the same central charge and twisted superpotential, but different Kähler potential, which nevertheless has the same asymptotics. One could rule out the existence of such a "fake" Liouville theory in the neighborhood of the ordinary Liouville theory by studying marginal deformations of the latter. We take an alternative route, which directly demonstrates that the dual of the GLSM flows to the $\mathcal{N}=2$ Liouville theory. Let us dualize the phase of P only, leaving Φ as it is. As explained above, the classical dualization is exact in this case. The resulting gauged linear sigma-model has both twisted and ordinary chiral fields and the following action

$$S=\frac{1}{2\pi}\int \mathrm{d}^2x\,\mathrm{d}^4\theta\left[\overline{\Phi}\,\mathrm{e}^V\Phi-\frac{1}{2k}|Y_P|^2-\frac{1}{2e^2}|\Sigma|^2\right]+\frac{1}{4\pi}\left(\int \mathrm{d}^2x\,\mathrm{d}^2\widetilde{\theta}\,\Sigma Y_P+h.c.\right). \tag{6.15}$$

Recall now that the twisted chiral superfield e^{Y_P} has axial R-charge 2 and vector R-charge 0. Hence we can deform the above theory by adding a twisted superpotential

$$\frac{\kappa}{4\pi}\int \mathrm{d}^2x\,\mathrm{d}^2\widetilde{\theta}\,e^{Y_P}+h.c. \tag{6.16}$$

without breaking the axial R-symmetry. It follows that this deformation results in an exactly marginal deformation of the IR fixed point which does not change the central charge and preserves $(2,2)$ supersymmetry. Furthermore, the asymptotic region in the undeformed theory corresponds to $|\Phi|\to\infty, \mathrm{Re}\,Y_P\to-\infty$. Since the twisted superpotential (6.16) is exponentially small in this region, this deformation does not change the asymptotic behavior of the model. Note also that after the twisted superpotential has been added, we cannot dualize back to the (Φ,P,V) variables.

Now we recall that the fermionic 2d Black Hole does not have non-trivial marginal deformations preserving $(2,2)$ supersymmetry. It follows that the model (6.15) deformed by the twisted superpotential (6.16) flows to the fermionic 2d black hole at level k for all κ. We can use this to our advantage by taking the limit $\kappa\to\infty$. To see what happens in this limit, we set $Y_P=\tilde{Y}_P-\log(\kappa/\kappa_0)$ so that in terms of $\tilde{Y}_P$ the twisted superpotential remains fixed. In terms of $\Phi, \tilde{Y}_P$ and V the action becomes

$$\begin{aligned}S=\frac{1}{2\pi}\int \mathrm{d}^2x\,\mathrm{d}^4\theta&\left[\overline{\Phi}\,\mathrm{e}^V\Phi-\frac{1}{2k}|\tilde{Y}_P|^2-\frac{1}{2e^2}|\Sigma|^2\right]+\\&+\left(\frac{1}{4\pi}\int \mathrm{d}^2x\,\mathrm{d}^2\widetilde{\theta}\left[\Sigma\left(-\log\frac{\kappa}{\kappa_0}+\tilde{Y}_P\right)+\kappa_0 e^{\tilde{Y}_P}\right]+h.c.\right).\end{aligned} \tag{6.17}$$

We see that $\mathrm{Re}\log(\kappa/\kappa_0)$ plays the role of the Fayet-Iliopoulos term. For $\kappa\to\infty$ the Fayet-Iliopoulos term breaks the gauge symmetry at a very high scale of order

$\log \kappa$, the gauge field eats Φ, and all the fields except $\tilde{Y}_P$ get a mass of order $\log \kappa$. Integrating them out classically, we are left with a twisted chiral superfield $\tilde{Y}_P$ with a twisted superpotential $e^{\tilde{Y}_P}$ and a Kähler potential

$$-\frac{1}{2k}|\tilde{Y}_P|^2 \,.$$

This is an $\mathcal{N}=2$ Liouville theory with central charge $c = 3 + 6/k$. As κ increases, the accuracy of the classical approximation becomes arbitrarily good. On the other hand, we know that the IR limit of the theory does not depend on κ at all. Hence the GLSM flows to the $\mathcal{N}=2$ Liouville theory for all κ, including $\kappa = 0$. This concludes the argument.

7. Some generalizations

In this section, we discuss a few generalizations of our setup. One generalization is to consider an orbifold of the fermionic 2d Black Hole background with respect to a discrete subgroup of the U(1) isometry.[7] Other generalizations are sigma-models on higher dimensional manifolds, some of which can be used to construct dilatonic superstring backgrounds, while others have a mass gap.

7.1 Orbifolds

In $\mathcal{N}=2$ Liouville theory (1.3), the form of the superpotential e^{-Y} constrains the periodicity of $\operatorname{Im} Y$ to be an integer multiple of 2π, and therefore the radius of the circle parametrized by $\operatorname{Im} Y$ is quantized in units of $1/\sqrt{k}$. As mentioned in Section 5, this is an important difference between the $\mathcal{N}=2$ Liouville theory and its bosonic relative, the sine-Liouville theory: in the latter the radius of the circle can be varied independently of k. We have shown that the $\mathcal{N}=2$ Liouville theory with the minimal radius $1/\sqrt{k}$ is mirror to the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ supercoset. What about the other values of the radius? Since asymptotically mirror transformation reduces to T-duality, the mirror for Liouville theory with radius $n/\sqrt{k}$ must be some generalization of the supercoset with asymptotic radius $\sqrt{k}/n$. An obvious guess is an orbifold of the supercoset by a $\mathbb{Z}_n$ subgroup of the momentum symmetry.

To show that this guess is correct, note that the orbifoldized supercoset can be obtained by orbifoldizing the GLSM (2.2) by the same symmetry. This means that one should take the period of $\mathrm{Im}\ P$ to be $2\pi/n$ instead of 2π. To derive the mirror of such a model, we use the approach explained in subsection 6.2: we T-dualize P to a twisted chiral multiplet Y_P and add a twisted superpotential e^{Y_P}. As we increase the coefficient of e^{Y_P}, the theory is smoothly deformed to $\mathcal{N}=2$ Liouville theory. The only difference is that the period of $\mathrm{Im}\ Y_P$ is now $2\pi n$ instead of 2π. This proves that the $\mathbb{Z}_n$ orbifold of the supercoset is mirror to the $\mathcal{N}=2$ Liouville theory with radius $n/\sqrt{k}$.

[7] This idea arose in a conversation with Juan Maldacena.

Note that the $\mathbb{Z}_n$ action on the fermionic 2d Black Hole has a fixed point at the tip of the cigar. Thus the orbifoldized sigma-model metric has a conical singularity with a deficit angle $2\pi(1-1/n)$. Nevertheless the conformal field theory is well-defined. In the bosonic case, FZZ duality suggests that the cigar with an arbitrary conical deficit leads to a well-defined CFT, but it is not known how to see this directly.

7.2 Multi-variable models

It is straightforward to generalize the story to theories with a larger number of fields and higher rank gauge groups. Let us consider a $\mathrm{U}(1)^M$ gauge theory with $N+M$ matter fields Φ_i $(i=1,\ldots,N)$, P_ℓ $(\ell=1,\ldots,M)$ where the gauge transformation is defined by $\Phi_i \to \mathrm{e}^{i\sum_{\ell=1}^M R_{i\ell}\Lambda_\ell}\Phi_i$ and $P_\ell \to P_\ell + i\Lambda_\ell$. The fields P_ℓ are periodic in the imaginary direction and we take all periodicities to be $2\pi i$. The action of the system is given by

$$S=\frac{1}{2\pi}\int \mathrm{d}^2x\,\mathrm{d}^4\theta\left[\sum_{i=1}^N \overline{\Phi}_i\,\mathrm{e}^{R_i\cdot V}\Phi_i+\sum_{\ell=1}^M\frac{k_\ell}{4}(P_\ell+\overline{P}_\ell+V_\ell)^2-\sum_{\ell=1}^M\frac{1}{2e_\ell^2}|\Sigma_\ell|^2\right], \quad (7.1)$$

where $R_i\cdot V=\sum_{\ell=1}^M R_{i\ell}V_\ell$. The chiral anomaly equation $\partial_\mu j_A^\mu = 2\sum_{i=1}^N R_i\cdot F_{+-}$ has a supersymmetric extension

$$\overline{D}_+\mathcal{J}^\circ=\frac{1}{2}\sum_{i,\ell}R_{i\ell}\overline{D}_-\Sigma_\ell\,, \quad (7.2)$$

where $\mathcal{J}^\circ$ is defined by

$$\begin{aligned}\mathcal{J}^\circ =&\sum_{i=1}^N D_-(\Phi_i\,\mathrm{e}^{R_i\cdot V})\,\mathrm{e}^{-R_i\cdot V}\overline{D}_-(\,\mathrm{e}^{R_i\cdot V}\overline{\Phi}_i)+\\ &+\sum_{\ell=1}^M\left\{\frac{k_\ell}{2}D_-(P_\ell+\overline{P}_\ell+V_\ell)\overline{D}_-(P_\ell+\overline{P}_\ell+V_\ell)+\frac{i}{2e_\ell^2}\Sigma_\ell(\partial_0-\partial_1)\overline{\Sigma}_\ell\right\}. \quad (7.3)\end{aligned}$$

The modified current

$$\mathcal{J}=\mathcal{J}^\circ+\frac{1}{2}\sum_{i,\ell}R_{i\ell}[\overline{D}_-,D_-](P_\ell+\overline{P}_\ell+V_\ell) \quad (7.4)$$

satisfies the right-chiral condition $\overline{D}_+\mathcal{J}=0$. The components of this current form an $\mathcal{N}=2$ superconformal algebra with central charge

$$c=3\left(N+\sum_{\ell=1}^M\frac{2b_\ell^2}{k_\ell}\right), \quad (7.5)$$

where $b_\ell := \sum_{i=1}^N R_{i\ell}$. If we make a natural assumption that for large $-\mathrm{Re}\,P_\ell$ the theory flows to a free theory, we can argue as before that the full theory flows to a

SCFT with central charge given by (7.5). The linear terms in (7.4) show that there is a linear dilaton in such an asymptotic region, with the components of the gradient proportional to the chiral anomaly coefficients b_ℓ. Thus the system describes a $2N$-dimensional background with a non-trivial dilaton profile (except in the case where all b_ℓ vanish).

As before, dualization of Φ_i and P_ℓ turns them into twisted chiral superfields Y_i and Y_{P_ℓ} of period $2\pi i$. The twisted superpotential is given by

$$\widetilde{W} = \sum_{\ell=1}^{M} \Sigma_\ell \left(\sum_{i=1}^{N} R_{i\ell} Y_i + Y_{P_\ell} \right) + \sum_{i=1}^{N} \mathrm{e}^{-Y_i}\,, \tag{7.6}$$

where the exponential terms are from Φ_i vortices. The Kähler potential for Y_i is vanishingly small in the continuum limit, but that for Y_{P_ℓ} remains finite and equal to

$$-|Y_{P_\ell}|^2/2k_\ell\,.$$

In the infrared limit $e_\ell \to \infty$, it is appropriate to integrate out the gauge multiplets, which imposes a constraint $\sum_{i=1}^{N} R_{i\ell} Y_i + Y_{P_\ell} = 0$. Thus we are left with a theory of N fields Y_i with the following Kähler potential and superpotential:

$$K = -\frac{1}{2} \sum_{i,j=1}^{N} g_{ij} \overline{Y}_i Y_j + \cdots\,, \tag{7.7}$$

$$\widetilde{W} = \sum_{i=1}^{N} \mathrm{e}^{-Y_i}\,, \tag{7.8}$$

where the terms denoted by dots are small in the asymptotic region, and g_{ij} is given by

$$g_{ij} = \sum_{\ell=1}^{M} R_{i\ell} \frac{1}{k_\ell} R_{j\ell}\,. \tag{7.9}$$

If we omit the terms denoted by dots in the Kähler potential, then the theory is conformally invariant, with the superconformal algebra generated by the supercurrent

$$\widetilde{\mathcal{J}} = \sum_{i,j} \frac{1}{2} g_{ij} D_- Y_i \overline{D}_- \overline{Y}_j + \sum_{ij} g_{ij} (\partial_0 - \partial_1) \mathrm{Im}\, Y_j\,. \tag{7.10}$$

Its central charge is given by (7.5). This suggests that the terms denoted by dots in (7.7) vanish in the IR limit.

7.3 Squashed Toric Sigma-Models

Including matter fields transforming inhomogeneously under the gauge group, like P in our theory, provides interesting generalizations of the standard linear sigma-models. In this way one can obtain not only new superconformal field theories, but also new massive $\mathcal{N} = 2$ field theories. Since this topic is beyond the scope of this paper, we shall only briefly comment on it.

Consider a $\mathrm{U}(1)^k$ gauge theory with N chiral superfields Φ_i of charge Q_i^a ($i = 1, \ldots, N$, $a = 1, \ldots, k$) and FI-Theta parameters $t^a = r^a - i\theta^a$. It has a flavor symmetry group $\mathrm{U}(1)^{N-k}$ acting on Φ_i with charges $R_{i\ell}$ ($\ell = 1, \ldots N-k$) complementary to Q_i^a. For a suitable choice of r^a, the space of classical vacua is a toric manifold X of dimension $N-k$ where the $\mathrm{U}(1)^{N-k}$ flavor group action determines the structure of the torus fibration. The metric on X is obtained by the standard Kähler reduction. For example, for $\mathrm{U}(1)$ gauge theory with two charge 1 chiral fields the classical moduli space is $X = \mathbb{CP}^1$ with the round (Fubini-Study) metric. At low energies the theory reduces to the non-linear supersymmetric sigma model on X.

Now let us consider the following deformation of this system. We gauge the $\mathrm{U}(1)^{N-k}$ flavor group and introduce for each $\mathrm{U}(1)$ factor a chiral superfield P_ℓ transforming inhomogeneously. The action of the system reads

$$S = \frac{1}{2\pi}\int \mathrm{d}^2x \left\{ \int \mathrm{d}^4\theta \Big[\sum_{i=1}^{N} \overline{\Phi}_i \, \mathrm{e}^{Q_i\cdot V + R_i\cdot V'} \Phi_i - \sum_{a=1}^{k} \frac{1}{2e_a^2}|\Sigma_a|^2 \Big] + \mathrm{Re}\int \mathrm{d}^2\widetilde{\theta} \sum_{a=1}^{k} t^a \Sigma_a + \right.$$
$$\left. + \int \mathrm{d}^4\theta \Big[\sum_{\ell=1}^{N-k} \frac{k_\ell}{4}(P_\ell + \overline{P}_\ell + V'_\ell)^2 - \sum_{\ell=1}^{N-k} \frac{1}{2e_\ell^2}|\Sigma'_\ell|^2 \Big] \right\}, \tag{7.11}$$

where $Q_i \cdot V = \sum_{a=1}^{k} Q_i^a V_a$ and $R_i \cdot V' = \sum_{\ell=1}^{N-k} R_{i\ell} V'_\ell$. The vacuum manifold X' is again a toric manifold with the same complex structure and the same Kähler class as X, but with a different Kähler metric. For large r^a's, deep in the interior of the base of the torus fibration, the sizes of the torus fibers are constants proportional to $\sqrt{k_\ell}$. We will say that X' is a "squashed version" of the toric manifold, and we obtain the sigma model on a squashed toric manifold at low energies. For $X = \mathbb{CP}^1$ (round 2-sphere), X' looks like a sausage, so we obtain a supersymmetric version of the "sausage model" of [32]. In the limit $k_\ell \to \infty$, the P_ℓ-Σ'_ℓ pairs decouple, and we recover the sigma-model on the "round toric manifold" X.

The theory is expected to flow to a non-trivial superconformal field theory when $\sum_{i=1}^{N} Q_i^a = 0$ for all a. If this condition is fulfilled, then the central charge of the IR fixed point is

$$c = 3\left(N - k + \sum_{\ell=1}^{N-k} \frac{2b_\ell^2}{k_\ell} \right), \tag{7.12}$$

where $b_\ell = \sum_{i=1}^{N} R_{i\ell}$. In the limit $k_\ell \to \infty$ (no squashing), $c/3$ becomes the complex dimension $N-k$ of the manifold X.

The dual theory is found as above, i.e. by dualizing Φ_i and P_ℓ, taking account of the Φ_i-vortices, and integrating out the gauge multiplets. We find that the dual

Kähler potential and twisted superpotential are

$$K = -\frac{1}{2}\sum_{i,j=1}^{N} g_{ij}\overline{Y}_i Y_j + \cdots\,, \qquad \widetilde{W} = \sum_{i=1}^{N} \mathrm{e}^{-Y_i}\,, \tag{7.13}$$

where g_{ij} is defined by (7.9) with $M = N - k$. This time, however, integration over the gauge multiplets Σ_a imposes a constraint

$$\sum_{i=1}^{N} Q_i^a Y_i = t^a\,. \tag{7.14}$$

This is the mirror of the sigma-model on the squashed toric manifold X'. It is the same as the mirror of the sigma-model on the "round toric" X, except that the Kähler potential is now finite whereas that for the mirror of X is vanishingly small in the continuum limit [1]. For example, when $X = \mathbb{CP}^1$, X' is sausage-shaped, and we find that the mirror of the supersymmetric sausage model is the $\mathcal{N} = 2$ sine-Gordon model with a finite Kähler potential. This equivalence has been conjectured by Fendley and Intriligator [21] as a natural generalization of [33].

The introduction of matter fields which transform inhomogeneously under the gauge group is analogous to the introduction of "magnetic" gauge fields with BF couplings in $2+1$ dimensional gauge theories [34]. In fact, in $2+1$ dimensions they are related by abelian electric-magnetic duality. Mirror symmetry between a squashed toric sigma-model and the Landau-Ginzburg model with a finite Kähler potential can also be derived from the all-scale $\mathcal{N} = 4$ mirror symmetry in $2+1$ dimensions [34] by an RG flow to an $\mathcal{N} = 2$ mirror [35] and further compactification to $1+1$ dimensions [22].

8. Concluding remarks

We have proved the equivalence of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ Kazama-Suzuki supercoset model and $\mathcal{N} = 2$ Liouville theory. We first argued that a super-renormalizable gauge theory flows to the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ supercoset model. The argument had three ingredients: the analysis of the RG flow in the one-loop approximation which is valid for $k \gg 1$, an exact computation of the infrared central charge of the gauge theory, and the analysis of marginal deformations of the supercoset. We then used the argument of [1] to find the dual description of the gauge theory. This dual theory flows in the IR limit to the $\mathcal{N} = 2$ Liouville theory. We also gave an alternative derivation of the mirror dual by showing that the gauge theory can be continuosly deformed to the $\mathcal{N} = 2$ Liouville theory while leaving the infrared fixed point unchanged.

This example teaches us an important lesson: a super-renormalizable gauge theory can flow to a background with a non-trivial dilaton profile, including a region with a linear dilaton. We have shown how the dilaton is generated in two different ways: by a one-loop analysis of the RG flow, and by computing the currents of the $\mathcal{N} = 2$ superconformal algebra in the topologically twisted gauge theory. In the first instance, we observed that the RG flow has a well-defined fixed point only if the target-space coordinates transform non-trivially under the Weyl rescaling. This is a signature of the dilaton gradient. In the second instance, we saw that naïve superconformal generators must be corrected by terms linear in fields because of the axial/Konishi anomaly of the gauge theory.

Another interesting aspect of this work is that it sheds some light on the relation between coset models and Landau-Ginzburg models. It has been known for a long time that many (super)conformal field theories can be realized as coset models as well as the IR limits of Landau-Ginzburg models, but the relation between the two descriptions has not been well understood. The present work is the first example where the two descriptions are connected in a rather transparent way. It would be interesting to see if the methods of this paper can be extended to other models, for example, the K-th $\mathcal{N} = 2$ unitary minimal model which can be realized as the $\mathrm{SU}(2)_K/\,\mathrm{U}(1)$ Kazama-Suzuki model or as the IR limit of the Landau-Ginzburg model with the superpotential $W = X^{K+2}$. In fact, the equivalence of the two models motivated the observation of [10, 11] that certain correlation functions of the $\mathrm{SL}(2,\mathbb{R})_{1+2}/\,\mathrm{U}(1)$ Kazama-Suzuki model and the $W = X^{-1}$ Landau-Ginzburg model agree. (As pointed out in the first reference of [30], for certain purposes $\mathrm{SL}(2,\mathbb{R})_K/\,\mathrm{U}(1)$ can be regarded as an analytic continuation of $\mathrm{SU}(2)_K/\,\mathrm{U}(1)$ to negative K.) More generally, it was proposed in [12] that there is a relation between the $\mathrm{SL}(2,\mathbb{R})_{k+2}/\,\mathrm{U}(1)$ Kazama-Suzuki model and $(W = X^{-k})/\mathbb{Z}_k$ Landau-Ginzburg orbifold (for integer k). As should be clear by now, these observations and conjectures can be regarded as a consequence of the supersymmetric FZZ duality in the special case $k \in \mathbb{N}$, if we identify e^{-Y} with X^{-k}.

Our research was partly motivated by the bosonic FZZ duality. In this paper, we have only considered the supersymmetric version, but it is important to understand the FZZ duality itself. One could attempt to apply the methods of this paper to gain some understanding of this duality. For example, one could try to find a super-renormalizable gauge theory which flows to the bosonic coset model, and then look for a dual description. Without supersymmetry, one may not be able to make an exact statement, but one may be able to see qualitatively how the FZZ duality emerges. Alternatively, one could start with the supersymmetric FZZ duality and consider a supersymmetry breaking perturbation which is relevant or marginally relevant and gives a mass to the fermions but not to the bosons. Then one should analyze the corresponding perturbation of the $\mathcal{N} = 2$ Liouville theory. In particular, it would be interesting to understand the origin of the restriction $k > 2$ in the bosonic FZZ duality.

Another interesting direction to pursue is to study D-branes in the supercoset/Liouville theory. Since this SCFT is relevant for both the deformed conifold and the ALE space [36, 12, 9, 13, 37], such a study should improve our understanding of Dbrane dynamics near the conifold and ALE singularities. The supercoset/Liouville theory also describes Little String Theories in a double scaling limit [9], so there should also be a connection with D-branes in the presence of NS 5-branes. For a discussion of D-branes in $\mathcal{N}=2$ Liouville theory and for references on D-branes in a linear dilaton background, see for example [38]. The relation between the descriptions of D-branes in the coset models and in the Landau-Ginzburg models also deserves study, and the present work may be useful in this regard.

A. Conventions

Here we record our conventions for superfields on $(2,2)$ superspace with coordinates x^0, x^1 (bosonic), $\theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-$ (fermionic). The bosonic coordinates span the flat Minkowski space (metric $\eta_{00}=-1$, $\eta_{11}=1$, and $\eta_{01}=0$) and we often use the light cone coordinates $x^\pm = x^0 \pm x^1$ and derivatives $\partial_\pm := \partial/\partial x^\pm = (\partial_0 \pm \partial_1)/2$. The fermionic coordinates are related by complex conjugation: $(\theta^\pm)^\dagger = \overline{\theta}^\pm$.

Supersymmetry transformation are represented on superfields by derivative operators

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm\,\partial_\pm\,, \tag{A.1}$$

$$\overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm\,\partial_\pm\,, \tag{A.2}$$

which obey $\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} = -2i\partial_\pm$. Another pair of derivatives

$$\begin{aligned} D_\pm &= \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm\,\partial_\pm\,, \\ \overline{D}_\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm\,\partial_\pm\,, \end{aligned} \tag{A.3}$$

anti-commutes with $\mathcal{Q}_\pm$, $\overline{\mathcal{Q}}_\pm$, and obeys $\{D_\pm, \overline{D}_\pm\} = 2i\partial_\pm$. Vector/axial R-rotations are

$$\begin{aligned} \mathrm{e}^{i\alpha F_V} &: \mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm) \mapsto \mathrm{e}^{i\alpha q_V}\mathcal{F}(x^\mu,\, \mathrm{e}^{-i\alpha}\theta^\pm,\, \mathrm{e}^{i\alpha}\overline{\theta}^\pm) \\ \mathrm{e}^{i\beta F_A} &: \mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm) \mapsto \mathrm{e}^{i\beta q_A}\mathcal{F}(x^\mu,\, \mathrm{e}^{\mp i\beta}\theta^\pm,\, \mathrm{e}^{\pm i\beta}\overline{\theta}^\pm)\,, \end{aligned} \tag{A.4}$$

where q_V/q_A are the vector/axial R-charges of $\mathcal{F}$. A chiral superfield Φ obeys $\overline{D}_\pm\Phi = 0$, while a twisted chiral superfield U obeys $\overline{D}_+U = D_-U = 0$. A supersymmetric action is constructed from D-terms, F-terms, and twisted F-terms which are given by the following superspace integrals respectively:

$$\int \mathrm{d}^2x\,\mathrm{d}^4\theta\,K(\mathcal{F}_i) = \int \mathrm{d}^2x\,\mathrm{d}\theta^+\mathrm{d}\theta^-\mathrm{d}\overline{\theta}^-\,\mathrm{d}\overline{\theta}^+\,K(\mathcal{F}_i), \tag{A.5}$$

$$\int \mathrm{d}^2x\mathrm{d}^2\theta\,W(\Phi_i) = \int \mathrm{d}^2x\,\mathrm{d}\theta^-\mathrm{d}\theta^+\,W(\Phi_i)\Big|_{\overline{\theta}^\pm=0}, \tag{A.6}$$

$$\int \mathrm{d}^2x\,\mathrm{d}^2\widetilde{\theta}\,\widetilde{W}(U_i) = \int \mathrm{d}^2x\,\mathrm{d}\overline{\theta}^-\,\mathrm{d}\theta^+\,\widetilde{W}(U_i)\Big|_{\overline{\theta}^+=\theta^-=0}. \tag{A.7}$$

Here $K(-)$ is an arbitrary differentiable function of arbitrary superfields $\mathcal{F}_i$, $W(\Phi_i)$ is a holomorphic function of chiral superfields Φ_i, and $\widetilde{W}(U_i)$ is a holomorphic function of twisted chiral superfields U_i.

The vector superfield in the Wess-Zumino gauge is expressed as

$$\begin{aligned} V = &\ \theta^-\overline{\theta}^-(v_0 - v_1) + \theta^+\overline{\theta}^+(v_0 + v_1) - \theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} + \\ &+ i\theta^-\theta^+(\overline{\theta}^-\overline{\lambda}_- + \overline{\theta}^+\overline{\lambda}_+) + i\overline{\theta}^+\overline{\theta}^-(\theta^-\lambda_- + \theta^+\lambda_+) + \theta^-\theta^+\overline{\theta}^+\overline{\theta}^- D\,. \end{aligned} \tag{A.8}$$

The field-strength superfield is given by

$$\Sigma := \overline{D}_+ D_- V = \sigma + i\theta^+\overline{\lambda}_+ - i\overline{\theta}^-\lambda_- + \theta^+\overline{\theta}^-(D - iv_{01}) + \cdots\,, \tag{A.9}$$

where $v_{01} = \partial_0 v_1 - \partial_1 v_0$.

Let us also fix a convention for the normalization of the sigma-model action. For a target space with metric g_{IJ} the sigma-model action on the two-dimensional Minkowski space will be

$$S = \frac{1}{4\pi}\int g_{IJ}(\partial_0 X^I \partial_0 X^J - \partial_1 X^I \partial_1 X^J)\mathrm{d}^2 x\,. \tag{A.10}$$

B. OPE of elementary fields

In this appendix we compute the short distance singularity of the product of two elementary fields of the GLSM (2.2), or (2.3).

The leading singularity for the matter fields is the standard one:

$$\begin{aligned} \phi(x)\overline{\phi}(0) &\sim -\frac{1}{2}\log(x^2)\,, & \psi_\pm(x)\overline{\psi}_\pm(0) &\sim \frac{-i}{x^\pm}\,, \\ p(x)\overline{p}(0) &\sim -\frac{1}{k}\log(x^2)\,, & \chi_\pm(x)\overline{\chi}_\pm(0) &\sim \frac{-2i/k}{x^\pm}\,, \\ \sigma(x)\overline{\sigma}(0) &\sim -e^2\log(x^2)\,, & \lambda_\pm(x)\overline{\lambda}_\pm(0) &\sim \frac{-2ie^2}{x^\pm}\,. \end{aligned} \tag{B.1}$$

More subtle is the subleading singularity and the OPE of gauge fields. To compute them we need to fix the gauge symmetry. We choose the standard Lorentz gauge. Namely, we add to the action (2.3) the term

$$-\frac{1}{2\pi}\int \mathrm{d}^2 x \frac{1}{8\alpha}(\partial^\mu v_\mu)^2\,, \tag{B.2}$$

where α is the gauge parameter that should not appear in any gauge-invariant physical observables. Then it is straightforward to derive the following OPE ($\varphi_P := \mathrm{Im}\,p$)

$$\begin{aligned} \partial_\pm\varphi_P(x)\,\partial_\pm\varphi_P(0) &\sim -\frac{1}{2k}\frac{1}{(x^\pm)^2} + \frac{\alpha}{2}\frac{x^\mp}{x^\pm}\,, \\ \partial_+\varphi_P(x)\,\partial_-\varphi_P(0) &\sim \frac{\pi i}{2k}\delta(x) + \frac{\alpha}{2}\log(x^2)\,, \end{aligned}$$

$$
\begin{aligned}
v_\pm(x)\, v_\pm(0) &\sim \left(\frac{\alpha}{2} - \frac{e^2}{8}\right) \frac{x^\mp}{x^\pm}\,, \\
v_+(x)\, v_-(0) &\sim \left(\frac{\alpha}{2} + \frac{e^2}{8}\right) \log(x^2)\,, \\
\varphi_P(x) v_\pm(0) &\sim -\frac{\alpha}{2} \partial_\mp^{-1} \log(x^2)\,.
\end{aligned}
\tag{B.3}
$$

From this we see that the gauge-invariant current $A_\pm := \partial_\pm \varphi_P + v_\pm$ has the following OPE:

$$
\begin{aligned}
A_\pm(x) A_\pm(0) &\sim -\frac{1}{2k} \frac{1}{(x^\pm)^2} - \frac{e^2}{8} \frac{x^\mp}{x^\pm}\,, \\
A_+(x) A_-(0) &\sim \frac{\pi i}{2k} \delta(x) + \frac{e^2}{8} \log(x^2)\,.
\end{aligned}
\tag{B.4}
$$

In this paper, we do not use the equations (B.3) and (B.4) that include delta-functions, which are convention-dependent contact terms.

C. Konishi anomaly

Let us define $:\psi_\mp(x_1)\overline{\psi}_\mp(x_2):$ by $\psi_\mp(x_1)\overline{\psi}_\mp(x_2) - \frac{-i}{x_1^\mp - x_2^\mp}$. By a one-loop computation, we find

$$
\langle\, :\psi_-(x_1)\overline{\psi}_-(x_2):\, \mathcal{O}\,\rangle \sim -\frac{i}{\pi} \int \frac{\mathrm{d}^2 z}{(x_1^- - z^-)(x_2^- - z^-)} \langle v_+(z) \mathcal{O} \rangle\,. \tag{C.1}
$$

In particular, we have

$$
\begin{aligned}
\langle\, :(\partial_+\psi_-(x_1)\overline{\psi}_-(x_2) + \psi_-(x_1)\partial_+\overline{\psi}_-(x_2)):\, \mathcal{O}\,\rangle &\sim -\left\langle \frac{v_+(x_1) - v_+(x_2)}{x_1^- - x_2^-}\, \mathcal{O} \right\rangle \\
&\sim -\left\langle \left\{ \partial_- v_+(x_2) + \right.\right. \\
&\qquad \left.\left. + \frac{x_1^+ - x_2^+}{x_1^- - x_2^-} \partial_+ v_+(x_2) \right\} \mathcal{O} \right\rangle.
\end{aligned}
$$

We see that the limit $x_1 \to x_2$ is ambiguous. This ambiguity is absent for the gauge-invariant current $\psi_-\overline{\psi}_-$ defined as

$$
\psi_-\overline{\psi}_-(x) := \lim_{x_1 \to x_2} \left(\psi_-(x_1) \exp\left(i \textstyle\int_{x_2}^{x_1} v\right) \overline{\psi}_-(x_2) - \frac{-i}{x_1^- - x_2^-} \right) \tag{C.2}
$$

$$
= \,:\psi_-(x)\overline{\psi}_-(x):\, + v_-(x) + \lim_{x_1 \to x_2} \frac{x_1^+ - x_2^+}{x_1^- - x_2^-} v_+(x)\,. \tag{C.3}
$$

Indeed, we see that

$$
\langle \partial_+(\psi_-\overline{\psi}_-)(x) \mathcal{O} \rangle \sim \langle F_{+-}(x) \mathcal{O} \rangle\,. \tag{C.4}
$$

Similarly, if we define $\psi_+\overline{\psi}_+$ as the limit of $\psi_+(x_1)\exp(i\int_{x_2}^{x_1} v)\overline{\psi}_+(x_2) - \frac{-i}{x_1^+ - x_2^+}$, we find

$$\langle \partial_-(\psi_+\overline{\psi}_+)(x)\mathcal{O}\rangle \sim -\langle F_{+-}(x)\mathcal{O}\rangle\,. \tag{C.5}$$

Gauge-invariant composites that appears in the text are always defined by a formula like (C.2). For instance, let us look at the axial current $j_A^+ = \psi_-\overline{\psi}_- + \cdots$ and $j_A^- = -\psi_+\overline{\psi}_+ + \ldots$ in (4.4). The OPE (C.4), (C.5) based on such a definition is consistent with the chiral anomaly equation $\partial_+ j_A^+ + \partial_- j_A^- = 2F_{+-}$.

Such quantum effects can modify the classical equation

$$\overline{D}_+\mathcal{J}^\circ \stackrel{?}{=} 0\,, \tag{C.6}$$

where $\mathcal{J}^\circ$ is the superfield defined in (4.6). Let us look at the lowest component of $\mathcal{J}^\circ$

$$j_-^\circ = \psi_-\overline{\psi}_- + \frac{k}{2}\chi_-\overline{\chi}_- + \frac{i}{e^2}\sigma\partial_-\overline{\sigma}\,. \tag{C.7}$$

The equation (C.6) would tell us that it commutes with $\overline{Q}_+$. However, when $\psi_-\overline{\psi}_-$ is defined as in (C.3), the commutator becomes

$$[\overline{Q}_+, j_-^\circ] = \frac{i}{2}\lambda_-\,, \tag{C.8}$$

where we have used $[\overline{Q}_+, v_-] = \frac{i}{2}\lambda_-$ and $[\overline{Q}_+, v_+] = 0$. The right-hand side of (C.8) is the lowest component of the superfield $\frac{1}{2}\overline{D}_-\Sigma$. Hence the supersymmetric completion of (C.8) is

$$\overline{D}_+\mathcal{J}^\circ = \frac{1}{2}\overline{D}_-\Sigma\,, \tag{C.9}$$

which can be regarded as the anomalous form of (C.6). One can also explicitly check other components of the superfield equation (C.9). For instance, the θ^--component equation $\{\overline{Q}_+, G_-^\circ\} = -i\partial_-\sigma$ follows from a one-loop computation, while the $\theta^-\overline{\theta}^-$-component $\{\overline{Q}_+, T_-^\circ\} = -\frac{1}{4}\partial_-\lambda_-$ is a consequence of a gauge-invariant definition like (C.2) plus one-loop effects.

D. Parity invariance of (gauged) WZW models

In this appendix we discuss the definition of world-sheet parity for (gauged) WZW models on a group manifold G. The WZW action is given by

$$S_{\rm WZW}(g) = \frac{1}{8\pi}\int_\Sigma \mathrm{Tr}\left[(g^{-1}\partial_0 g)^2 - (g^{-1}\partial_1 g)^2\right]\mathrm{d}^2x + \frac{1}{12\pi}\int_B \mathrm{Tr}\left[(g^{-1}\mathrm{d}g)^3\right], \tag{D.1}$$

where B is a three-dimensional manifold bounded by the two-dimensional world-sheet Σ over which the field g is extended. The WZ term depends on the orientation and is flipped under parity. This can be compensated by the transformation $g \to g^{-1}$, since $g^{-1}\mathrm{d}g \to g\mathrm{d}g^{-1} = -g(g^{-1}\mathrm{d}g)g^{-1}$. The kinetic term is invariant under both parity and $g \to g^{-1}$. Thus, the WZW model is parity-invariant if accompanied by $g \to g^{-1}$.

Gauging by $g \to h^{-1}gh$ for h in a subgroup $H \subset G$ leads to a vector gauged WZW model, where $g^{-1}\partial_\mu g$ in the kinetic term is replaced by $g^{-1}D^{\mathrm{v}}_\mu g = g^{-1}\partial_\mu g + g^{-1}A_\mu g - A_\mu$ and the WZ term is modified by adding

$$\Gamma^{\mathrm{v}}(A,g) = -\frac{1}{4\pi}\int_\Sigma \mathrm{Tr}\left[A(g^{-1}\mathrm{d}g + \mathrm{d}gg^{-1}) + Ag^{-1}Ag\right]. \tag{D.2}$$

Under $g \to g^{-1}$ the covariant derivative transforms as $g^{-1}D^{\mathrm{v}}g \to -g(g^{-1}D^{\mathrm{v}}g)g^{-1}$ and thus the kinetic term is invariant. Furthermore, it is easy to see that (D.2) flips sign under this transformation, $\Gamma^{\mathrm{v}}(A,g^{-1}) = -\Gamma^{\mathrm{v}}(A,g)$. Thus, the vector gauged WZW model is parity invariant, again if accompaied by $g \to g^{-1}$.

Gauging by $g \to h^{-1}gh^{-1}$ for h in an abelian subgroup $H \subset G$ is another possibility called axial gauging. The kinetic term is obtained by replacing $g^{-1}\partial_\mu g \to g^{-1}D^{\mathrm{a}}_\mu g = g^{-1}\partial_\mu g + g^{-1}A_\mu g + A_\mu$, and the WZ term is modified by

$$\Gamma^{\mathrm{a}}(A,g) = -\frac{1}{4\pi}\int_\Sigma \mathrm{Tr}\left[A(g^{-1}\mathrm{d}g - \mathrm{d}gg^{-1}) - Ag^{-1}Ag\right]. \tag{D.3}$$

Under $g \to g^{-1}$ the covariant derivative transforms as $g^{-1}D^{\mathrm{a}}g \to -g(g^{-1}\mathrm{d}g - g^{-1}Ag - A)g^{-1}$, and thus the kinetic term is invariant only if the sign of A is flipped. Also, it is straightforward to see that (D.3) flips sign under $g \to g^{-1}, A \to -A$. Thus the axially gauged WZW model is parity-invariant if accompanied by $g \to g^{-1}$ and $A \to -A$.

Axially gauged $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$**.** The euclidean (bosonic or fermionic) 2d Black Hole is associated with the axial gauging of $\mathrm{SL}(2,\mathbb{R})$ by the $\mathrm{U}(1)$ generated by $i\sigma_2$. Thus the parity should act on the fields as $g \to g^{-1}$ and $A \to -A$. Setting $g = \mathrm{e}^{i\sigma_2(t+\varphi)/2}\,\mathrm{e}^{\rho\sigma_3}\,\mathrm{e}^{i\sigma_2(t-\varphi)/2}$, we see that the transformation $g \to g^{-1}$ corresponds to $\rho \to -\rho, \varphi \to \varphi, t \to -t$. (The last one is compatible with $A \to -A$.) The sign flip of ρ can actually be undone by a π-shift of φ. Hence world-sheet parity can be defined to act on the coordinates as

$$\rho \to \rho\,, \qquad \varphi \to \varphi\,, \qquad t \to -t\,, \qquad A \to -A\,. \tag{D.4}$$

Next let us describe the action of parity on the current algebra. Left and right current algebras are associated with the transformation of the group elements of the form $g \to g_L g g_R$. Under $g \to g^{-1}$ this becomes $g \to g_R^{-1} g g_R^{-1}$. Thus the right-moving currents J^+, J^3, J^- are transformed to the left-moving currents $\tilde{J}^-, \tilde{J}^3, \tilde{J}^+$ and vice versa. In particular, the right-moving lowest-weight representations are transformed to the left-moving highest-weight representations. For example, the representation $\widehat{\mathcal{D}}_j^+ \times \widehat{\mathcal{D}}_j^+$ is exchanged with $\widehat{\mathcal{D}}_j^- \times \widehat{\mathcal{D}}_j^-$.[8]

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