Anisotropy of the Lundgren–Townsend model of fine-scale turbulence

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The effect of a statistically anisotropic distribution of stretched vortices in the Lundgren–Townsend model of the fine-scale structure of homogeneous turbulence is considered. Lundgren's argument that anisotropy does not affect the three-dimensional energy spectrum is confirmed. Examples of velocity derivative moments and one-dimensional vorticity spectra are worked out for the case of an axisymmetric probability distribution. It is found that scaling of three-dimensional vorticity spectra may not be visible in the one-dimensional spectra.

I. INTRODUCTION

An extension of the Lundgren–Townsend model of the fine scales of homogeneous turbulence was recently carried out by Pullin and Saffman,[1] henceforth referred to as PS. This model contains two essential elements. The first, introduced by Townsend,[2] assumes that the fine scales of turbulence can be modeled by a superposition of local solutions of the Navier–Stokes equations for the vorticity in straight rod-like structures. The structures are each subject to vortex stretching supposed to be provided by the local strain field of the global vorticity. The second element refers to the detailed vorticity in the rods. Townsend[2] used Burgers vortices while Lundgren[3] utilized an evolutionary distribution describing a nearly axisymmetric spiral vortex.

Pullin and Saffman[1] concentrated on the vorticity and velocity derivative moments, and made predictions about the dependence of these quantities on Reynolds number and the order of the moment. Assumptions had to be made about the parameters in the model, and, in particular, it was supposed that the stretched vortex structures are distributed isotropically in space. The purpose of this note is to examine possible consequences of an anisotropic distribution of the structures. Lundgren,[3] Eq. (50), points out that the result for the power spectrum $E(k)$ is independent of the anisotropy assumption, since an integration over a shell in wave number space is made, which is equivalent to an integral over a random isotropic orientation of the vortex structures. This statement is confirmed below; see Eq. (22).

We take axes $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ fixed in space, and introduce spherical polar coordinate angles θ and ϕ, with $\mathbf{k}$ the axis of symmetry. The angle ϕ is longitude, and the angle θ is colatitude (see Fig. 1). A vector $\boldsymbol{\rho}$ is expressible as

$$\boldsymbol{\rho}=\rho(\mathbf{k}\cos\theta+\mathbf{i}\sin\theta\cos\phi+\mathbf{j}\sin\theta\sin\phi). \tag{1}$$

The orientation of the structure will be defined by unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, with $\mathbf{e}_3$ parallel to the vorticity in the structure, i.e.,

$$\boldsymbol{\omega}=\omega(r_1,r_2)\mathbf{e}_3, \tag{2}$$

where r_1,r_2 are coordinates in the plane of the cross section in the directions $\mathbf{e}_1$, $\mathbf{e}_2$ (see Fig. 2). The unit vectors $\mathbf{e}_i$ are expressible in terms of the fixed coordinate system by introducing Euler angles, α, β, γ, where α is colatitude, β is longitude, and γ is spin (see, e.g., Jeffreys and Jeffreys,[4] Sec. 4.034). The relations are

$$\begin{aligned}\mathbf{e}_1&=E_{11}\mathbf{i}+E_{12}\mathbf{j}+E_{13}\mathbf{k},\\ \mathbf{e}_2&=E_{21}\mathbf{i}+E_{22}\mathbf{j}+E_{23}\mathbf{k},\\ \mathbf{e}_3&=E_{31}\mathbf{i}+E_{32}\mathbf{j}+E_{33}\mathbf{k},\end{aligned} \tag{3}$$

where the E_{ij} are the coefficients of the matrix

$$\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma-\sin\beta\sin\gamma & \cos\alpha\sin\beta\cos\gamma+\cos\beta\sin\gamma & -\sin\alpha\cos\gamma \\ -\cos\alpha\cos\beta\sin\gamma-\sin\beta\cos\gamma & -\cos\alpha\sin\beta\sin\gamma+\cos\beta\cos\gamma & \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{pmatrix}. \tag{4}$$

When $\gamma=0$, the unit vectors $\mathbf{k}$, $\mathbf{e}_1$, and $\mathbf{i}$ are coplanar.

In Sec. II, we shall consider the effect of anisotropy on the average values of the squares and cubes of the velocity gradients at a single point. This will be followed in Sec. III by consideration of the energy spectrum, and in Sec. IV we shall derive expressions for the one-dimensional vorticity spectra and calculate some examples.

II. VELOCITY GRADIENTS

In the fixed coordinate system, the Cartesian components of velocity are denoted by (u,v,w) or $(u_{1'},u_{2'},u_{3'})$, and the axes are (x,y,z) or $(x_{1'},x_{2'},x_{3'})$, used interchangeably for convenience. The components of the velocity gradient tensor relative to the fixed axes are $\partial u_{i'}/\partial x_{j'}$. When the vorticity in the structures is given by the Lundgren[3]

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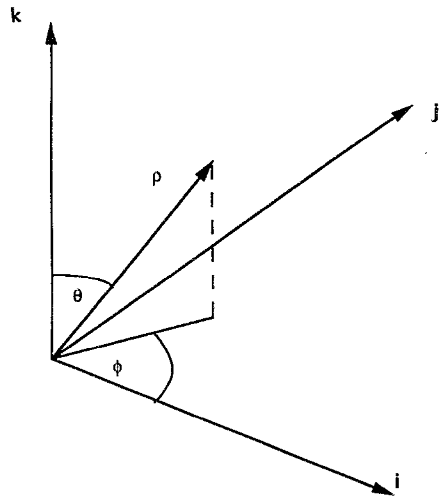

FIG. 1. Coordinate axes fixed in space.

stretched spiral vortex, the components of the velocity gradient tensor are to, leading order (see PS, Sec. V B),

$$\mathscr{D}=\begin{pmatrix} \frac{1}{2}\omega-\frac{1}{2}a & -\frac{1}{2}\omega & 0 \\ \frac{1}{2}\omega & -\frac{1}{2}\omega-\frac{1}{2}a & 0 \\ 0 & 0 & a \end{pmatrix}, \tag{5}$$

where a is the external stretching rate of strain in the model (assumed currently to be independent of time) and ω is a function of r_1 and r_2, and also of t. The components of the tensors are related by

$$\frac{\partial u_{i'}}{\partial x_{j'}}=E_{pi}E_{qj}\mathscr{D}_{pq} \tag{6}$$

(summation over repeated suffices is implied).

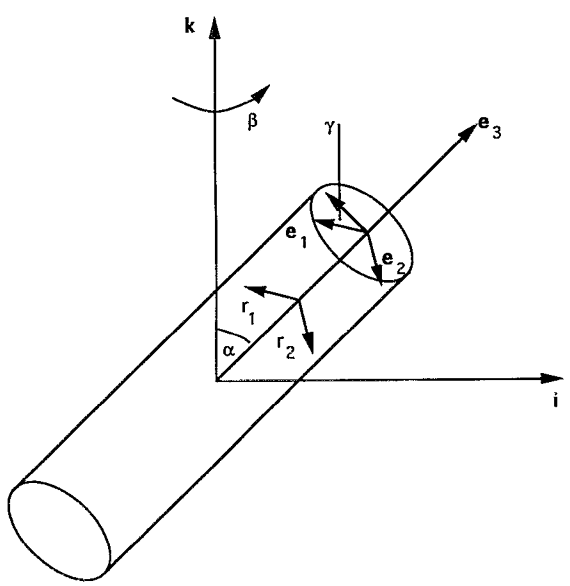

FIG. 2. Orientation of the vortex structure. The vorticity is parallel to e_3.

The ensemble average of moments of the velocity gradients are obtained by averaging powers and products of the terms in (6) over the Euler angles and the cross section of the structure. For example,

$$\overline{\overline{\left(\frac{\partial w}{\partial z}\right)^2}}=\langle E_{p3}E_{q3}E_{r3}E_{s3}\rangle\overline{\mathscr{D}_{pq}\mathscr{D}_{rs}}, \tag{7}$$

where the double overbar denotes an ensemble average, the single overbar denotes an average over the cross section and life of the structure (as evaluated in PS, Sec. VI A), and the angle brackets are averages over the Euler angles,

$$\langle f(E_{ij})\rangle = \int_0^\pi \int_0^{2\pi} \int_0^{2\pi} f(E_{ij})P(\alpha,\beta,\gamma)\sin\alpha \frac{d\alpha}{2}\frac{d\beta}{2\pi}\frac{d\gamma}{2\pi}. \tag{8}$$

If $P\equiv 1$, the distribution is isotropic.

It is expected that the probability density function $P(\alpha,\beta,\gamma)$ of the Euler angles is determined by interaction between large-scale properties of the flow and the vortex structures. This is currently an unsolved problem. As an example, however, of the type of behavior that might occur, consider the case when

$$P=1+\tfrac{1}{3}\delta+\delta\cos 2\alpha, \tag{9}$$

where δ is an anisotropy parameter in the range $-\frac{3}{4}<\delta<\frac{3}{2}$, required to make P non-negative. The turbulence is then statistically axisymmetric. Substituting (9) into (7) and (8) and similar expressions for other velocity derivatives, we obtain

$$\overline{\overline{\left(\frac{\partial w}{\partial z}\right)^2}}=\frac{\overline{\omega^2}}{15}\left(1-\frac{8}{21}\delta\right),\quad \overline{\overline{\left(\frac{\partial w}{\partial x}\right)^2}}=\frac{2\overline{\omega^2}}{15}\left(1-\frac{5}{21}\delta\right), \tag{10}$$

$$\overline{\overline{\left(\frac{\partial u}{\partial x}\right)^2}}=\frac{\overline{\omega^2}}{15}\left(1+\frac{4}{21}\delta\right),\quad \overline{\overline{\left(\frac{\partial u}{\partial y}\right)^2}}=\frac{2\overline{\omega^2}}{15}\left(1+\frac{10}{21}\delta\right), \tag{11}$$

$$\overline{\overline{\left(\frac{\partial u}{\partial z}\right)^2}}=\frac{2\overline{\omega^2}}{15}\left(1-\frac{5}{21}\delta\right).$$

The other components follow from the assumed symmetry.

The ensemble averages of the squares of the vorticity components are found from the relations $\omega_x=E_{31}\omega$, $\omega_y=E_{32}\omega$, and $\omega_z=E_{33}\omega$. On evaluating the averages, we obtain

$$\overline{\overline{\omega_x^2}}=\frac{\overline{\omega^2}}{3}\left(1-\frac{4}{15}\delta\right),\quad \overline{\overline{\omega_z^2}}=\frac{\overline{\omega^2}}{3}\left(1+\frac{8}{15}\delta\right). \tag{12}$$

Cubes of the velocity derivatives (related to the skewness coefficients) are

$$\overline{\overline{\left(\frac{\partial u}{\partial x}\right)^3}}=-\frac{2}{35}a\overline{\omega^2}\left(1+\frac{1}{3}\delta\right),$$
$$\overline{\overline{\left(\frac{\partial w}{\partial z}\right)^3}}=-\frac{2}{35}a\overline{\omega^2}\left(1-\frac{2}{3}\delta\right). \quad (13)$$

We emphasize that the results (10)–(13) hold for rod-like structures with vorticity given by the Lundgren spiral vortex.

III. ENERGY SPECTRUM

We follow the argument in PS, Appendix B. The function $f(r_1,r_2)$ denotes a scalar function of position relative to the structure, which generates a stationary random function of position $\tilde{f}(\mathbf{x})$. We require the ensemble average

$$\overline{\overline{\tilde{f}(\mathbf{x})\tilde{f}(\mathbf{x}+\boldsymbol{\rho})}}\equiv F(\boldsymbol{\rho}). \quad (14)$$

Then by the argument leading from PS, (B2)–(B7), modified here by the explicit inclusion of the spin Euler angle γ, which introduces an extra factor $1/2\pi$ into (B7) and includes the anisotropy function P, we have

$$F(\boldsymbol{\rho})=\frac{1}{2}N\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}|\hat{f}(\kappa_1,\kappa_2)|^2 \times e^{-i\rho_1\kappa_1-i\rho_2\kappa_2}P(\alpha,\beta,\gamma)d\kappa_1\,d\kappa_2\sin\alpha\,d\alpha\,d\beta\,d\gamma, \quad (15)$$

where

$$\rho_1=\boldsymbol{\rho}\cdot\mathbf{e}_1=\rho(\cos\theta\,E_{13}+\sin\theta\cos\phi)E_{11}+\sin\theta\sin\phi\,E_{12}$$
$$\rho_2=\boldsymbol{\rho}\cdot\mathbf{e}_2=\rho(\cos\theta\,E_{23}+\sin\theta\cos\phi\,E_{21}+\sin\theta\sin\phi\,E_{22}), \quad (16)$$

and $\hat{f}(\kappa_1,\kappa_2)$ is the Fourier transform of $f(r_1,r_2)$. The shell-averaged autocorrelation function is defined by (Batchelor,[5] Sec. 3.1)

$$F(\rho)=\iint F(\boldsymbol{\rho})\frac{dS(\rho)}{4\pi\rho^2}, \quad (17)$$

and the power spectrum $\Gamma(k)$ of $\tilde{f}(\mathbf{x})$ is

$$\Gamma(k)=\frac{2}{\pi}\int_0^{\infty}k\rho\sin(k\rho)F(\rho)d\rho$$
$$=\frac{2}{\pi}\int_0^{\infty}k^2\rho^2\frac{\sin k\rho}{k\rho}\iint\frac{dS(\rho)}{4\pi\rho^2}F(\boldsymbol{\rho})d\rho. \quad (18)$$

We substitute into (18) the identity

$$\frac{\sin k\rho}{k\rho}=\frac{1}{4\pi}\iint e^{i(k_1\rho_1+k_2\rho_2+k_3\rho_3)}\frac{dS(k)}{k^2}, \quad (19)$$

and the expression (15), to give

$$\Gamma(k)=\frac{N}{16\pi^3}\int k^2\rho^2\,d\rho\iint e^{i\mathbf{k}\cdot\boldsymbol{\rho}}\frac{dS(k)}{k^2}\iint\frac{dS(\rho)}{\rho^2} \times\iiint\iint|\hat{f}|^2e^{-i\boldsymbol{\rho}\cdot\boldsymbol{\kappa}}P\,d\kappa_1\,d\kappa_2\sin\alpha\,d\alpha\,d\beta\,d\gamma. \quad (20)$$

We now note that we can write $d\rho\,dS(\rho)=d\rho_1\,d\rho_2\,d\rho_3$, and integrate with respect to ρ_1, ρ_2, ρ_3, the integration of (20), replacing terms involving ρ by $8\pi^3\delta(k_1-\kappa_1)\delta(k_2-\kappa_2)\delta(k_3)$, to obtain

$$\Gamma(k)=\frac{N}{2}\iint dS(k)\iiint\iint|\hat{f}(\kappa_1,\kappa_2)|^2\delta(k_1-\kappa_1) \times\delta(k_2-\kappa_2)\delta(k_3)d\kappa_1\,d\kappa_2\,P\sin\alpha\,d\alpha\,d\beta\,d\gamma. \quad (21)$$

The integration of (21) with respect to the Euler angles gives $8\pi^2$, and on carrying out the integration with respect to κ_1, κ_2, and k, we obtain [since $\delta(k_3)dS(k)=k\delta(k_3)dk_3\,d\vartheta$, where $k_1=k\cos\vartheta$, $k_2=k\sin\vartheta$]

$$\Gamma(k)=4\pi^2Nk\oint|\hat{f}(k\cos\vartheta,k\sin\vartheta)|^2\,d\vartheta, \quad (22)$$

which is PS Eq. (B16) or Eq. (B19).

When we put $f\equiv\omega(r_1,r_2)$, where ω appears in (2), then (22) gives the three-dimensional power spectrum of the vorticity, from which the shell-averaged three-dimensional energy spectrum can be calculated directly. Since the vorticity distribution in the structures is assumed independent of their orientation, we have the important corollary that when $\omega(r_1,r_2)$ is given by the Lundgren spiral vortex, the resulting $k^{-5/3}$ behavior of $E(k)$ does not depend, in the Lundgren–Townsend model, on the isotropy of the small scales.

IV. ONE-DIMENSIONAL VORTICITY SPECTRA

We consider the longitudinal and transverse spectra of the vorticity field, defined as the Fourier transforms $\Gamma_{3'3'}(k)$ and $\Gamma_{1'1'}(k)$, or equivalently, $\Gamma_{zz}(k)$ and $\Gamma_{xx}(k)$, with respect to ρ of the autocorrelations,

$$W_{3'3'}(\rho)=\overline{\overline{\omega_{3'}(\mathbf{x})\omega_{3'}(\mathbf{x}+\rho\mathbf{e}_{3'})}}, \quad (23)$$

$$W_{1'1'}(\rho)=\overline{\overline{\omega_{1'}(\mathbf{x})\omega_{1'}(\mathbf{x}+\rho\mathbf{e}_{3'})}}, \quad (24)$$

where $\mathbf{e}_{3'}$ is equivalent to the unit vector $\mathbf{k}$. Then, proceeding as above to express the ensemble averages in terms of the structures, we have

$$W_{3'3'}(\rho)=\frac{N}{2}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}P(\alpha,\beta,\gamma)E_{33}^2\sin\alpha\,d\alpha\,d\beta\,d\gamma \times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{f}(\kappa_1,\kappa_2)|^2e^{-i\rho(E_{13}\kappa_1+E_{23}\kappa_2)} \times d\kappa_1\,d\kappa_2, \quad (25)$$

and

$$W_{1'1'}(\rho)=\frac{N}{2}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}P(\alpha,\beta,\gamma)E_{31}^2\sin\alpha\,d\alpha\,d\beta\,d\gamma$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{f}(\kappa_1,\kappa_2)|^2e^{-i\rho(E_{13}\kappa_1+E_{23}\kappa_2)}$$
$$\times d\kappa_1\,d\kappa_2. \qquad (26)$$

To obtain the spectra, we multiply by $(1/2\pi)e^{-ik\rho}$ and integrate with respect to ρ from $-\infty$ to ∞, giving

$$\Gamma_{zz}(k)=\frac{N}{2}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}P(\alpha,\beta,\gamma)E_{33}^2\sin\alpha\,d\alpha\,d\beta\,d\gamma$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{f}(\kappa_1,\kappa_2)|^2\delta(k+\kappa_1E_{13}+\kappa_2E_{23})$$
$$\times d\kappa_1\,d\kappa_2, \qquad (27)$$

and

$$\Gamma_{xx}(k)=\frac{N}{2}\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}P(\alpha,\beta,\gamma)E_{31}^2\sin\alpha\,d\alpha\,d\beta\,d\gamma$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{f}(\kappa_1,\kappa_2)|^2\delta(k+\kappa_1E_{13}+\kappa_2E_{23})$$
$$\times d\kappa_1\,d\kappa_2, \qquad (28)$$

where Γ_{zz} is the longitudinal one-dimensional vorticity spectrum and Γ_{xx} is the lateral spectrum. According to (4),

$$E_{33}=\cos\alpha,\quad E_{31}=\sin\alpha\cos\beta,$$
$$E_{13}=-\sin\alpha\cos\gamma,\quad E_{23}=\sin\alpha\sin\gamma. \qquad (29)$$

Note that if $P\equiv1$, Γ_{xx} and Γ_{zz} should satisfy the isotropic relations (see Batchelor[4] Sec. 3.4; note the different orientation of the axes)

$$\Gamma_{zz}(k)=\frac{1}{4}\int_k^{\infty}\left(1-\frac{k^2}{\kappa^2}\right)\frac{\Gamma(\kappa)}{\kappa}\,d\kappa, \qquad (30)$$

$$\Gamma_{xx}(k)=\frac{1}{2}\Gamma_{zz}(k)-\frac{k}{2}\frac{d\Gamma_{zz}}{dk}, \qquad (31)$$

where $\Gamma(k)$ is the power spectrum of the vorticity and is given by (22). The evaluation of the integrals in (27) and (28) is a nontrivial task. A partial simplification can be made when P is independent of γ by using the result (see Appendix A)

$$\iiint G(\gamma)\delta(k-\kappa_1\sin\alpha\cos\gamma+\kappa_2\sin\alpha\sin\gamma)F(\kappa_1,\kappa_2)$$
$$\times d\kappa_1\,d\kappa_2\,d\gamma$$
$$=\sum_{\gamma_0}\frac{1}{\sin\alpha}\iint_{\kappa_1^2+\kappa_2^2>k^2/\sin^2\alpha}\frac{F(\kappa_1,\kappa_2)G(\gamma_0)d\kappa_1\,d\kappa_2}{(\kappa_1^2+\kappa_2^2-k^2/\sin^2\alpha)^{1/2}}, \qquad (32)$$

where the γ_0 are the roots of

$$\frac{k}{\sin\alpha}=\kappa_1\cos\gamma_0-\kappa_2\sin\gamma_0. \qquad (33)$$

When there is no explicit dependence on γ, there are two roots of (33) in the range of γ, so the sum is replaced by a multiplication by 2. Then substituting into (27) and (28), and carrying out the integration with respect to β, we obtain ($\kappa_1=\kappa\cos\vartheta$, $\kappa_2=\kappa\sin\vartheta$)

$$\Gamma_{zz}(k)=N\int_0^{\pi}P_1(\alpha)\cos^2\alpha$$
$$\times\int_{k/\sin\alpha}^{\infty}\int_0^{2\pi}\frac{|\hat{f}|^2\kappa\,d\kappa\,d\vartheta}{(\kappa^2-k^2/\sin^2\alpha)^{1/2}}\,d\alpha \qquad (34)$$

and

$$\Gamma_{xx}(k)=N\int_0^{\pi}P_2(\alpha)\sin^2\alpha$$
$$\times\int_{k/\sin\alpha}^{\infty}\int_0^{2\pi}\frac{|\hat{f}|^2\kappa\,d\kappa\,d\vartheta}{(\kappa^2-k^2/\sin^2\alpha)^{1/2}}\,d\alpha, \qquad (35)$$

where

$$P_1(\alpha)=\int_0^{2\pi}P(\alpha,\beta)d\beta,\quad P_2(\alpha)=\int_0^{2\pi}P(\alpha,\beta)\cos^2\beta\,d\beta. \qquad (36)$$

Using (22), we can write these in terms of the power spectrum

$$\Gamma_{zz}(k)=\frac{1}{4\pi^2}\int_0^{\pi}P_1(\alpha)\cos^2\alpha$$
$$\times\int_{k/\sin\alpha}^{\infty}\frac{\Gamma(\kappa)d\kappa}{(\kappa^2-k^2/\sin^2\alpha)^{1/2}}\,d\alpha \qquad (37)$$

and

$$\Gamma_{xx}(k)=\frac{1}{4\pi^2}\int_0^{\pi}P_2(\alpha)\sin^2\alpha$$
$$\times\int_{k/\sin\alpha}^{\infty}\frac{\Gamma(\kappa)d\kappa}{(\kappa^2-k^2/\sin^2\alpha)^{1/2}}\,d\alpha. \qquad (38)$$

It is shown in Appendix B that (30)–(31) are satisfied when $P\equiv1$.

To our knowledge, there does not exist a simple relation between the one-dimensional vorticity spectra and the one-dimensional velocity spectra that are usually measured. The latter are a special case of the velocity structure functions, whose calculation from the Lundgren–Townsend model is a topic for future work.

In order to illustrate the present results, we calculate Γ_{zz} and Γ_{xx} for a model $\Gamma(k)$, given by

$$\Gamma(k)=B(\epsilon^3/\nu)^{1/4}(k\eta)^{\mu}e^{-Ak\eta}, \qquad (39)$$

where ϵ is the dissipation, $\eta=(\nu^3/\epsilon)^{1/4}$ is the Kolmogorov scale, ν is the kinematic viscosity, $\mu>0$ is a parameter, and A and B are dimensionless constants. We remark that (39) is not the specific form obtained from either the Townsend[2] or Lundgren[3] $\omega(r_1,r_2)$ in the structures, but since the right-hand sides of (37)–(38) are functionals only of $\Gamma(k)$ and $P(\alpha)$, we make the choice (39) first for simplicity and

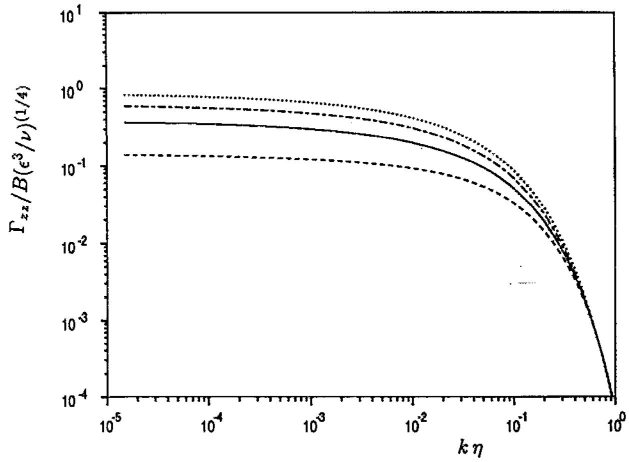

FIG. 3. Longitudinal one-dimensional vorticity spectrum for statistically axisymmetric anisotropy. Key: dash; $\delta=-0.75$. Solid; $\delta=0$. Chain dash; $\delta=0.75$. Dot; $\delta=1.5$

second because, with $\mu=\frac{1}{3}$, this is the dissipation spectrum suggested by some three-dimensional numerical simulations. Kida and Murakami[6] use a least-squares fit to the equivalent form of (39) for the energy spectrum found from numerical simulations of decaying turbulence, and find $B\approx 16.8\pm 1.2$, $\mu\approx 0.4\pm 0.1$, and $A\approx 4.9\pm 0.1$, while Kerr[7] assumes $\mu=\frac{1}{3}$ and finds $B\approx 13.0\pm 0.2$ and $A\approx 5.1 \pm 0.1$ for forced isotropic turbulence at Taylor microscale Reynolds numbers of $O(80)$. Note that the stated values of the factor B in (39) are twice those found for the energy spectra by virtue of $\Gamma(k)=2k^2E(k)$.

We consider statistically axisymmetric turbulence with P given by (9). For isotropic turbulence, $\delta=0$, substitution of (39) into (30) and (31) gives

$$\frac{\Gamma_{zz}}{B(\epsilon^3/\nu)^{(1/4)}}=\frac{1}{4A^\mu}\left(\Gamma[\mu,Ak\eta]-A^2k^2\eta^2\Gamma[\mu-2,Ak\eta]\right), \tag{40}$$

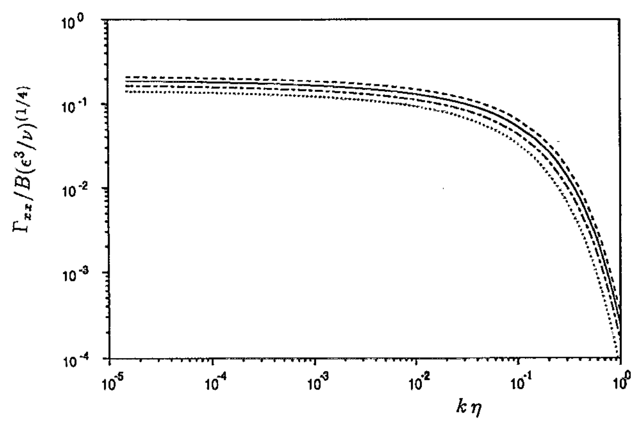

FIG. 4. Lateral one-dimensional vorticity spectrum for statistically axisymmetric anisotropy. For the key see the caption of Fig. 3.

$$\frac{\Gamma_{xx}}{B(\epsilon^3/\nu)^{(1/4)}}=\frac{1}{8A^\mu}\left(\Gamma[\mu,Ak\eta]+A^2k^2\eta^2\Gamma[\mu-2,Ak\eta]\right), \tag{41}$$

where

$$\Gamma[\mu,z]=\int_z^\infty t^{\mu-1}e^{-t}\,dt, \tag{42}$$

is the incomplete gamma function. We use square brackets to distinguish this from the vorticity spectra. When $k\eta$ is large, the asymptotic behavior of the incomplete gamma function gives leading-order dissipation range one-dimensional spectra of the form

$$\frac{\Gamma_{zz}}{B(\epsilon^3/\nu)^{(1/4)}}\sim\frac{(k\eta)^{\mu-2}}{2A^2}\,e^{-Ak\eta}, \tag{43}$$

$$\frac{\Gamma_{xx}}{B(\epsilon^3/\nu)^{(1/4)}}\sim\frac{(k\eta)^{\mu-1}}{4A}\,e^{-Ak\eta}. \tag{44}$$

For $\delta\neq 0$ we calculate the one-dimensional spectra by evaluation of the double integrals (37) and (38), using (39) with $\mu=\frac{1}{3}$ and $A=5.1$. The inner integrals (with respect to k) can be evaluated analytically in terms of generalized hypergeometric functions, but the integral with respect to the Euler angle α was evaluated numerically for fixed δ. This procedure was checked by comparison of the results so obtained with $\delta=0$ with a numerical evaluation of (40) and (41), and a six figure agreement was found over five decades in $k\eta$.

Figures 3 and 4 show the effect of anisotropy on Γ_{zz} and on Γ_{xx}, respectively. When $k\rightarrow 0$, the one-dimensional spectra approach finite limits [these may be evaluated explicitly for $\delta=0$ and $\mu>0$ using (40) and (41)], unlike the three-dimensional spectrum, which vanishes like $k^{1/3}$. Thus, it may be expected that in measurement from experiment or from numerical simulation, the scaling of three-dimensional vorticity spectra may not be visible in the one-dimensional spectra. In the dissipation range it may be seen that the effect of anisotropy is greater on Γ_{xx} than on Γ_{zz}, while outside this range the effect is reversed.

Finally, we remark that if the exponential in (39) is replaced by $\exp(-A_1k^2\eta^2)$, then the dissipation range one-dimensional spectra are accordingly altered but for small $k\eta$ the slow approach to a finite limit at $k=0$ is retained.

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APPENDIX A: EVALUATION OF INTEGRALS

Using the relation

$$\delta[f(x)]=\frac{1}{f'(x_0)}\,\delta(x-x_0), \tag{A1}$$

where $f(x_0)=0$, we can write the left-hand side of (32) after integration with respect to γ as

$$\frac{1}{\sin\alpha}\iint \frac{G(\gamma_0)F(\kappa_1,\kappa_2)d\kappa_1\,d\kappa_2}{\kappa_1\sin\gamma_0+\kappa_2\cos\gamma_0}, \tag{A2}$$

where γ_0 is defined as a function of κ_1 and κ_2 by (33). Squaring the denominator of (A2) and adding to the square of (33), we find that

$$\kappa_1^2+\kappa_2^2-\frac{k^2}{\sin^2\alpha}=(\kappa_1\sin\gamma_0+\kappa_2\cos\gamma_0)^2, \tag{A3}$$

from which follows (32).

APPENDIX B: ISOTROPIC RELATIONS

When $P=1$, we can write (35) and (34) as

$$\Gamma_{xx}=\int_0^\pi \sin^2\alpha\,F\left(\frac{k}{\sin\alpha}\right)d\alpha,$$
$$\Gamma_{zz}=2\int_0^\pi \cos^2\alpha\,F\left(\frac{k}{\sin\alpha}\right)d\alpha. \tag{B1}$$

Then

$$\Gamma_{xx}-\frac{1}{2}\Gamma_{zz}-\frac{k}{2}\frac{d\Gamma_{zz}}{dk}$$
$$=\int_0^\pi\left(\sin^2\alpha\,F-\cos^2\alpha\,F-\frac{k\cos^2\alpha}{\sin\alpha}F'\right)d\alpha$$
$$=-\int_0^\pi\frac{d}{d\alpha}\left[F\left(\frac{k}{\sin\alpha}\right)\sin\alpha\cos\alpha\right]d\alpha=0. \tag{B2}$$

This is relation (31).

For (30), we have from (37),

$$\Gamma_{zz}(k)=\frac{1}{2\pi}\int_0^\pi\cos^2\alpha\,d\alpha\int_{k/\sin\alpha}^\infty\frac{\Gamma(\kappa)d\kappa}{\sqrt{\kappa^2-k^2/\sin^2\alpha}}. \tag{B3}$$

Changing variables by introducing $u=k/\sin\alpha$ and taking α from 0 to $\pi/2$, we can rewrite (B3) as

$$\Gamma_{zz}(k)=\frac{k}{\pi}\int_k^\infty\frac{du}{u^3}\frac{\sqrt{u^2-k^2}}{\sqrt{\kappa^2-u^2}}\int_u^\infty\Gamma(\kappa)d\kappa. \tag{B4}$$

Drawing a plot in a κ-u plane, we see that we can interchange the order of integration with $k<u<\kappa$ and $k<\kappa<\infty$. Then

$$\Gamma_{zz}(k)=\frac{k}{\pi}\int_k^\infty\Gamma(\kappa)d\kappa\int_k^\kappa\frac{du}{u^3}\frac{\sqrt{u^2-k^2}}{\sqrt{\kappa^2-u^2}}. \tag{B5}$$

The integral with respect to u can be evaluated in closed form and is $(\pi/4)(\kappa^2-k^2)/k\kappa^3$. Hence

$$\Gamma_{zz}(k)=\frac{1}{4}\int_k^\infty\left(1-\frac{k^2}{\kappa^2}\right)\frac{\Gamma(\kappa)}{\kappa}d\kappa. \tag{B6}$$

This agrees with (30).

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