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LARGE AMPLITUDE WHIRLING OF AN ELASTIC STRING – A NONLINEAR EIGENVALUE PROBLEM*

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1. Introduction. The whirling of a string is a problem of some antiquity, the linearized solutions to a number of such problems having been derived by D. Bernoulli (1700–1782) and L. Euler (1707–1783). The nonlinear theory of whirling of inextensible strings has been discussed by Kolodner [1] and Caughey [2], [3]. Carrier [4], [5] has discussed the theory of nonlinear vibration of an elastic string using perturbation methods. In the present paper exact solutions will be presented for the whirling of an elastic string with fixed ends, and it will be shown that the modes of free whirling are orbitally stable.

2. Formulation. Consider the problem of the free whirling of an elastic string about a fixed axis OZ, one end of the string being fixed at the origin while the other end is fixed on the Z-axis at a distance L from the origin. It will be assumed that no external forces act on the string. See Fig. 1.

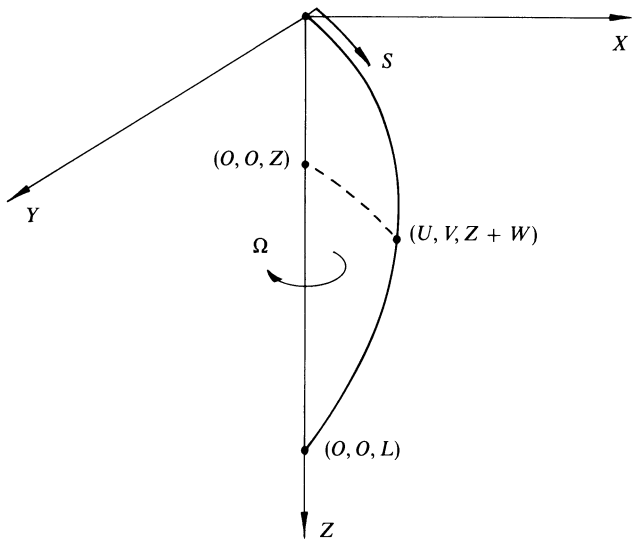

FIG. 1. *Large amplitude whirling of an elastic string*

3. Equations of motion. Denote by
L—the distance between the points of attachment of the string,
S—the arc length measured from the origin,
γ—the constant linear density of the string in the rest position,
A—the rest area of the string,
E—elastic modulus,
$T(S, t)$—tension at a point distance S from the origin,

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T_0—the rest tension in the string,

$\mathbf{X}(S, t) = \{U(Z, t), V(Z, t), Z + W(Z, t)\}$—the position vector of a point in the string with rest coordinates $(0, 0, Z)$,

$U(Z, t), V(Z, t), W(Z, t)$—the Lagrangian displacements of that point.

Consider the motion of the string in a set of rectangular coordinate axes rotating about the Z-axis with angular velocity $\Omega(t)$. The equations of motion are

$$(T\mathbf{X}_S)_Z = (\gamma\mathbf{X}_t)_t, \tag{1}$$

where

$$\mathbf{X} = \{U, V, Z + W\}, \qquad \mathbf{U} = \{U, V, W\}. \tag{2}$$

We shall assume with Carrier [4], [5] that the string obeys the stress-strain law

$$T - T_0 = EA\{[(1 + W_Z)^2 + V_Z^2 + U_Z^2]^{1/2} - 1\} \tag{3}$$

This law is probably as close to reality as we could postulate for most elastic materials. The boundary conditions for (1) are

$$\mathbf{U}(0, t) = \mathbf{0}, \qquad \mathbf{U}(L, t) = \mathbf{0}. \tag{4}$$

Let us introduce the dimensionless variables

$$\begin{gathered} s = \frac{S}{L}, \qquad \mathbf{x} = \frac{\mathbf{X}}{L}, \qquad b = \frac{T}{T_0}, \\ \frac{L^2\gamma\Omega^2}{T_0} = \omega^2, \qquad \tau = \left(\frac{T_0}{\gamma}\right)^{1/2} t, \qquad \mathbf{u} = \frac{\mathbf{U}}{L}, \\ \alpha^2 = \frac{T_0}{EA} = \beta^{-1} \ll 1. \end{gathered} \tag{5}$$

We write out the components of (1) in the dimensionless variables

$$[(1 + b)u_s]_z = u_{\tau\tau} - \omega^2 u - 2\omega v_\tau - \omega_\tau v, \tag{6}$$

$$[(1 + b)v_s]_z = v_{\tau\tau} - \omega^2 v + 2\omega u_\tau + \omega_\tau u, \tag{7}$$

$$[(1 + b)(z + w)_s]_z = w_{\tau\tau}, \tag{8}$$

with the boundary conditions

$$u(0, \tau) = v(0, \tau) = w(0, \tau) = 0,$$

$$u(1, \tau) = v(1, \tau) = w(1, \tau) = 0.$$

4. Steady whirling. We shall investigate the possibility of steady whirling at a constant angular velocity ω. Without loss of generality we shall seek solutions in which the deformations are confined to the x, z-plane.

For steady whirling, we have

$$v = u_\tau = v_\tau = w_\tau = \omega_\tau = u_{\tau\tau} = v_{\tau\tau} = w_{\tau\tau} = 0. \tag{9}$$

Equations (6) and (8) now become

$$[(1 + b)\sin\theta]' + \omega^2 u = 0, \tag{10}$$

$$[(1 + b)\cos\theta]' = 0. \tag{11}$$

In the above equations the prime denotes total differentiation with respect to z, and

$$\sin\theta = \frac{u'}{[(u')^2 + (1 + w')^2]^{1/2}} = \frac{u'}{1 + \alpha^2 b} = \varphi, \tag{12}$$

$$b = \beta[[(u')^2 + (1 + w')^2]^{1/2} - 1]. \tag{13}$$

Integrating (11), we have

$$(1 + b)\cos\theta = H = \text{const.} \tag{14}$$

Since $w(0) = w(1) = 0$,

$$\int_0^1 w'\,dz = 0. \tag{15}$$

However, from (12)

$$\tan\theta = \mu = \frac{u'}{1 + w} = \frac{\varphi}{\sqrt{1 - \varphi^2}}. \tag{16}$$

Thus

$$w' = \frac{u'}{\mu} - 1. \tag{17}$$

Using (12), (14) and (16), we obtain

$$w' = \left[(1 - \alpha^2) + \frac{\alpha^2 H}{\sqrt{1 - \varphi^2}}\right]\sqrt{1 - \varphi^2} - 1. \tag{18}$$

Using (15), we obtain

$$H = \frac{1}{\alpha^2}\left[1 - (1 - \alpha^2)\int_0^1 \sqrt{1 - \varphi^2}\,dz\right] = \frac{1}{\alpha^2}\left[1 - (1 - \alpha^2)\int_0^1 \frac{dz}{\sqrt{1 + \mu^2}}\right]. \tag{19}$$

Differentiating (10), we obtain

$$[(1 + b)\varphi]'' + \omega^2 u' = 0. \tag{20}$$

Using (12), (14) and (16), we obtain

$$\mu'' + \omega^2\left[\frac{1 - \alpha^2}{H}\frac{\mu}{\sqrt{1 + \mu^2}} + \alpha^2\mu\right] = 0. \tag{21}$$

From (10) it follows that

$$\mu'(0) = \mu'(1) = 0. \tag{22}$$

Equations (21) and (22) constitute a nonlinear eigenvalue problem similar to the class of problems studied by Pimbley [6]. Unfortunately the parameter H in (21) depends on the global properties of the solution and so the existence and uniqueness of solutions of the nonlinear eigenvalue problem cannot be established by direct application of Pimbley's general theory.

5. Properties of solutions. We note that $f(\mu) = [(1-\alpha^2)/H]\cdot[\mu/\sqrt{1+\mu^2}] + \alpha^2\mu$ satisfies the global Lipschitz condition

$$|f(\mu) - f(\delta)| \leqq \left(\frac{1-\alpha^2}{H} + \alpha^2\right)|\mu - \delta|. \tag{23}$$

Thus the initial value problem

$$\mu'' + \omega^2\left(\frac{1-\alpha^2}{H}\frac{\mu}{\sqrt{1+\mu^2}} + \alpha^2\mu\right) = 0, \tag{24}$$

$$\mu(0) = a, \quad \mu'(0) = 0, \quad H, \alpha, \omega, \text{constant},$$

has a unique solution for all z. This solution is continuous in the parameters a, ω, α, H.

Multiplying (25) by μ' and integrating the limits of zero and z, we have

$$\mu'^2 = C - \alpha^2\omega^2\left[\frac{2(1-\alpha^2)}{\alpha^2 H}\sqrt{1+\mu^2} + \mu^2\right]. \tag{25}$$

However,

$$\mu(0) = a, \qquad \mu'(0) = a.$$

Therefore

$$\mu' = \pm\alpha\omega\left[\frac{2(1-\alpha^2)}{\alpha^2+1}(\sqrt{1+a^2} - \sqrt{1+\mu^2}) + a^2 - \mu^2\right]^{1/2}. \tag{26}$$

Thus the trajectories of μ in the μ–μ' phase plane are closed ovals, symmetric with respect to both the μ and μ' axes, and passing through the points $(a, 0)$, $(-a, 0)$. Furthermore, solutions of (25) are oscillating and periodic with period Z^*, where

$$\underset{a<\infty}{Z^*} = \frac{4}{\alpha\omega}\int_0^a \frac{d\mu}{\left[\frac{2(1-\alpha^2)}{\alpha^2 H}(\sqrt{1+a^2} - \sqrt{1+\mu^2}) + a^2 - \mu^2\right]^{1/2}} < \infty. \tag{27}$$

(This follows since the only singularity in the integrand is at $\mu = a$, and this singularity is integrable.)

Since the trajectories of μ in the phase plane are symmetric with respect to both the μ and μ' axes, the solution $\mu(z)$ has the following properties:

(i) $\mu(0) = \mu(nZ^*) = a$,
(ii) $\mu(Z^*/2) = \mu((2n+1)Z^*/2) = -a$,
(iii) $\mu(Z^*/4) = \mu((2n+1)Z^*/4) = 0$,
(iv) $\mu(z) = \mu(-z) = \mu(nZ^* - z)$,
(v) $\mu(z) = -\mu(Z^*/2 - z) = -\mu((2n+1)Z^*/2 - z)$,
(vi) $\mu'(0) = \mu'(nZ^*/2) = 0$, where $n = 1, 2, 3 \cdots$.

6. Existence and uniqueness of solutions of nonlinear eigenvalue problem.

THEOREM 1. *For each value of* $[a, n]$, $n = 1, 2, 3 \cdots$, $0 < a < \infty$, *there exist a unique solution* $\mu_n(a, z)$ *and a unique rotational speed* $\omega_n(a)$ *for the nonlinear eigenvalue problem* (21), (22).

Proof. The proof of Theorem 1 is given in Appendix A.

7. **Linearized versus nonlinear problem.** The linearized equation is obtained by setting $u \sim \varepsilon\tilde{u} + O(\varepsilon)$, $b \sim O(\varepsilon)$ as $\varepsilon \to 0$ and $|\tilde{u}| < \infty$. For our present purposes it is more convenient to let $\mu \sim \varepsilon\tilde{\mu} + O(\varepsilon)$ as $\varepsilon \to 0$ and $|\tilde{\mu}| < \infty$.

Using (19) it is easily seen that $H \sim 1 + O(\varepsilon)$ as $\varepsilon \to 0$. Thus

$$\begin{aligned} &\tilde{\mu}'' + \omega^2\tilde{\mu} = 0, \qquad 0 < z < 1, \\ &\tilde{\mu}'(0) = \tilde{\mu}'(1) = 0. \end{aligned} \tag{28}$$

This is a well-posed eigenvalue problem which has the solution

$$\begin{aligned} &\tilde{\mu}_n = A_n \cos, \\ &\omega_n = n\pi, \qquad n = 1, 2, 3 \cdots, \end{aligned} \tag{29}$$

where the A_n are arbitrary constants.

It should be noted that the eigenvalue and eigenfunction corresponding to $n = 0$ are inadmissible since for $n = 0$,

$$\tilde{\mu} = \tilde{u}' = A_0.$$

However, $\int_0^1 \tilde{u}'\, dz = 0 = A_0$. Thus $A_0 \equiv 0$, i.e., $\tilde{\mu} \equiv 0$, the trivial solution.

The linear theory thus predicts that the string will whirl at any one of the eigenvalues, ω_n, but not at any other angular velocity. Experiments on nonlinear whirling appear to show that steady whirling can occur at almost any rotational speed. We shall now show that steady whirling cannot occur below $\omega = \omega_1$.

Let $\mu(a, z)$ be a nontrivial solution of (21) satisfying (22) and with $\mu(a, 0) = a$. Denote the corresponding rotational speed by $\omega(a)$. Multiplying through by $\mu(a, z)$ and integrating by parts, we see that

$$\omega^2(a) = \frac{\int_0^1 \mu'^2\, dz}{\int_0^1 \left\{ \frac{1 - \alpha^2}{H} \frac{\mu^2}{\sqrt{1 + \mu^2}} + \alpha^2\mu^2 \right\} dz}, \tag{30}$$

$$\omega^2(a) \geqq \min_{\substack{\mu \in C_2 \\ \mu' \not\equiv 0 \\ \mu'(0) = \mu'(1) = 0}} \left[\frac{\int_0^1 \mu'^2\, dz}{\int_0^1 \mu^2\, dz} \right] \cdot \frac{\int_0^1 \mu^2\, dz}{\int_0^1 \left\{ \frac{1 - \alpha^2}{H} \frac{\mu^2}{\sqrt{1 + \mu^2}} + \alpha^2\mu^2 \right\} dz}, \tag{31}$$

where C_2 is the class of continuous functions with continuous first derivatives and piecewise continuous second derivatives.

Now the first term on the right-hand side of (31) is simply the lowest nonzero eigenvalue of the eigenvalue problem

$$\begin{aligned} &\mu'' + \lambda^2\mu = 0, \\ &\mu'(0) = \mu'(1) = 0. \end{aligned}$$

Thus

$$\min_{\substack{\mu \in C_2 \\ \mu \not\equiv 0 \\ \mu'(0) = \mu'(1) = 0}} \frac{\int_0^1 \mu'^2\, dz}{\int_0^1 \mu^2\, dz} = \pi^2. \tag{32}$$

From (19) it is easily seen that

$$1 \leqq H \leqq \frac{1}{\alpha^2}.$$

Therefore

$$\int_0^1 \left\{ \frac{1-\alpha^2}{H} \frac{\mu^2}{\sqrt{1+\mu^2}} + \alpha^2\mu^2 \right\} dz \leqq \int_0^1 (1-\alpha^2+\alpha^2)\mu^2\, dz.$$

Thus

$$\omega^2(a) \geqq \pi^2. \tag{33}$$

Thus, nontrivial solutions of the nonlinear eigenvalue problem can occur only if $\omega(a) \geqq \omega_1 = \pi$. Furthermore, since $\mu(a, z)$ is continuous in a

$$\liminf_{a\to 0} \omega(a) = \omega_1 = \pi. \tag{34}$$

Since by Schwarz's inequality

$$\int_0^1 \frac{\mu^2}{\sqrt{1+\mu^2}}\, dz \leqq \left[\int_0^1 \mu^2\, dz\right]^{1/2} \left[\int_0^1 \frac{\mu^2}{1+\mu^2}\, dz\right]^{1/2},$$

thus

$$\omega^2(a) \geqq \pi^2 \frac{\left(\int_0^1 \mu^2\, dz\right)^{1/2}}{\frac{1-\alpha^2}{H}\left(\int_0^1 \frac{\mu^2}{1+\mu^2}\, dz\right)^{1/2} + \alpha^2\left(\int_0^1 \mu^2\, dz\right)^{1/2}}. \tag{35}$$

Since $\mu(a, z)$ is continuous in a and z, and $\mu(a, 0)$ is equal to a,

$$\lim_{a\to\infty} \int_0^1 \mu^2\, dz = \infty, \tag{36a}$$

$$\lim_{a\to\infty} \int_0^1 \frac{\mu^2}{1+\mu^2}\, dz < \infty. \tag{36b}$$

Thus

$$\lim_{a\to\infty} \omega^2(a) \geqq \frac{\pi^2}{\alpha^2}. \tag{37}$$

Consider now the two eigenvalue problems

$$\theta'' + \lambda^2\alpha^2\theta = 0, \qquad \theta'(0) = \theta'(1) = 0, \tag{38}$$

and

$$\mu'' + \omega^2(a)q(z)\mu = 0, \qquad \mu'(0) = \mu'(1) = 0, \tag{39}$$

where

$$q(z) = \left[\frac{1-\alpha^2}{H} \frac{1}{\sqrt{1+\mu^2(z)}} + \alpha^2\right] \geqq \alpha^2. \tag{40}$$

Then by the Sturm comparison theorem, we have

$$\omega_n^2(a) \leqq \lambda_n^2 = \left(\frac{n\pi}{\alpha}\right)^2, \qquad n = 1, 2, 3, \cdots, \tag{41}$$

where $\omega_n(a)$ is the rotational speed associated with a solution of (39) having n internal zeros. In particular, $\omega_1(a)$ is the eigenvalue associated with the solution having one internal zero. Using (37) and (41) we see that

$$\lim_{a \to \infty} \omega_1(a) = \frac{\pi}{\alpha}. \tag{42}$$

Thus

$$\pi \leqq \omega_1(a) \leqq \frac{\pi}{\alpha}. \tag{43}$$

If we integrate (21) over the interval zero to unity and make use of the boundary conditions (22), then

$$\omega^2(a) \int_0^1 \left[\frac{1 - \alpha^2}{H} \frac{\mu}{\sqrt{1 + \mu^2}} + \alpha^2 \mu\right] dz = 0. \tag{44}$$

Since $\omega^2(a) \neq 0$,

$$\int_0^1 \left[\frac{1 - \alpha^2}{H} \frac{\mu}{\sqrt{1 + \mu^2}} + \alpha^2 \mu\right] dz = 0. \tag{45}$$

Thus $\mu(a, z)$ must have at least one zero on the interval $0 \leqq z \leqq 1$. If we select $\omega_1(a)$ such that $\mu_1(a, z)$ satisfies (21) and (22) and has exactly one zero on the interval $0 \leqq z \leqq 1$, then we shall call $\mu_1(a, z)$ the first eigenfunction and $\omega_1(a)$ the first eigenvalue of the nonlinear eigenvalue problem (21), (22). The eigenvalue $\omega_1(a)$ is continuous in a and satisfies (43).

As previously shown, the solutions of the initial value problem (24) are periodic in z of period Z^*, and vanish at the odd multiples of $Z^*/4$. Since $\mu_1(a, z)$ satisfies (24) it is also periodic in z. Furthermore, since $\mu_1'(a, 0) = \mu_1'(a, 1) = 0$, and since $\mu_1(a, z)$ vanishes only once on the interval $0 \leqq z \leqq 1$, it follows that $\mu_1(a, z)$ is of period $Z^* = 2$. Hence

$$\mu_1(a, Z^*/4) = \mu_1(a, 1/2) = 0. \tag{46}$$

From (10) and (14)

$$[(1 + b) \sin \theta]' = -\omega^2 u, \qquad (1 + b) \cos \theta = H.$$

Therefore

$$H[\tan \theta]' = H\mu' = -\omega^2 u.$$

Hence

$$u_1(a, z) = -\frac{H_1(a)}{\omega_1^2(a)} \mu_1'(a, z), \tag{47}$$

where $H_1(a)$ is the unique value of the constant H in (14) which allows $\mu_1(a, z)$ to

satisfy the boundary conditions (22). It then follows from (46) and (47) and the properties of solutions of the initial value problem (24) that $u_1(a, z)$ has no zeros on the interval $0 < z < 1$ and is symmetric about $z = 1/2$.

8. Higher modes. The structure of (21) is such that if $\mu_1(a, z)$ is a solution of (21) with $\omega = \omega_1(a)$, then

$$\mu_n(a, z) = \mu_1(a, nz) \tag{48}$$

is a solution of (21) with ω given by

$$\omega_n(a) = n\omega_1(a). \tag{49}$$

Furthermore, since $\mu_1(a, z)$ is periodic of period 2 and is symmetric about the points $z = n$, $n = 1, 2, 3, \cdots$,

$$\begin{aligned} \mu_n'(a, 0) &= n\mu_1'(a, 0) = 0, \\ \mu_n'(a, 1) &= n\mu_1'(a, n) = 0. \end{aligned} \tag{50}$$

Thus $\mu_n(a, z)$ is a solution of the nonlinear eigenvalue problem. Since $\mu_1(a, z)$ has one zero on the interval $0 \leqq z \leqq 1$, $\mu_n(a, z)$ has exactly n zeros on the same interval. Hence $\mu_n(a, z)$ is the nth eigenfunction of the nonlinear eigenvalue problem and $\omega_n(a)$ is the corresponding eigenvalue.

Using (10), (14) and (16) in the same way as was done for the first eigenfunction $\mu_1(a, z)$, we can easily show that

$$u_n(a, z) = -\frac{H_n(a)}{\omega_n^2(a)}\mu_n'(a, z), \tag{51}$$

where $H_n(a)$ is the unique value of the constant H in (15) which permits $\mu_n(a, z)$ to satisfy the boundary conditions (22). Using (51) and the properties of solutions of the initial value problem (24) we can easily show that $u_n(a, z)$ has exactly $n - 1$ zeros on the interval $0 < z < 1$. It may easily be shown, using similar arguments to those advanced for the first mode, that the nth mode of whirling for rotational speeds in the range

$$\omega_n = n\pi \leqq \omega \leqq \frac{n\pi}{\alpha}. \tag{52}$$

By using (52) the number of distinct modes of whirling in a given range of rotational speeds is easily established.

(53)

(a) If $p\pi/\alpha < n\pi < \omega < (n + 1)\pi < (p + 1)\pi/\alpha$; $p < n$, $a \ll 1$, then

(i) $p\pi/\alpha < \omega < (n + 1)\pi$. It follows from (52) that there exist no modes higher than the nth and no modes lower than the $(p + 1)$st.

(ii) $q\pi < \omega < q\pi/\alpha$; $q = p + 1, p + 2, \cdots, n$. It follows from (52) that the qth mode exists for $q \in [p - 1, n]$. Therefore, if $p\pi/\alpha < n\pi < \omega < (n + 1)\pi < (p + 1)\pi/\alpha$, there exist exactly $n - p$ distinct modes of whirling. Similarly it may be shown that

(b) If $p\pi/\alpha < \omega < n\pi$, there exist exactly $n - p - 1$ distinct modes of whirling.

(c) If $n\pi < \omega < p\pi/\alpha$, there exist exactly $n - p + 1$ distinct modes of whirling.

9. Solution of nonlinear eigenvalue problem. Let us now return to the problem of solving the nonlinear eigenvalue problem (21), (22). In particular, let us consider the first mode of whirling $\mu_1(a, z)$, where

$$\mu_1'' + \omega_1^2(a)\frac{1-\alpha^2}{H_1}\frac{\mu_1}{\sqrt{1+\mu^2}} + \alpha^2\mu_1 = 0, \qquad 0 < z < 1, \tag{53}$$

$$\mu_1'(a, 0) = \mu_1'(a, 1) = 0, \quad \mu_1(a, 0) = a = -\mu_1(a, 1), \quad \mu_1\left(a, \frac{1}{2}\right) = 0, \tag{54}$$

$$H_1 = H_1(a) = \frac{1}{\alpha^2}\left[1 - (1-\alpha^2)\int_0^1 \frac{dz}{\sqrt{1+\mu_1^2}}\right].$$

From (26), we have

$$\mu_1'(a, z) = \pm\alpha\omega_1(a)[f(\sqrt{1+a^2} - \sqrt{1+\mu_1^2}) + a^2 - \mu_1^2]^{1/2}, \tag{55}$$

where

$$f = \frac{2(1-\alpha^2)}{\alpha^2 H_1}.$$

Since

$$z = \int_0^z d\eta = \int_a^{\mu_1(a,z)} \frac{d\mu}{\mu_1'}$$

we have

$$\frac{1}{2} = \int_a^0 d\mu/\mu_1'. \tag{56}$$

Since $\mu_1(a, z) > 0$ for $0 < z < 1/2$, it follows that $\mu_1'(a, z)$ is less than zero on this interval. Therefore

$$\frac{1}{2}\alpha\omega_1(a) = \int_0^1 \frac{d\mu}{[f(\sqrt{1+a^2} - \sqrt{1+\mu^2}) + a^2 - \mu^2]^{1/2}}. \tag{57}$$

The transformation $q = \sqrt{1+\mu_1^2}$ reduces (57) to

$$\frac{1}{2}\alpha\omega_1(a) = \int_1^{q_0=\sqrt{1+a^2}} \frac{q\,dq}{[q^2-1]^{1/2}[f(q_0-q) + q_0^2 - q^2]^{1/2}}, \tag{58}$$

which is of the form

$$\frac{1}{2}\alpha\omega_1(a) = \int_1^{q_0} \frac{q\,dq}{\{a_0(q-\alpha_1)(q-\alpha_2)(q-\alpha_3)(q-\alpha_4)\}^{1/2}}, \tag{59}$$

where $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$,

$$a_0 = -1, \quad \alpha_1 = q_0, \quad \alpha_2 = 1, \quad \alpha_3 = -1, \quad \alpha_4 = -(q_0 + f). \tag{60}$$

This is a standard integral (see, for example, Gröbner and Hofrieter [7, p. 84]).

Therefore

$$\tfrac{1}{2}\alpha\omega_1(a) = \frac{2}{[(q_0+1)(q_0+1+f)]^{1/2}}\left\{2\Pi\left(\frac{\pi}{2} - \frac{q_0-1}{q_0+1}, k\right) - F\left(\frac{\pi}{2}, k\right)\right\},$$

where

$$k^2 = \left[\frac{q_0-1}{q_0+1}\right]\cdot\left[\frac{q_0+f-1}{q_0+f+1}\right],$$

(61)

$$\Pi(\varphi, \rho, k) = \int_0^{\varphi} \frac{d\theta}{(1+\rho\sin^2\theta)\sqrt{1-h^2\sin^2\theta}}$$

is the incomplete[1] elliptic integral of the third kind;

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-h^2\sin^2\theta}}$$

is the incomplete elliptic integral of the first kind. From (55), we have

$$(1-\alpha^2)\int_0^1 \frac{dz}{(1+\mu_1^2)^{1/2}} = 1 - \alpha^2 H_1. \tag{62}$$

If we regard μ_1 as the independent variable, we have

$$(1-\alpha^2)\int_{\mu=a}^{-a} \frac{d\mu}{\mu_1'(1+\mu_1^2)^{1/2}} = 1 - \alpha^2 H_1. \tag{63}$$

Since $\mu_1(a, 0) = -\mu_1(a, 1) = a$, $\mu_1'(a, 0) = \mu_1'(a, 1) = 0$, and $\mu_1'(a, \frac{1}{2}) = 0$, therefore $\mu_1'(a, z) < 0$ for $0 < z < 1$. Hence

$$\frac{2(1-\alpha^2)}{\alpha\omega_1(a)}\int_0^a \frac{d\mu_1}{(1+\mu_1^2)[f(\sqrt{1+a^2}-\sqrt{1+\mu_1^2})+a^2-\mu_1^2]^{1/2}} = 1 - \alpha^2 H_1. \tag{64}$$

The substitution $q = \sqrt{1+\mu_1^2}$, reduces (64) to

$$\frac{2(1-\alpha^2)}{\alpha\omega_1(a)}\int_1^{q_0=\sqrt{1+a^2}} \frac{dq}{[q^2-1]^{1/2}[f(q_0-q)+q_0^2-q^2]^{1/2}} = 1 - \alpha^2 H_1, \tag{65}$$

which is of the form

$$\frac{2(1-\alpha^2)}{\alpha\omega_1(a)}\int_1^{q_0} \frac{dq}{\{a_0(q-\alpha_1)(q-\alpha_2)(q-\alpha_3)(q-\alpha_4)\}^{1/2}} = 1 - \alpha^2 H_1, \tag{66}$$

where a_0, α_1, α_2, α_3, α_4 are given by (60). This is again a standard integral [7]. Therefore

$$\frac{(1-\alpha^2)}{2\omega_1(a)}\cdot\frac{4}{[(q_0+1)(q_0+1+f)]^{1/2}}\cdot F\left(\frac{\pi}{2}, k\right) = 1 - \alpha^2 H_1, \tag{67}$$

where k is given by (61).

[1] For $\varphi = \pi/2$ the functions $\Pi(\varphi, \rho, h)$ and $F(\varphi, h)$ are said to be complete elliptic integrals.

Eliminating $\omega_1(a)$ between (61) and (67), we determine the constant H_1 by the transcendental equation

$$\frac{1-\alpha^2 H_1}{1-\alpha^2} = \frac{1}{1-\alpha^2} - \frac{2}{f} = \frac{F(\pi/2, k)}{2\Pi\left(\frac{\pi}{2}, -\frac{q_0-1}{q_0+1}, k\right) - F\left(\frac{\pi}{2}, k\right)}. \tag{68}$$

Having solved for $H_1 = H_1(a)$, the eigenvalue $\omega_1(a)$ is given by

$$\omega_1(\mathrm{a}) = \frac{4}{\alpha} \cdot \frac{1-\alpha^2}{1-\alpha^2 H_1} \cdot \frac{F(\pi/2, k)}{[(q_0+1)(q_0+1+f)]^{1/2}}. \tag{69}$$

From (10), (14) and (16), we have

$$u_1(a, z) = -\frac{H_1(a)}{\omega_1^2(a)} \mu_1'(a, z) \tag{70}$$

$$= \frac{\alpha}{\omega_1(a)} [f(q_0 - q) + q^2 - q^2]^{1/2} H_1(a), \tag{71}$$

where

$$\begin{aligned} q_0 &= \sqrt{1+a^2} = \sec\theta_0, \qquad q = \sqrt{1+\mu^2} = \sec\theta, \\ -\theta_0 &\leqq \theta \leqq \theta_0, \qquad \tan\theta_0 = a. \end{aligned} \tag{72}$$

Now

$$z + w_1(a, z) = \int_0^z (1 + w_1')\, dz \tag{73}$$

$$= \alpha^2 H_1 z + (1-\alpha^2) \int_0^z \frac{dz}{\sqrt{1+\mu_1^2}}. \tag{74}$$

Using (26), we may integrate (74) to give

$$z + w_1(a, z) = \frac{2}{\alpha\omega_1(a)} \cdot \frac{1}{[(q_0+1)(q_0+1+f)]^{1/2}} \tag{75}$$

$$\{\alpha^2 H_1 [2\Pi(\phi, \rho, k) - F(\phi, k)]_{\phi=\Phi}^{\pi/2} + [(1-\alpha^2) F(\phi, k)]_{\phi=\Phi}^{\pi/2}\},$$

where

$$\begin{aligned} &\sin\Phi = \left[\frac{(q_0+1)(q-1)}{(q_0-1)(q+1)}\right]^{1/2} \operatorname{sgn}\mu, \\ &q = \sqrt{1+\mu^2} = \sec\theta, \\ &-\theta_0 < \theta < \theta_0, \qquad \tan\theta_0 = a, \\ &\rho = -\frac{q_0-1}{q_0+1}, \qquad k^2 = \frac{q_0-1}{q_0+1} \cdot \frac{q_0+f-1}{q_0+f+1}, \\ &f = \frac{2(1-\alpha^2)}{\alpha^2 H_1}. \end{aligned} \tag{76}$$

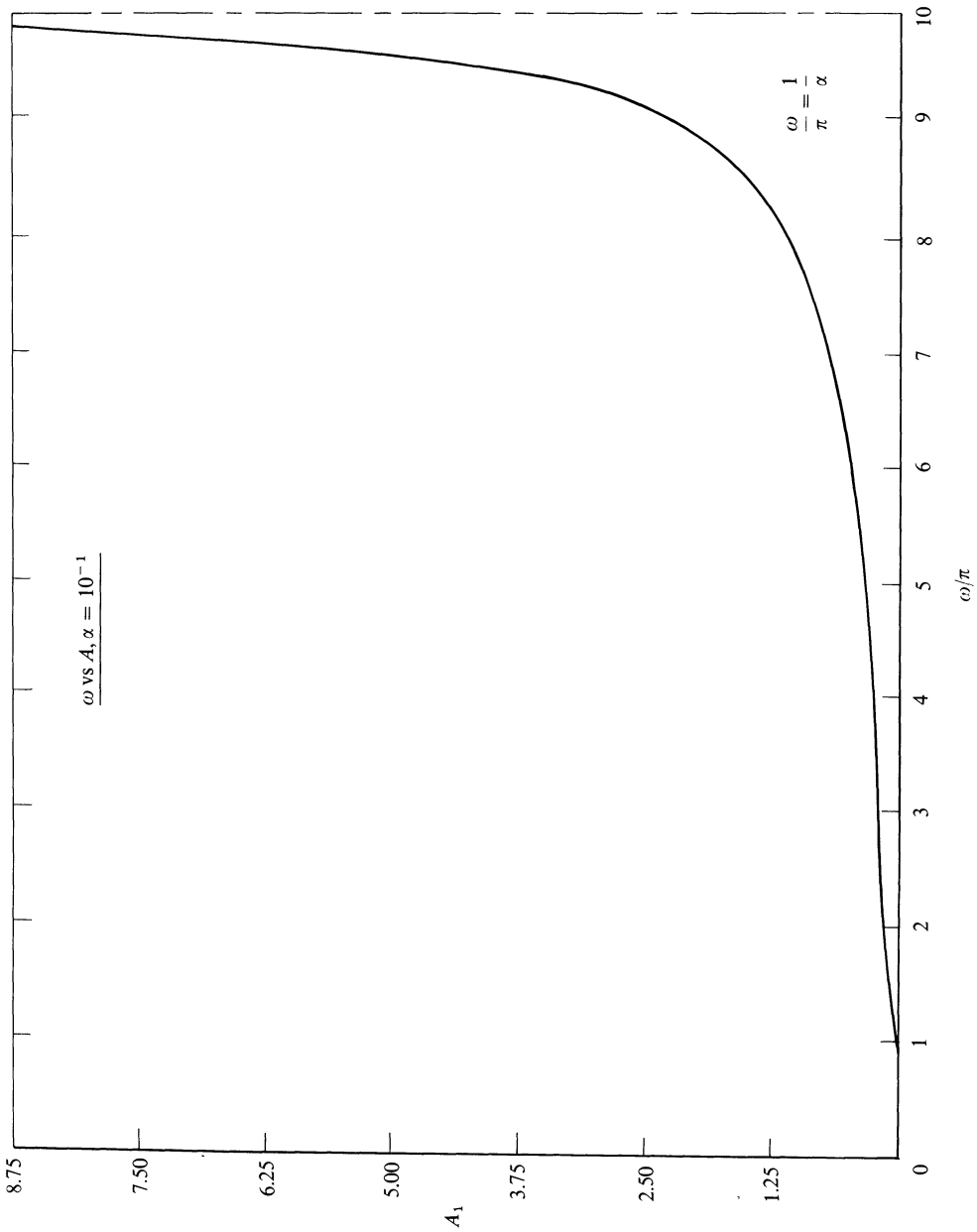
ω vs $A, \alpha = 10^{-1}$
$\frac{\omega}{\pi} = \frac{1}{\alpha}$
8.75
7.50
6.25
5.00
3.75
2.50
1.25
A_1
0
1
2
3
4
5
6
7
8
9
10
ω/π

FIG. 2

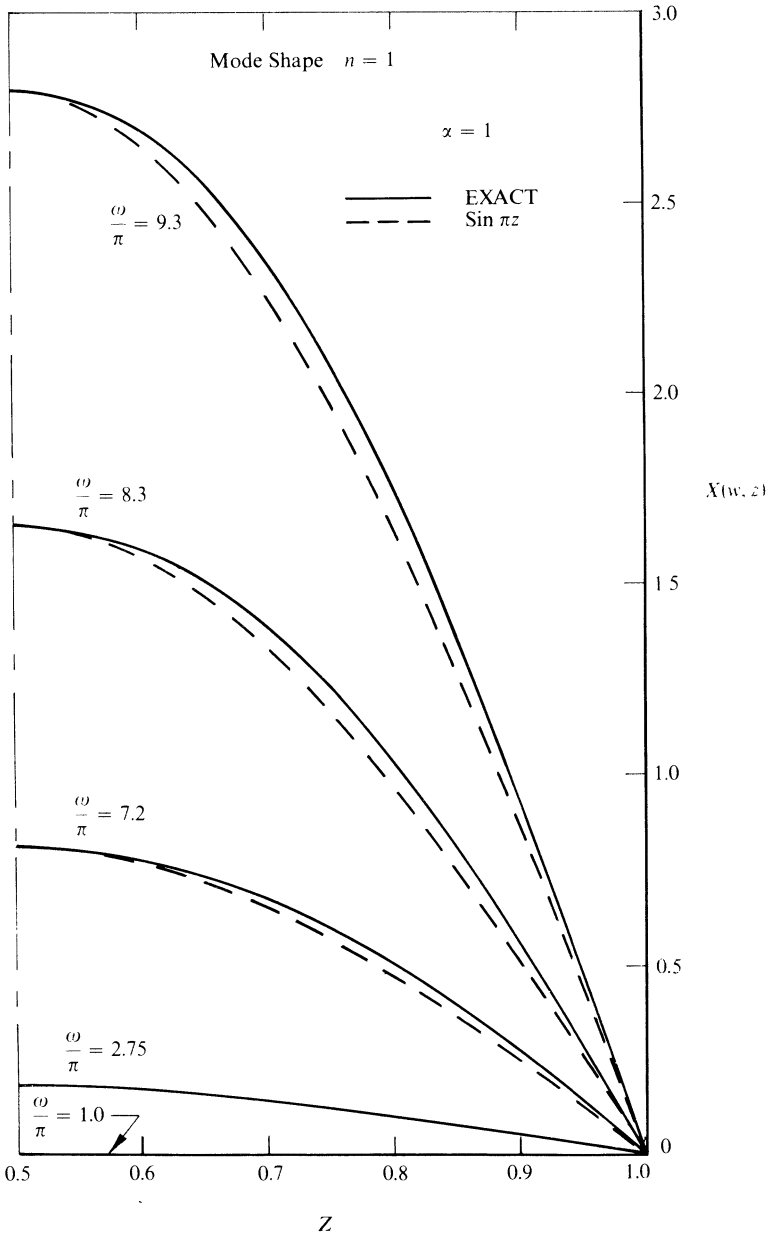
Mode Shape $n = 1$
$\alpha = 1$
EXACT
Sin πz
$\frac{\omega}{\pi} = 9.3$
$\frac{\omega}{\pi} = 8.3$
$\frac{\omega}{\pi} = 7.2$
$\frac{\omega}{\pi} = 2.75$
$\frac{\omega}{\pi} = 1.0$
3.0
2.5
2.0
1.5
1.0
0.5
0
$X(w, z)$
0.5
0.6
0.7
0.8
0.9
1.0
Z

FIG. 3

From (72) the amplitude of the first whirling mode is

$$A_1 = |u_1(a, z)|_{\max} = \frac{\alpha H_1(a)}{\omega_1(a)}(f(\sec\theta_0 - 1) + \sec^2\theta_0 - 1)^{1/2}. \tag{77}$$

Figure 2 shows a plot of $A_1(\omega)$ versus ω for $\alpha = 10^{-1}$. It is seen that the results are in agreement with the predictions of the quantitative theory, and that the first mode of whirling exists for $\pi < \omega < \pi/\alpha$.

Let $X_1(\omega, z)$ denote the geometrical shape of the first mode of whirling for angular velocity ω. $X_1(\omega, z)$ is obtained by plotting $u_1(a, z)$ versus $z + w_1(a, z)$ for $\omega = \omega_1(a)$. Figure 3 shows a plot of $X_1(\omega, z)$ for various angular velocities ω. Since $X_1(\omega, z)$ is symmetric about $z = 1/2$, $X_1(\omega, z)$ has been plotted for $1/2 \leqq z \leqq 1$. It will be observed that $X_1(\omega, z)$ is approximated quite closely by $A_1 \sin \pi z$, the eigenfunction of the linearized problem.

10. Higher modes. The eigenfunctions and eigenvalues for the higher modes may be obtained in exactly the same way as the first mode; they are given by

$$X_n(\omega, z) = \frac{1}{n} X_1\left(\frac{\omega}{n}, nz\right),$$

$$A_n(\omega) = \frac{A_1(\omega/n)}{n}, \tag{78}$$

$$\omega = \omega_n(a) = n\omega_1(a), \qquad n = 1, 2, 3, \cdots.$$

Figure 4 shows $A_n(\omega)$ versus ω for $n = 1, 2, 3$. Figure 5 shows a plot of the three distinct modes of whirling for $\omega = 3.5\pi$.

11. Stability. Though we have demonstrated the existence of unique solutions to the nonlinear eigenvalue problem, we do not know if they are stable against small perturbations.

We shall first show that the free whirling modes cannot be Lyapunov stable. To show this, suppose that the system is whirling in the first mode with an amplitude $A_1(\omega)$ at an angular velocity ω. Suppose that the string lies initially in the z, x-plane. Let us impart the following small perturbations to the system:

$$\xi = \varepsilon X_1(\omega, z), \qquad \dot{\xi} = 0,$$

$$\eta = 0, \qquad \dot{\eta} = \Delta\omega X_1(\omega, z), \qquad \Delta\omega/\omega \ll 1,$$

where ε is so chosen that $(1 + \varepsilon)A_1(\omega) = A_1(\omega + \Delta\omega)$. After perturbation the system will continue to whirl in the first mode, but with an increased amplitude $(1 + \varepsilon)A_1$ and an increased angular velocity $\omega + \Delta\omega$. Relative to the initial configuration, the perturbation ξ oscillates between $\xi = \varepsilon X_1(\omega, z)$ and $\xi = (2 + \varepsilon)X_1(\omega, z)$ with a frequency $\Delta\omega/\pi$. It is therefore obvious that the system cannot be Lyapunov stable. Thus we see that orbital stability is the best we can expect.

In order to show that the free modes of whirling are orbitally stable we shall first show that the perturbed motions are Lyapunov stable with respect to a special nonuniformly rotating coordinate system $\tilde{X}, \tilde{Y}, \tilde{Z}$. It is then a simple matter to infer orbital stability.

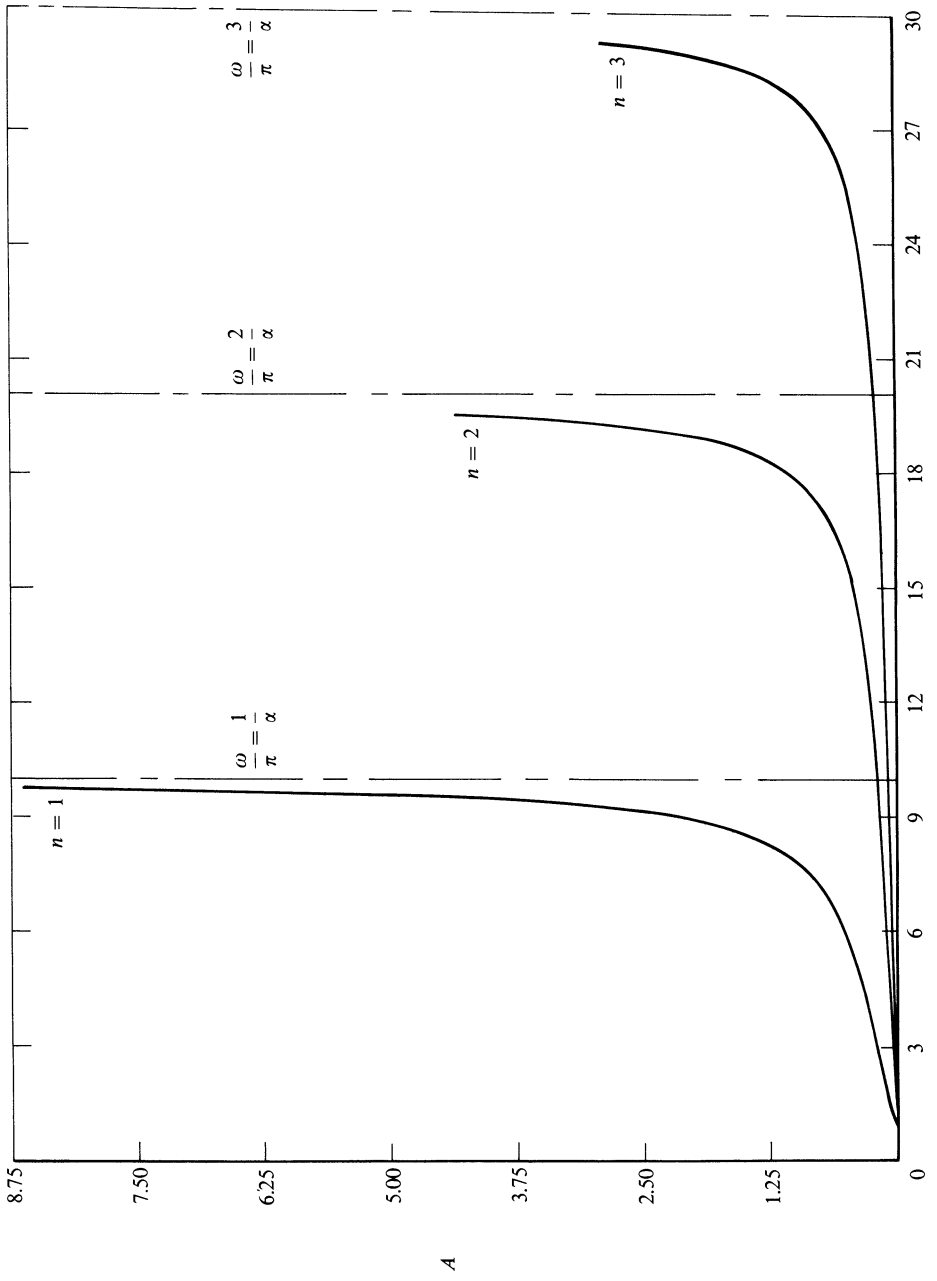
8.75
7.50
6.25
5.00
3.75
2.50
1.25
0
A
3
6
9
12
15
18
21
24
27
30
ω/π
n = 1
n = 2
n = 3
ω/π = 1/α
ω/π = 2/α
ω/π = 3/α

FIG. 4(a)

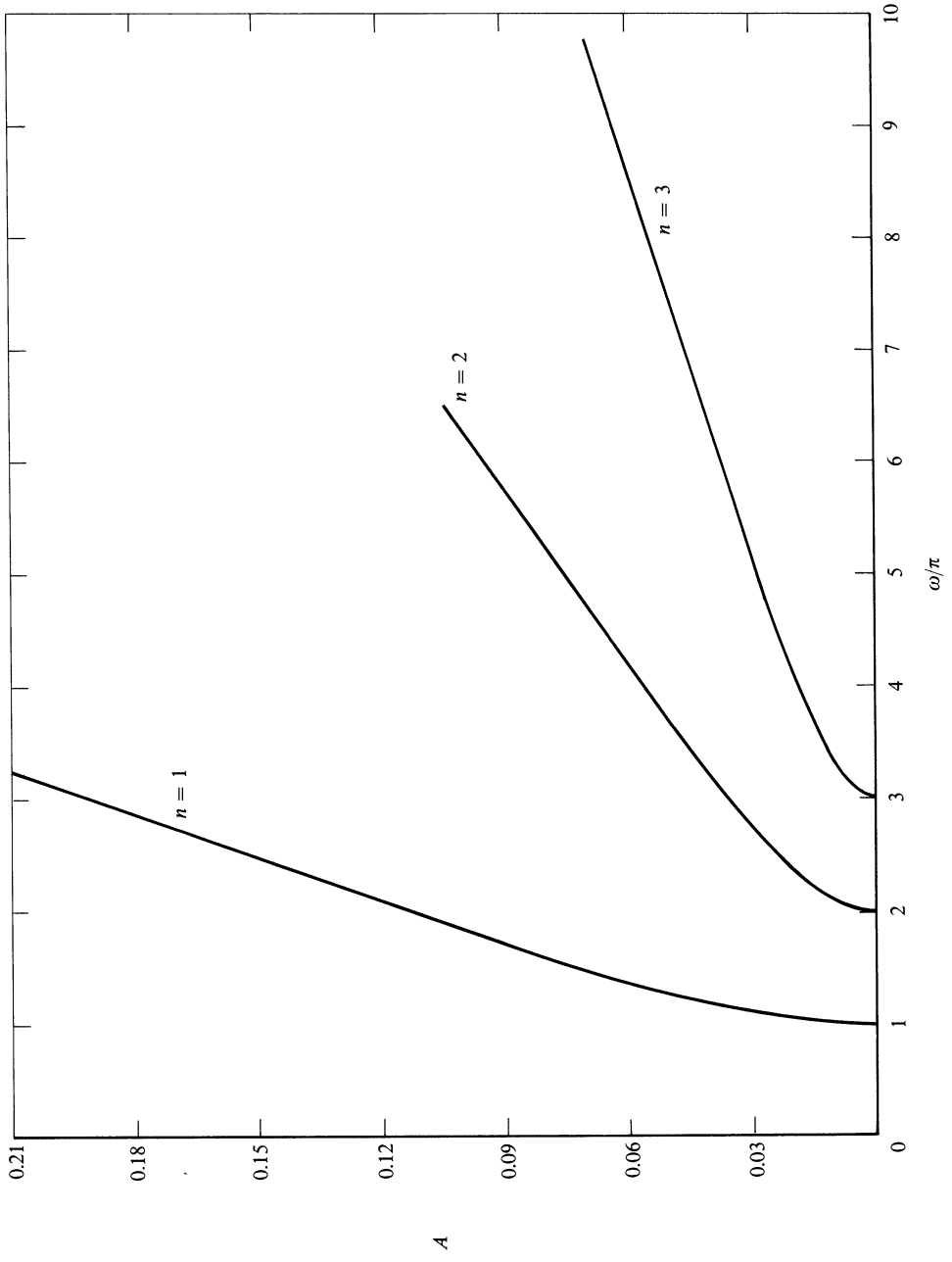

FIG. 4(b)

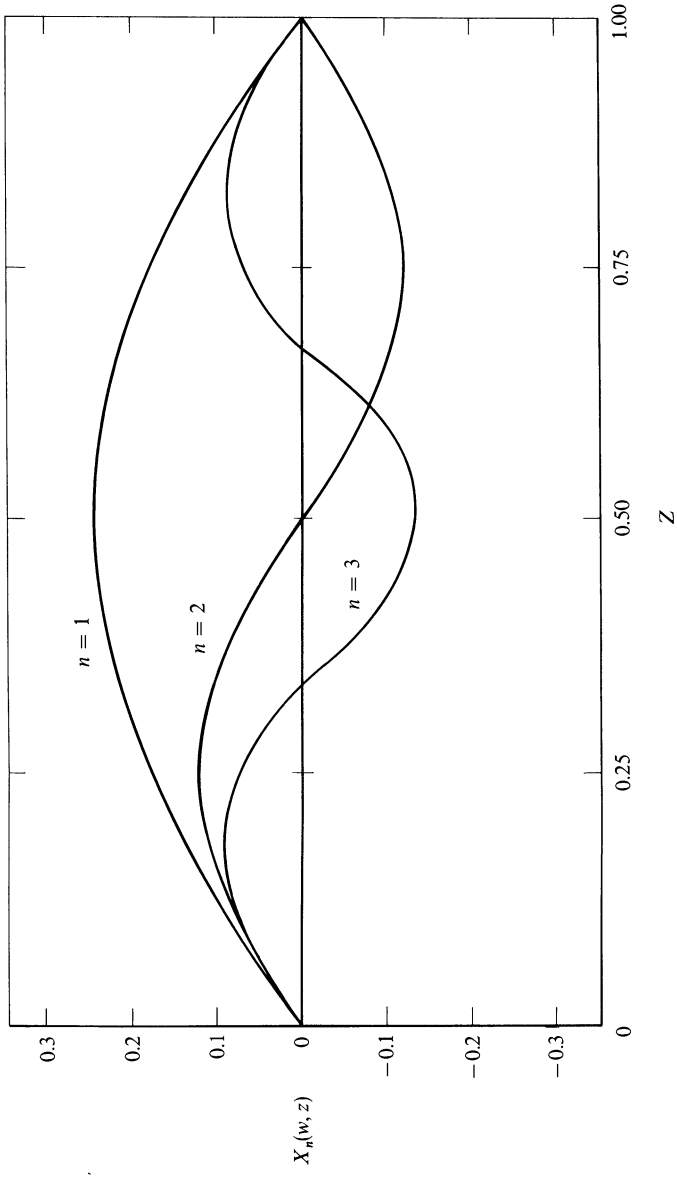

FIG. 5. *Distinct modes of whirling*; $\omega = 3.5\pi$

Consider a new set of coordinate axes $\tilde{X}, \tilde{Y}, \tilde{Z}$ rotating with dimensionless angular velocity $\lambda(\tau)$, the $\tilde{X}$-axis coinciding with the X-axis at $\tau = 0$. Let $u_n(a, z)$ and $\omega_n(a, z)$ be the eigenfunctions for the nth mode of whirling having an eigenvalue $\omega_n(a)$.

Let[2]

$$u = u_n + \xi, \quad v = 0 + \eta, \quad \omega = \omega_n + \zeta, \tag{79}$$

where u_n, ω_n correspond to the nth mode of free whirling, and ξ, η, ζ are perturbations, not necessarily small, on the steady whirling. In terms of the perturbation variables, the equations of motion in the $\tilde{X}, \tilde{Y}, \tilde{Z}$ frame may be written

$$\left[\beta\xi_z - (\beta - 1)\left(\frac{u_n' + \xi_z}{\Delta_2} - \frac{u_n'}{\Delta_1}\right)\right]_z = \xi_{\tau\tau} - \lambda^2(\xi + u_n) - 2\lambda\eta_\tau - \lambda_\tau\eta - \omega_n^2 u_n, \tag{80}$$

$$\left[\beta\eta_z - (\beta - 1)\frac{\eta_z}{\Delta_2}\right]_z = \eta_{\tau\tau} - \lambda^2\eta + 2\lambda\xi_\tau + (u_n + \xi)\lambda_\tau, \tag{81}$$

$$\left[\beta\zeta_z - (\beta - 1)\left(\frac{1 + w_n' + \zeta_z}{\Delta_2} - \frac{1 + w_n'}{\Delta_1}\right)\right]_z = \zeta_{\tau\tau}, \tag{82}$$

$$\begin{aligned} &\xi(0, \tau) = \xi(1, \tau) = 0, \qquad \eta(0, \tau) = \eta(1, \tau) = 0, \\ &\zeta(0, \tau) = \zeta(1, \tau) = 0, \end{aligned} \tag{83}$$

where

$$\begin{aligned} \Delta_2 &= [\{(u_n' + \xi_z)^2 + \eta_z^2 + (1 + w_n' + \zeta_z)^2\}]^{1/2}, \\ \Delta_1 &= [\{(u_n')^2 + (1 + w_n')^2]^{1/2}. \end{aligned} \tag{84}$$

Since the angular velocity $\lambda(\tau)$ is completely arbitrary, let us choose it in such a way that[3]

$$\lambda(0) = \omega_n(a), \tag{85a}$$

$$\int_0^1 [(\omega_n^2 + \lambda^2)\xi + \lambda\eta_\tau + \tfrac{1}{2}\lambda^2 u_n]u_n\, dz = C, \tag{85b}$$

C = const. given by

$$C = \int_0^1 \omega_n[2\omega_n\xi(0) + \eta_\tau(0) + \tfrac{1}{2}\omega_n u_n]u_n\, dz.$$

Consider the functional

$$\begin{aligned} V = \frac{1}{2}\int_0^1 \Big\{ &(\xi_\tau - \lambda\eta)^2 + (\eta_\tau + \lambda\xi)^2 + \zeta_\tau^2 \\ &+ (\xi_z^2 + \eta_z^2 + \zeta_z^2)\frac{\beta[\Delta_1 - 1] + 1}{\Delta_1} + \frac{(\beta - 1)}{\Delta_1}[\Delta_2 - \Delta_1]^2\, dz\Big\}. \end{aligned} \tag{86}$$

[2] It will be assumed that ξ, η, ζ exist and are sufficiently smooth so that $\xi_{\tau\tau}, \xi_{zz}$, etc., are continuous.

[3] If $|\xi|, |\eta_\tau|$ are assumed to be small, $\lambda(\tau)$ is given by

$$\lambda(\tau) = \omega_n - \int_0^1 (\eta_\tau + 2\omega_n\xi)u_n\, dz\Big|_0^\tau \Big/ \int_0^1 [u_n]^2\, dz \quad \text{(see Appendix B)}$$

Since

$$\beta = \alpha^{-2} >> 1, \qquad \frac{\beta[\Delta_1 - 1] + 1}{\Delta_1} \geqq \frac{1}{2 - \alpha^2} \qquad \text{(see 100))},$$

$V > 0$, unless ξ, η, ζ are identically zero when $V = 0$.

Differentiating V with respect to τ and evaluating along a trajectory of the perturbed motion, we have

$$(87)\frac{dV}{d\tau} = \int_0^1 \left\{ [(\lambda^2 - \omega_n^2)(\xi_\tau - \lambda\eta) - \lambda_\tau(\eta_\tau + \lambda\xi)]u_n - \lambda(\beta - 1)\left[\frac{u_n'\eta_z}{\Delta_2} - \frac{u_n'\eta_z}{\Delta_1}\right]\right\} dz.$$

Using (81) and (10) we have

$$(88) \int_0^1 (\beta - 1)u_n'\eta_z\left(\frac{1}{\Delta_2} - \frac{1}{\Delta_1}\right) dz = \int_0^1 [\eta_{\tau\tau} + 2\lambda\xi_\tau - (\lambda^2 - \omega_n^2)\eta + (\xi + u_n)\lambda_\tau]u_n\, dz.$$

Therefore

$$(89) \qquad \frac{dV}{d\tau} = -\int_0^1 \{[\omega_n^2 + \lambda^2]\xi_\tau + 2\lambda\lambda_\tau\xi + \lambda_\tau\eta_\tau + \lambda\eta_{\tau\tau} + \lambda\lambda_\tau u_n\}u_n\, dz.$$

Differentiating (60) with respect to τ, we have

$$(90) \qquad \int_0^1 \{(\omega_n^2 + \lambda^2)\xi_\tau + 2\lambda\lambda_\tau\xi + \lambda_\tau\eta_\tau + \lambda\lambda_\tau u_n\}u_n\, dz \equiv 0.$$

Hence

$$(91) \qquad \frac{dV}{d\tau} \equiv 0.$$

Therefore

$$(92) \qquad V(\tau) = V(0) = \text{const.}$$

Now

$$(93) \qquad \xi = \int_0^z \xi_z\, dz.$$

Therefore

$$(94) \qquad |\xi| \leqq \int_0^z |\xi_z|\, dz \leqq \int_0^1 |\xi_z|\, dz.$$

Application of the Schwarz inequality gives

$$(95) \qquad |\xi| \leqq \left[\int_0^1 \xi_z^2\, dz\right]^{1/2}.$$

Thus

$$(96) \qquad \sup_z |\xi| \leqq \left[\int_0^1 \xi_z^2\, dz\right]^{1/2}.$$

Similarly

$$(97) \qquad \sup_z |\eta| \leqq \left[\int_0^1 \eta_z^2\, dz\right]^{1/2}, \qquad \sup_z |\zeta| \leqq \left[\int_0^1 \zeta_z^2\, dz\right]^{1/2}.$$

From (86) we see that

$$\int_0^1 \left(\frac{\beta[\Delta_1 - 1] + 1}{\Delta_1}\right) \xi_z^2 \, dz \leqq 2V(0). \tag{98}$$

Using (13), (14), (18) and (19), we have $1 \leqq H_n \leqq 1/\alpha^2$. Thus

$$\frac{\beta[\Delta_1 - 1] + 1}{\Delta_1} = \frac{H_n}{(1 - \alpha^2)\cos\theta + \alpha^2 H_n} \geqq \frac{1}{(1 - \alpha^2) + \alpha^2/\alpha^2}.$$

Hence

$$\frac{\beta[\Delta_1 - 1] + 1}{\Delta_1} \geqq \frac{1}{2 - \alpha^2}. \tag{99}$$

Therefore

$$\int_0^1 \xi_z^2 \, dz \leqq 2V(0)(2 - \alpha^2). \tag{100}$$

Substituting (101) into (97), we have

$$\sup_z |\xi| \leqq \sqrt{2V(0)(2 - \alpha^2)}. \tag{101}$$

This holds similarly for η and ζ.

Hence

$$\begin{aligned} &\sup_z |\xi| \leqq \sqrt{2V(0)(2 - \alpha^2)} && \text{for every } \tau \geqq 0; \\ &\sup_z |\eta| \leqq \sqrt{2V(0)(2 - \alpha^2)} && \text{for every } \tau \geqq 0; \\ &\sup_z |\zeta| \leqq \sqrt{2V(0)(2 - \alpha^2)} && \text{for every } \tau \geqq 0. \end{aligned} \tag{102}$$

Thus we see that, in the specially chosen rotating coordinate system, the perturbed motions are Lyapunov stable. Let us construct a tube of cross sectional radius $2[V(0)(2 - \alpha^2)]^{1/2}$, mean radius $u_n(a, z)$, and axial distance $z + w_n(a, z)$ from the origin. We see that since the coordinates of a point on the string with rest coordinates $(0, 0, z)$ lie inside the tube at $\tau = 0$ and remain inside the tube for all τ greater than zero, the perturbed motions are orbitally stable. Hence the nth mode of free whirling is orbitally stable. Since the stability analysis did not depend on the index n, we see that all modes of free whirling are orbitally stable.

12. Further problems in the large amplitude whirling of an elastic string. If the string is constrained by two parallel frictionless plates, so that the motion is constrained to lie in a plane passing through the Z-axis, then two new problems may be formulated:

(i) The constraining plates are free to rotate about the Z-axis.

(ii) The constraining plates are forced to rotate about the Z-axis with a fixed angular velocity ω.

In the first case the mode shapes and eigenvalues are identical with those of the unconstrained problem. The stability analysis for this case follows along similar lines to that for the unconstrained problem and it may be shown that all modes are orbitally stable.

In the second problem the mode shapes are identical to those in free whirling, the eigenvalues are of course constrained to be equal to the impressed rotational speed ω. The stability analysis in this case is different from that of the previous cases and shows that the first mode is Lyapunov stable, while all the higher modes are unstable. In this case the first mode of whirling exists for $\pi \leqq \omega \leqq \pi/\alpha$; there are no solutions for $\omega > \pi/\alpha$.

Appendix A.

A.1. Existence and uniqueness of solutions of the nonlinear eigenvalue problem. For given values of a, ω, H, the initial value problem (25) possesses a unique periodic solution, symmetric about the even multiples of a quarter period, and antisymmetric about the odd multiples of a quarter period. The period Z^* is given by

$$Z^* = \frac{4}{\alpha\omega}\int_0^4 \frac{d\mu}{[(2(1-\alpha^2)/\alpha^2 H)(\sqrt{1+a^2}-\sqrt{1+\mu^2})+a^2-\mu^2]^{1/2}}. \tag{A.1}$$

Thus, for given values of a and H, a unique value of ω can be found such that

$$Z^* = 2. \tag{A.2}$$

With this value of ω, we have

$$\begin{aligned} \mu(1) &= -\mu(0) = -a, \\ \mu'(1) &= \mu'(0) = 0, \\ \mu(\tfrac{1}{2}) &= 0. \end{aligned} \tag{A.3}$$

Since this solution satisfies (21) and the boundary conditions (22), and since it vanishes at only one point in the interval $0 < z < 1$, it is the first eigenfunction of the nonlinear eigenvalue problem. If it can be established that for a given value of a there exists a unique value of H, then the uniqueness of the solution of the nonlinear eigenvalue problem will be assured. From (58), we have

$$\frac{\alpha\omega}{2} = \int_0^a \frac{d\mu}{\Delta^{1/2}}. \tag{A.4}$$

Similarly from (65), we have

$$\frac{2(1-\alpha^2)}{\alpha\omega}\int_0^a \frac{d\mu}{\sqrt{1+\mu^2}\,\Delta^{1/2}} = 1-\alpha^2 H, \tag{A.5}$$

where

$$\begin{aligned} \Delta &= f(\sqrt{1+a^2}-\sqrt{1+\mu^2})+a^2-\mu^2, \\ f &= \frac{2(1-\alpha^2)}{\alpha^2 H}. \end{aligned} \tag{A.6}$$

Eliminating ω between (A.4) and (A.5), we have

$$\frac{1}{1-\alpha^2} - \frac{2}{f} = \frac{\displaystyle\int_0^a \frac{d\mu}{\sqrt{1+\mu^2}\Delta^{1/2}}}{\displaystyle\int_0^a \frac{d\mu}{\Delta^{1/2}}}. \tag{A.7}$$

Let us define a function $G(f)$, where

$$G(f) = \int_0^a \frac{d\mu}{\sqrt{1+\mu^2}\Delta^{1/2}} + \frac{2}{f} - \frac{1}{1-\alpha^2}. \tag{A.8}$$

From (20) we have

$$1 \leqq H \leqq 1/\alpha^2.$$

Therefore

$$2(1-\alpha^2) \leqq f \leqq \frac{2(1-\alpha^2)}{\alpha^2}. \tag{A.9}$$

Thus the proof of the uniqueness of solutions of the nonlinear eigenvalue problem is reduced to proving that the equation

$$G(f) = 0; \qquad 2(1-\alpha^2) \leqq f \leqq 2(1-\alpha^2)/\alpha^2 \tag{A.10}$$

has a unique solution.

A.2. Existence and uniqueness of solutions of (A.10). We observe that $G(f)$ is continuous in f, and that

$$G(f_1) > 0, \qquad \text{where } f_1 = 2(1-\alpha^2), \tag{A.11a}$$

$$G(f_2) < 0, \qquad \text{where } f_2 = 2(1-\alpha^2)/\alpha^2. \tag{A.11b}$$

Since $G(f)$ is continuous in f and $G(f_1)$ and $G(f_2)$ are of opposite sign, there exists at least one zero of $G(f_1)$ on the interval $f_1 < f < f_2$. If we can show that $G(f)$ is monotone decreasing on this interval we will have shown that there exists a unique solution of (A.10). Alternatively, if we can show that $G'(\hat{f})$ is negative at any point $f = \hat{f}$ which is a solution of (A.10), then we will have shown that (A.10) has a unique solution. This follows from the fact that, since $G(f)$ is continuous in f, $G'(f)$ must be of opposite sign at consecutive zeros of $G(f)$.

Differentiating $G(f)$ with respect to f, we have

$$G'(f) = \frac{\displaystyle\int_0^a \frac{d\mu}{\Delta_1\Delta^{1/2}} \int_0^a \frac{d\mu}{\sqrt{1+\mu^2}\Delta^{1/2}} - \int_0^a \frac{d\mu}{\Delta^{1/2}} \int_0^a \frac{d\mu}{\sqrt{1+\mu^2}\Delta_1\Delta^{1/2}}}{2\left(\displaystyle\int_0^a \frac{d\mu}{\Delta^{1/2}}\right)^{1/2}} - \frac{2}{f^2}, \tag{A.12}$$

where

$$\Delta_1 = f + \sqrt{1+\mu^2} + \sqrt{1+a^2}.$$

Since

$$f_1 + 1 + \sqrt{1-a^2} \leqq \Delta_1 \leqq f + 2\sqrt{1+a^2},$$

therefore

$$\text{(A.13)} \qquad G'(f) \leqq \frac{\sqrt{1+a^2}-1}{2(f+\sqrt{1+a^2}+1)(f+2\sqrt{1+a^2})} \cdot \frac{\displaystyle\int_0^a \frac{d\mu}{\sqrt{1+\mu^2}\Delta^{1/2}}}{\displaystyle\int_0^a \frac{d\mu}{\Delta^{1/2}}} - \frac{2}{f^2}.$$

Since

$$\text{(A.14)} \qquad \frac{\displaystyle\int_0^a \frac{d\mu}{\sqrt{1+\mu^2}\Delta^{1/2}}}{\displaystyle\int_0^a \frac{d\mu}{\Delta^{1/2}}} \leqq \sqrt{\frac{(2\sqrt{1+a^2})(1+\sqrt{1+a^2}+f)}{(1+\sqrt{1+a^2})(2\sqrt{1+a^2}+f)}} \cdot \frac{\displaystyle\int_0^a \frac{d\mu}{\sqrt{1+\mu^2}\sqrt{a^2-\mu^2}}}{\displaystyle\int_0^a \frac{d\mu}{\sqrt{a^2-\mu^2}}},$$

the change of variable $\mu = a\cos\theta$ reduces (A.14) to

$$\text{(A.15)} \qquad \frac{\displaystyle\int_0^a \frac{d\mu}{\sqrt{1+\mu^2}\Delta^{1/2}}}{\displaystyle\int_0^a \frac{d\mu}{\Delta^{1/2}}} \leqq \sqrt{2}\frac{1}{\sqrt{1+a^2}}\frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{2\sqrt{2}}{\pi}\frac{F\left(\frac{\pi}{2},k\right)}{\sqrt{1+a^2}},$$

where

$$k^2 = \frac{a^2}{1+a^2}.$$

Substituting (A.15) into (A.13), we have

$$\text{(A.16)} \qquad G'(f) \leqq \frac{[\sqrt{1+a^2}-1]\dfrac{2\sqrt{2}}{\pi}\dfrac{F(\pi/2,k)}{\sqrt{1+a^2}} - 4\left(1+\dfrac{\sqrt{1+a^2}+1}{f}\right)\left(1+\dfrac{2\sqrt{1+a^2}}{f}\right)}{2(f+\sqrt{1+a^2}+1)(f+2\sqrt{1+a^2})}.$$

If f is a solution of (A.10), we have

$$\text{(A.17)} \qquad \frac{1}{1-\alpha^2} - \frac{2}{f} = \frac{\displaystyle\int_0^a \frac{d\mu}{\sqrt{1+\mu^2}\Delta^{1/2}}}{\displaystyle\int_0^a \frac{d\mu}{\Delta^{1/2}}} \leqq \frac{2\sqrt{2}}{\pi}\frac{F\left(\frac{\pi}{2},k\right)}{\sqrt{1+a^2}}.$$

Hence

$$\frac{2}{f} \geqq \frac{1}{1-\alpha^2} - \frac{2\sqrt{2}}{\pi}\frac{F(\pi/2,k)}{\sqrt{1+a^2}}. \tag{A.18}$$

Substituting (A.18) into (A.16) and blunting the inequality, we have

$$G'(f) \leqq \tag{A.19}$$

$$\frac{[\sqrt{1+a^2}-1]\frac{2\sqrt{2}}{\pi}\frac{F(\pi/2,k)}{\sqrt{1+a^2}} - 4\left[1+\sqrt{1+a^2}\left(\frac{1}{1-\alpha^2} - \frac{2\sqrt{2}}{\pi}\frac{F(\pi/2,k)}{\sqrt{1+a^2}}\right)\right]^2}{4(1+a)^2}.$$

The right-hand side is now a function of a independent of f. Using the tabulated values of $F(\pi/2, k)$ together with its asymptotic properties, we can easily show that

$$G'(f) \leqq -\varepsilon < 0 \quad \text{for } 0 < a < \infty, \quad f_1 < f < f_2. \tag{A.20}$$

Thus there exists only one solution to (A.10). Hence for a given value of a there exists a unique value of H_1 and hence a unique solution of the nonlinear eigenvalue problem (22), (23).

A.3. Higher modes. If the above arguments are repeated with $Z^* = 2/n$, $n = 2, 3, 4 \cdots$, then the existence and uniqueness of higher mode solutions of the nonlinear eigenvalue problem are assured. Thus we have the following theorem.

THEOREM 1. *For each value of* $[a, n]$, $n = 1, 2, 3 \cdots$, $0 < a < \infty$, *there exists a unique solution* $\mu_n(a, z)$ *and a unique rotational speed* $\omega_n(a)$ *for the nonlinear eigenvalue problem* (22), (23).

Appendix B. In developing the stability analysis it was implicitly assumed that (85b) admits a real positive solution $\lambda(\tau)$. It will now be shown that this is the case and that $\lambda(\tau)$ is given by

$$\lambda(\tau) = \omega_n - \frac{\int_0^1 (\eta_\tau + 2\omega_n\xi)u_n\,dz|_0^\tau}{\int_0^1 u_n^2\,dz}, \tag{B.1}$$

provided that

$$V(0) \ll \int_0^1 u_n^2\,dz. \tag{B.2}$$

Proof. Let us write (85b) in the form

$$\begin{aligned}\lambda^2\int_0^1 (u_n^2 + 2u_n\xi)\,dz + 2\lambda\int_0^1 \eta_\tau u_n\,dz - \omega_n^2\int_0^1 (u_n^2 - 2u_n\xi)\,dz \\ - 2\omega_n^2\int_0^1 \xi(0)\,u_n\,dz - 2\omega_n\int_0^1 [\omega_n\xi(0) + \eta_\tau(0)]u_n\,dz = 0.\end{aligned} \tag{B.3}$$

Solving for λ, we have

$$\lambda = -\frac{\int_0^1 \eta_\tau u_n\,dz \pm \left[\left(\int_0^1 \eta_\tau u_n\,dz\right)^2 + \varphi\right]^{1/2}}{\int_0^1 (u_n^2 + 2u_n\xi)}, \tag{B.4}$$

where

$$\begin{aligned}\varphi = {} & \omega_n^2\left[\left(\int_0^1 u_n^2\,dz\right)^2 - 4\left(\int_0^1 u_n\xi\,dz\right)^2\right] \\ & + 2\omega_n\left[\int_0^1 u_n^2\,dz + 2\int_0^1 u_n\xi\,dz\right]\left[\omega_n\int_0^1 u_n\xi(0)\,dz + \int_0^1 [\omega_n\xi(0) + \eta_\tau(0)]u_n\,dz\right].\end{aligned} \tag{B.5}$$

Sufficient conditions guaranteeing the existence of a real positive solution are

$$\int_0^1 (u_n^2 + 2u_n\xi)\,dz > 0, \tag{B.6a}$$

$$\varphi > 0. \tag{B.6b}$$

Using (86), (92) and (101) we have

$$\begin{aligned}&\int_0^1 (\eta_\tau + \lambda\xi)^2\,dz \leqq 2V(0) < 2V(0)(2 - \alpha^2), \\ &\sup_z |\xi| \leqq \sqrt{2V(0)(2 - \alpha^2)}.\end{aligned} \tag{B.7}$$

Now

$$\left|\int_0^1 u_n\xi\,dz\right| \leqq \int_0^1 |u_n\xi|\,dz \leqq \left[\int_0^1 u_n^2\,dz\int_0^1 \xi^2\,dz\right]^{1/2}. \tag{B.8}$$

Using (B.7), we have

$$\left|\int_0^1 u_n\xi\,dz\right| \leqq \int_0^1 u_n^2\,dz\Gamma, \tag{B.9}$$

where

$$\Gamma = \left[2V(0)(2 - \alpha^2)\Big/\int_0^1 u_n^2\,dz\right]^{1/2}.$$

Now

$$\int_0^1 (u_n^2 + 2u_n\xi)\,dz \geqq \int_0^1 u_n^2\,dz - 2\left|\int_0^1 u_n\xi\,dz\right| \geqq \left(\int_0^1 u_n^2\,dz\right)(1 - 2\Gamma).$$

Hence, if $\Gamma < 1/2$, (B.6a) is satisfied. Now

$$\int_0^1 [\omega_n\xi(0) + \eta_\tau(0)]^2\,dz = \int_0^1 [\lambda\xi + \eta_\tau]^2\,dz|_{\tau=0}. \tag{B.10}$$

Therefore

$$\int_0^1 [\omega_n\xi(0) + \eta_\tau(0)]^2\, dz \leqq 2V(0) < 2V(0)(2 - \alpha^2). \tag{B.11}$$

Thus

$$\left|\int_0^1 (\omega_n\xi(0) + \eta_\tau(0))u_n\, dz\right| \leqq \int_0^1 |\omega_n\xi(0) + \eta_\tau(0)|\,|u_n|\, dz$$
$$\leqq \left[\int_0^1 u_n^2\, dz \int_0^1 [\omega_n\xi(0) + \eta_\tau(0)]^2\, dz\right]^{1/2}.$$

Using (B.11), we have

$$\left|\int_0^1 [\omega_n\xi(0) + \eta_\tau(0)]u_n\, dz\right| \leqq \Gamma \int_0^1 u_n^2\, dz. \tag{B.12}$$

From (B.5), we have

$$\varphi \geqq \omega_n^2\left[\left(\int_0^1 u_n^2\, dz\right)^2 - 4\left(\int_0^1 u_n\xi\, dz\right)^2\right]$$
$$- 2\omega_n\left[\int_0^1 u_n^2\, dz + 2\left|\int_0^1 u_n\xi\, dz\right|\right]\left[\omega_n\left|\int_0^1 u_n\xi(0)\, dz\right|\right.$$
$$\left. + \left|\int_0^1 [\omega_n\xi(0) + \eta_\tau(0)]u_n\, dz\right|\right].$$

Using (B.9), (B.12), we have

$$\varphi \geqq \omega_n^2\left(\int_0^1 u_n^2\, dz\right)^2\left[1 - 4\Gamma^2 - 2\Gamma(1 + 2\Gamma)\left(1 + \frac{1}{\omega_n}\right)\right].$$

Hence, if $\Gamma < 1/6$, (B.6b) is satisfied. Thus (85b) possesses a real positive solution $\lambda(\tau)$ whenever $V(0) \ll \int_0^1 u_n^2\, dz$. Since $V(0)$ is a measure of the initial perturbations $(\xi, \eta, \zeta, \xi_\tau, \eta_\tau, \zeta_\tau)$, we see that the perturbations need not be infinitesimally small; all that is required is that they be small compared to the modal displacement u_n.

B.1. Boundedness of $\lambda(\tau)$. If (B.3) is rearranged, we have

$$\lambda^2 \int_0^1 u_n^2\, dz \leqq 2|\lambda| \int_0^1 |\lambda\xi + \eta_\tau|\,|u_n|\, dz$$
$$+ \omega_n^2\left[\int_0^1 u_n^2\, dz + 2\int_0^1 |\xi u_n|\, dz + 2\int_0^1 |\xi(0)u_n|\, dz\right] \tag{B.13}$$
$$+ 2\omega_n \int_0^1 |\omega_n\xi(0) + \eta_\tau(0)|\,|u_n|\, dz.$$

Using (B.7), (B.8), (B.11), we have

$$\lambda^2 \leqq [2|\lambda| + 4\omega_n^2 + 2\omega_n]\Gamma + \omega_n{}^2, \tag{B.14}$$

where

$$\Gamma = \left[2V(0)(2 - \alpha^2) \Big/ \int_0^1 u_n^2 \, dz \right]^{1/2}.$$

Since

$$2|x|\,|y| \leqq \varepsilon x^2 + \frac{1}{\varepsilon} y^2 \qquad \text{for all } \varepsilon > 0,$$

therefore, taking $\varepsilon = 1/2$, we have

$$\lambda^2 \leqq 2\omega_n^2 \left[1 + \left(4 + \frac{1}{\omega_n} \right) \Gamma + \frac{4\Gamma^2}{\omega_n^2} \right]. \tag{B.15}$$

Now, if $V(0) \ll \int_0^1 u_n^2 \, dz$, then $\Gamma < 1$. Therefore

$$\lambda^2 \sim O(\omega_n^2), \qquad \omega_n = n\pi, \quad n = 1, 2, \cdots. \tag{B.16}$$

Hence $\lambda(\tau)$ is bounded. Now

$$\int_0^1 \eta_\tau u_n \, dz = \int_0^1 (\eta_\tau + \lambda\xi) u_n \, dz - 2\lambda \int_0^1 \xi u_n \, dz. \tag{B.17}$$

Therefore,

$$\left| \int_0^1 \eta_\tau u_n \, dz \right| \leqq \int_0^1 |\eta_\tau + \lambda\xi| \, |u_n| \, dz + 2|\lambda| \int_0^1 |\xi u_n| \, dz. \tag{B.18}$$

Using (B.7), (B.9), we have

$$\left| \int_0^1 \eta_\tau u_n \, dz \right| \leqq \int_0^1 u_n^2 \, dz \{ \Gamma(1 + 2|\lambda|) \}. \tag{B.19}$$

Since $\lambda(\tau)$ is bounded, it follows that if $\Gamma < 1/1 + 2|\lambda|$, then

$$\left| \int_0^1 \eta_\tau u_n \, dz \right| \leqq \int_0^1 u_n^2 \, dz. \tag{B.20}$$

Using (B.7) and (B.20) in (B.4) and (B.5), we see that the positive solution of (B.3) is given by

$$\lambda(\tau) \doteq \frac{\left\{ \omega_n^2 + 2\omega_n \dfrac{\int_0^1 [2\omega_n \xi(0) + \eta_\tau(0)] u_n \, dz}{\int_0^1 u_n^2 \, dz} \right\}^{1/2} - \dfrac{\int_0^1 \eta_\tau u_n \, dz}{\int_0^1 u_n^2 \, dz}}{1 + \int_0^1 2u_n \xi \, dz \Big/ \int_0^1 u_n^2 \, dz}. \tag{B.21}$$

Expanding (B.21) and retaining only first order terms, we have

$$\lambda(\tau) \doteq \omega_n - \frac{\int_0^1 (\eta_\tau + 2\omega_n\xi)u_n\, dz|_0^\tau}{\int_0^1 u_n^2\, dz}. \tag{B.22}$$

Furthermore we see that

$$|\lambda(\tau) - \omega_n| \leqq \frac{\int_0^1 |\eta_\tau + 2\omega_n\xi|\, |u_n|\, dz + \int_0^1 |\eta_\tau(0) + 2\omega_n\xi(0)|\, |u_n|\, dz}{\int_0^1 u_n^2\, dz}. \tag{B.23}$$

Using (B.7), we have

$$|\lambda(\tau) - \omega_n| \leqq 2\Gamma. \tag{B.24}$$

Thus for perturbations which are small compared to the modal displacements, the proper choice of $\lambda(\tau)$ is close to ω_n in the sense that

$$\lim_{V(0)\to 0} \lambda(\tau) = \omega_n. \tag{B.25}$$

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