el is, of course, consistent with our neglect of the μ mass here, and this is also consistent with Goldstone's theorem. On the other hand, if we take the mass formulas literally, then $\mu = 0.412$ GeV and thus R lies below the (μB) threshold in the "exact-symmetry limit." This was noted by R. J. Oakes and C. N. Yang, Phys. Rev. Letters 11, 174 (1963). In any case, we may defend the plausibility of the width Γ_R calculated here as corresponding to the most commonly observed magnitude of (μB)-resonance widths. Furthermore the value of the factor $\pi m_R \Gamma_R \sigma^R(\max)$ computed in our way turns out to be nearly the same as that computed from the observed values of the corresponding quantities for $\Delta(1236)$. Therefore it seems likely that we have not *under*estimated this factor in the (μB)-resonance contribution.

[18]From our choice of the values of α and g^2, and using the observed masses and phase space, the calculated width of $\Delta(1236)$ is $\Gamma_{\Delta \to N\pi} \approx 0.1$ GeV. The decay channel $\Delta \to \Sigma K$, which represents half of the total decay probability with SU(3) symmetry, is closed.

[19]V. Barger *et al.*, Nucl. Phys. B5, 411 (1968); V. Barger and D. Cline, *Phenomenological Theories of High Energy Scattering* (Benjamin, New York, 1969).

[20]See, for example, J. D. Jackson, Rev. Mod. Phys. 37, 484 (1965).

[21]J. Benecke and H. P. Dürr, Nuovo Cimento 56, 269 (1968).

[22]G. Wolf, Phys. Rev. 182, 1538 (1969).

[23]Aachen-Berlin-Birmingham-Bonn-Hamburg-London-München Collaboration, Nuovo Cimento 35, 659 (1965).

[24]Aachen-Berlin-Birmingham-Bonn-Hamburg-London-München Collaboration, Nuovo Cimento 31, 729 (1964).

[25]J. D. Jackson, J. T. Donohue, K. Gottfried, R. Keyser, and B. E. Y. Svensson, Phys. Rev. 139, B428 (1965).

[26]P. K. Williams, Phys. Rev. 181, 1963 (1969). The author gave an on-shell prediction, not for the differential cross section, but for the density-matrix element of the ρ in the reaction $\pi^- p \to \rho^- p$ at 3 GeV/c.

[27]In our convention, the relations between the invariant amplitudes and the conventional amplitudes f_1, f_2 remain the same as that in G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957).

PHYSICAL REVIEW D VOLUME 4, NUMBER 11 1 DECEMBER 1971

Chiral-Symmetry-Breaking Effects on Neutrino Scattering

Benjamin W. Lee* and Jeffrey E. Mandula†
California Institute of Technology, Pasadena, California 91109
(Received 24 May 1971)

The ideas of approximate scale invariance and operator light-cone expansions are used to relate inelastic neutrino scattering form factors to the one-nucleon matrix elements of double commutators of chiral (weak) charges with the Hamiltonian, $\langle N|[Q,[Q^\dagger,\mathcal{H}]]|N\rangle$. Two kinds of sum rules are given, one expressing the finiteness of the commutator, the other its connection to neutrino scattering. It is pointed out that these sum rules can, in principle, be directly evaluated from measurable form factors.

I. INTRODUCTION

It has been proposed that the strong-interaction Hamiltonian density can be divided into two terms,

$$\mathcal{H} = \mathcal{H}_{\text{sym}} + \mathcal{H}_{\text{break}}, \tag{1.1}$$

where $\mathcal{H}_{\text{sym}}$ is invariant under a chiral SU(3)⊗SU(3) group whose generators include the integrals of the electromagnetic and weak charge densities, and $\mathcal{H}_{\text{break}}$ is responsible for the breaking of SU(3) symmetry and for the nonzero masses of the pseudoscalar mesons.[1] It would be very interesting to know how $\mathcal{H}_{\text{break}}$ transforms under SU(3)⊗SU(3), i.e., what representation it belongs to, but it has turned out to be very difficult to find features of the strong interactions that unambiguously distinguish between possibilities. We would like to point out that neutrino scattering experiments can, in principal, partially measure both the magnitude and representation of $\mathcal{H}_{\text{break}}$. A great wealth of information is available if one separates events with total hadron strangeness 0 from those with strangeness 1, and if one measures neutrino scattering from protons separately from scattering from neutrons. Measurements made in this detail can disentangle the Clebsch-Gordan coefficients for SU(3)⊗SU(3), and would restrict the possible transformation properties of $\mathcal{H}_{\text{break}}$.

The connection between neutrino scattering and the transformation properties of $\mathcal{H}_{\text{break}}$ is provided by the expected scaling of deep-inelastic neutrino scattering.[2] If the light-cone analysis of the SLAC electron scattering experiments is correct, and if it can be extended to neutrino scattering, then integrals of the neutrino scaling functions are related to the equal-time commutators of the weak current J_μ with its adjoint and the divergence $\partial^\mu J_\mu^\dagger$. The equal-time commutator of J_0 with $\partial^\mu J_\mu^\dagger$ is essentially just the double commutator of weak (chiral) charges with the Hamiltonian $[Q,[Q^\dagger,\mathcal{H}_{\text{break}}]]$, and reflects the SU(3)⊗SU(3) transformation properties of $\mathcal{H}_{\text{break}}$.

Recently Cheng and Dashen[3] have estimated the size of the $I_t = 0$ nucleon matrix element of the double commutator of $\mathcal{H}_{\text{break}}$ with the strangeness-conserving axial charges, the so-called σ term, by an ingenious extrapolation of the pion-nucleon scattering amplitude to an unphysical point. The part of $\mathcal{H}_{\text{break}}$ which commutes with the strangeness-conserving axial charges eludes this analysis. While it may be thought that a similar analysis could be made on the KN scattering data, the unphysical point to which the continuation would be made is $s=0$, $t=2m_K^2$, beyond the t-channel threshold, and the attendant technical difficulties are presumably insurmountable. The method we present for measuring similar double commutators does not depend on analytic extrapolations of measured data, but does involve a very difficult measurement.

In this paper we present two types of sum rules. One type measures the one-nucleon matrix elements of the double commutator

$$\langle N(p)|[Q^i, [Q^j, \mathcal{H}_{\text{break}}]]|N(p)\rangle ,$$

and these are the sum rules that distinguish between possibilities for the transformation properties of $\mathcal{H}_{\text{break}}$. The other sum rules express the requirement that the equal-time commutator of J_0^i and $\partial^\mu J_\mu^j$ has its expected form, $\delta^3(\vec{x})$ times a finite operator. These assert that certain integrals of scaling functions vanish.

The assumptions underlying this analysis are:

(a) Broken scale invariance. It is expressed by the requirement that light-cone (small x^2) commutators have a smooth scale-invariant limit when all masses are set to zero.[4]

(b) Operator light-cone expansions[5-7] of current commutators for small x^2.

(c) The chiral-symmetry-breaking Hamiltonian density has only terms that break scale invariance, and has scale dimension less than 4.[8]

(d) Equal-time singularities of current commutators are the smooth (analytic) limits of the light-cone singularities.

(e) The conventional structure of the equal-time singularity of the commutator of a current with a divergence. Specifically, that the quantity

$$\int d^3x \langle N(p)|[J_0^i(x,0),\ \partial^\mu J_\mu^j(0)]|N(p)\rangle$$

is finite.

The results of our analysis which can be tested experimentally are:

(I) In the deep-inelastic region of neutrino scattering, the asymptotic behavior of the form factors W_4 and W_5, which measure the longitudinal (nonconserved) part of the weak current, are

$$W_{4,5}(\nu, q^2) \to \sum_a \frac{1}{(2m\nu)^{\zeta_a}} F_{4,5}(\alpha; [\zeta_a]) . \tag{1.2}$$

In this limit, $m\nu = p\cdot q$ and $-q^2$ are both large, but the ratio $\alpha = -q^2/2m\nu$ is held fixed. The values of ζ are connected with the strength of the light-cone singularity of the commutator of the weak current with its adjoint. The minimum value of ζ is[9]

$$\zeta_{\min} = \min[2, 3-\tfrac{1}{2}d] , \tag{1.3}$$

where d is the maximum scale dimension of $\mathcal{H}_{\text{break}}$. Since d is expected to be less than four, $\zeta_{\min}$ will exceed one.

(II) Whatever $\zeta_{\min}$, there will be a ν^{-2} term in the asymptotic expansion of W_4 and its coefficient is related to the multiple equal-time commutator $\langle N(p)|[Q, [Q^\dagger, \mathcal{H}]]|N(p)\rangle$, where Q is the chiral $(V-A)$ charge associated with the weak current, and $|N(p)\rangle$ denotes a proton or neutron state of momentum p. The relation is that

$$\int_0^1 \alpha\, d\alpha [F^{(N\nu)}(\alpha;[2]) + F^{(N\bar\nu)}(\alpha;[2])] = m^2 \langle N(p)|[Q,[Q^\dagger, \mathcal{H}]]|N(p)\rangle . \tag{1.4}$$

The superscripts $N\nu$ or $N\bar\nu$ denote neutrino or antineutrino scattering from the nucleon N.

(III) To control the equal-time singularity of the commutator of the weak current with its adjoint when $\zeta_{\min} < 2$, certain integrals of $F_{4,5}(\alpha;[\zeta])$ must vanish for each value of $\zeta < 2$. The following sum rules result:

$$\int_0^1 d\alpha\, \alpha^{\zeta-1}[F_4^{(N\nu)}(\alpha;[\zeta]) + F_4^{(N\bar\nu)}(\alpha;[\zeta])] = 0 \quad \text{if} \quad \zeta < 2 ,$$

$$\int_0^1 d\alpha\, \alpha^{\zeta}[F_4^{(N\nu)}(\alpha;[\zeta]) - F_4^{(N\bar\nu)}(\alpha;[\zeta])] = 0 \quad \text{if} \quad \zeta \le \tfrac{3}{2} ,$$

$$\int_0^1 d\alpha\, \alpha^{\zeta-1}[F_5^{(N\nu)}(\alpha;[\zeta]) - F_5^{(N\bar\nu)}(\alpha;[\zeta])] = 0 \quad \text{if} \quad \zeta \le \tfrac{3}{2} . \tag{1.5}$$

These are also valid for strange and nonstrange hadronic final states separately, and for either nucleon as target.

In Sec. III we review the conventional definitions of form factors and the light-cone expansions of current commutators, in order to agree on notation. Section III is a discussion of the equal-time limit of the light-cone expansion. There we deduce the consequences of the demand that the equal-time singularities of current commutators are the smooth limits of the light-cone singularities. In Sec. IV we translate these consequences into sum rules on the scaling functions of W_4 and W_5. In Sec. V we comment on the results of our analysis.

II. PRELIMINARIES

A. Definition of Form Factors

We define the inelastic form factors $W_i^{(\nu)}$, $i=1,2,\ldots,5$ for the reaction

$$N(p)+\nu(k_i)\to l(k_f)+\text{anything} \tag{2.1}$$

by[10,11]

$$\frac{1}{2\pi}\int d^4x\, e^{iq\cdot x}\langle p|[J_\mu^\dagger(x),J_\nu(0)]|p\rangle = W_{\mu\nu}^{(\nu)}(p,q)$$
$$=(g_{\mu\nu}-q^{-2}q_\mu q_\nu)W_1^{(\nu)}-\frac{P_\mu P_\nu}{m^2}W_2^{(\nu)}+\frac{i}{m^2}\epsilon_{\mu\nu\rho\sigma}p_\rho q_\sigma W_3^{(\nu)}+\frac{q_\mu q_\nu}{m^2}W_4^{(\nu)}+\frac{p_\mu q_\nu+p_\nu q_\mu}{m^2}W_5^{(\nu)}, \tag{2.2}$$

where $q=k_i-k_f$, $P=p-(p\cdot q)q/q^2$, and m is the nucleon mass. The form factors $W_i^{(\nu)}$ are functions of two invariants

$$W_i^{(\nu)}=W_i^{(\nu)}(\nu,q^2),\quad i=1,2,\ldots,5 \tag{2.3}$$

with $\nu=p\cdot q/m$. In Eq. (2.2), the matrix element of the current commutator is to be understood as averaged over the nucleon spin, and J_μ is the Cabibbo current:

$$J_\mu=\cos\theta\,(V_\mu^1+iV_\mu^2+A_\mu^1+iA_\mu^2)$$
$$+\sin\theta\,(V_\mu^4+iV_\mu^5+A_\mu^4+iA_\mu^5). \tag{2.4}$$

For $(p+q)^2\geq m^2$, we have

$$W_{\mu\nu}(p,q)=\sum_n\langle p|J_\mu^\dagger|n\rangle\langle n|J_\nu|p\rangle(2\pi)^3\delta^4(p+q-p_n). \tag{2.5}$$

For the antineutrino-initiated reactions

$$N(p)+\bar\nu(k_i)\to\bar l(k_f)+\text{anything} \tag{2.6}$$

we decompose the matrix element

$$W_{\mu\nu}^{(\bar\nu)}(p,q)=\frac{1}{2\pi}\int d^4x\,e^{iq\cdot x}\langle p|[J_\mu(x),J_\nu^\dagger(0)]|p\rangle$$
$$=\sum_n\langle p|J_\mu|n\rangle\langle n|J_\nu^\dagger|p\rangle(2\pi)^3\delta^4(p+q-p_n),$$
$$\text{for}\quad(p+q)^2\geq m^2 \tag{2.7}$$

into $W_i^{(\bar\nu)}(\nu,q^2)$ just as we did for $W_{\mu\nu}^{(\nu)}$ in Eq. (2.2). The two sets of form factors $W_i^{(\nu)}$ and $W_i^{(\bar\nu)}$ are related by analytic continuation. We see from Eqs. (2.2) and (2.7) that

$$W_{\mu\nu}^{(\bar\nu)}(p,q)=-W_{\nu\mu}^{(\nu)}(p,-q) \tag{2.8}$$

which implies, for example,

$$W_4^{(\bar\nu)}(\nu,q^2)=-W_4^{(\nu)}(-\nu,q^2),$$
$$W_5^{(\bar\nu)}(\nu,q^2)=+W_5^{(\nu)}(-\nu,q^2). \tag{2.9}$$

B. Light-Cone Expansion

The light-cone expansion of the commutator of two currents has been discussed by Brandt and Preparata,[5] Frishman,[6] Fritzsch and Gell-Mann,[7] and Mandula, Schwimmer, Weyers, and Zweig.[9] The light-cone expansion, suitable only when spin-averaged matrix elements between states of the same four-momentum are taken, may be written as[9]

$$[\partial^\mu J_\mu^\dagger(x),J_\nu(0)]=-\frac{1}{m^2}\partial_\nu\partial^2\mathcal{O}_4(x,0)$$
$$+\frac{i}{m}(\partial_\mu\partial_\nu+g_{\mu\nu}\partial^2)\mathcal{O}_5^\mu(x,0), \tag{2.10}$$

where $\mathcal{O}_4$ and $\mathcal{O}_5^\mu$ take the form

$$\mathcal{O}_4(x,0)=\sum_\alpha E(x;c_\alpha)A^{(\alpha)}(x,0),$$
$$\mathcal{O}_5^\mu(x,0)=\sum_\beta E(x;c_\beta)B^{(\beta)\mu}(x,0). \tag{2.11}$$

In Eq. (2.11), $E(x;c)$ represents the causal invariant function which vanishes for $x^2<0$ and which scales as

$$E(\lambda x;c)=\lambda^{-c}E(x;c).$$

We shall standardize $E(x;c)$ by the definition[6]

$$E(x;c)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\left[\left(\frac{1}{-k^2-i\epsilon k_0}\right)^{2-c/2}-\left(\frac{1}{-k^2+i\epsilon k_0}\right)^{2-c/2}\right]$$
$$=\frac{-i}{2^{4-c}\pi^2}\frac{\Gamma(c/2)}{\Gamma(2-c/2)}[(-x^2-i\epsilon x_0)^{-c/2}-(-x^2+i\epsilon x_0)^{-c/2}]. \tag{2.12}$$

It is important to bear in mind that $E(x;c)$ is a generalized function and that Eq. (2.12) must be understood by its analytic continuation in the parameter c when it is not defined in the classical sense. The operators $A^{(\alpha)}(x,0)$ and $B^{(\beta)\mu}(x,0)$ are "bilocal," in the same sense that any local operator at y, which lies outside the light cones with tips at points 0 and x, commutes with these operators. The operators $A^{(\alpha)}(x,0)$ and $B^{(\beta)\mu}(x,0)$ have Taylor expansions about $x_\mu=0$:

$$\begin{aligned} A^{(\alpha)}(x,0)&=\sum_{n=0}^{\infty}a^{(\alpha)\mu_1\cdots\mu_n}x_{\mu_1}\cdots x_{\mu_n}\,,\\ B^{(\beta)\mu}(x,0)&=\sum_{n=0}^{\infty}b^{(\beta)\mu\mu_1\cdots\mu_n}x_{\mu_1}\cdots x_{\mu_n}\,. \end{aligned} \tag{2.13}$$

It was shown in Ref. 9 that the most singular parts of $\mathcal{O}_4$ and $\mathcal{O}_5^\mu$ on the light cone are given by the terms proportional to $E(x;c_\alpha)$ with $c_\alpha=c_{\max}=\max[d-2,0]$, so that for $d>2$,

$$\begin{aligned} \mathcal{O}_4(x,0)&\sim\frac{\epsilon(x_0)\theta(x^2)}{(x^2)^{c_{\max}/2}}A\,,\\ \mathcal{O}_5^\mu(x,0)&\sim\frac{\epsilon(x_0)\theta(x^2)}{(x^2)^{c_{\max}/2}}B^\mu\,, \end{aligned} \tag{2.14}$$

where d is the physical dimension of the chiral $SU(3)\otimes SU(3)$ symmetry-breaking part of the Hamiltonian.

As we shall discuss in the next section, $\mathcal{O}_4$ and possibly $\mathcal{O}_5$ contain terms proportional to $E(x;c)$ with $c=0$, in order that the light-cone commutator Eq. (2.10) have a smooth limit as $x_0\to 0$ and converge to the equal-time commutator. Thus, for $d\le 2$, the leading singularity on the light cone is given by

$$\mathcal{O}_4(x,0)\sim\epsilon(x_0)\theta(x^2)(\ln x^2)A\,, \tag{2.15}$$

for $\mathcal{O}_4$ with nonvanishing A, and for $\mathcal{O}_5^\mu$,

$$\mathcal{O}_5^\mu(x,0)\sim\epsilon(x_0)\theta(x^2)(\ln x^2)B^\mu \tag{2.16}$$

if B^μ is nonvanishing. The operator A has the dimension d, and belongs to the same $SU(3)\otimes SU(3)$ multiplet as the symmetry-breaking Hamiltonian.

The form factors $W_4^{(\nu)}$ and $W_5^{(\nu)}$ are given by

$$\begin{aligned} W_4^{(\nu)}(\nu,q^2)&=\int\frac{d^4x}{2\pi}e^{iq\cdot x}\langle p|\mathcal{O}_4(x,0)|p\rangle\,,\quad \nu>0\\ W_5^{(\nu)}(\nu,q^2)&=\int\frac{d^4x}{2\pi}e^{iq\cdot x}\langle p|\mathcal{O}_5^\mu(x,0)|p\rangle\frac{p_\mu}{m}\,,\quad \nu>0\,. \end{aligned} \tag{2.17}$$

In the Bjorken limit,

$$\begin{aligned} &\nu,\,-q^2\to\infty\,,\\ &\alpha=-q^2/2m\nu\ \text{fixed}\,, \end{aligned} \tag{2.18}$$

their scaling behavior can be deduced from Eqs. (2.14)–(2.16). The result is

$$W_{4,5}^{(\nu)}(\nu,q^2)\sim\frac{1}{(2m\nu)^\zeta}F_{4,5}^{(\nu)}(\alpha)\,, \tag{2.19}$$

where the minimum value of ζ is $\min[3-\frac{1}{2}d,2]$. The same relations hold for $W_4^{(\bar\nu)}$ and $W_5^{(\bar\nu)}$, *mutatis mutandis*. The derivation of Eq. (2.19) will be given in Sec. IV.

III. EQUAL-TIME LIMIT

We shall consider now the limit $x_0\to 0$ in Eq. (2.10) and demand that the light-cone expansion of the commutator of two currents have a smooth, analytic equal-time limit. In particular, we will demand that the equal-time limit cannot depend on whether the limit $x_0\to 0_+$ or $x_0\to 0_-$ is taken.

The integrated equal-time commutator

$$\int d^3x[J_0(\vec{x},0),\partial^\mu J_\mu^\dagger(0)]=-iS(0) \tag{3.1}$$

defines the scalar density $S(x)$, which can be written in terms of the formal charge Q associated with the Cabibbo current,

$$Q=\int d^3x\,J_0(\vec{x},0)\,,$$

as

$$S(\vec{x},0)=[Q,[Q^\dagger,\mathcal{H}'(\vec{x},0)]]\,, \tag{3.2}$$

where $\mathcal{H}'(x)$ is the chiral $SU(3)\otimes SU(3)$ symmetry-breaking Hamiltonian density.[1] Since we want the commutator $[J_0(x),\partial^\mu J_\mu^\dagger(0)]$ to be local, we must have

$$[J_0(\vec{x},0),\partial^\mu J_\mu^\dagger(0)]=-i\delta^3(\vec{x})S(0)+\text{S.T.} \tag{3.3}$$

where S.T. stands for possible Schwinger terms which vanish upon integration over $\vec{x}$. In the following, we shall assume that the physical scale dimension d of $\mathcal{H}'$ is less than four. Under this assumption, the diagonal matrix elements of the Schwinger terms are absent in Eq. (3.3), as we shall show.

Let us consider the matrix elements of $A^{(\alpha)}$ and $B^{(\beta)\mu}$ between nucleon states of momentum p. From Eq. (2.13) we find

$$\begin{aligned} \langle p|A^{(\alpha)}(x,0)|p\rangle&=G^{(\alpha)}(p\cdot x)\\ &=\sum_{n=0}^{\infty}g_n^{(\alpha)}(p\cdot x)^n\,,\\ \langle p|B^{(\beta)\mu}(x,0)|p\rangle&=(p^\mu/m)H^{(\beta)}(p\cdot x)\\ &=(p^\mu/m)\sum_{n=0}^{\infty}h_n^{(\beta)}(p\cdot x)^n\,, \end{aligned} \tag{3.4}$$

where[12]

$$\begin{aligned} \langle p|a^{(\alpha)\mu_1\cdots\mu_n}|p\rangle&=g_n^{(\alpha)}p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}\,,\\ \langle p|b^{(\beta)\mu\mu_1\cdots\mu_n}|p\rangle&=h_n^{(\beta)}(p^\mu/m)p^{\mu_1}\cdots p^{\mu_n}\,. \end{aligned} \tag{3.5}$$

We next combine Eqs. (2.10), (2.11), (3.4), and (3.5) and express the matrix element

$$\langle N(p)|[J_0(x), \partial^\mu J_\mu^\dagger(0)]|N(p)\rangle$$

as a sum of terms of the form

$$\text{const}(x_0)^m E(x; c), \quad m \text{ integral} \tag{3.6}$$

by making use of the identities

$$\partial^2 E(x, c) = E(x; c+2),$$

$$\partial_\mu E(x, c) = 2(2-c)^{-1} x_\mu E(x; c+2).$$

[Note that $(2-c)^{-1}E(x; c+2)$ is finite at $c=2$.]

Let us consider the limit $x_0 \to 0$. The terms (3.6) have the limit

$$\lim_{x_0\to 0} x_0{}^m E(x; c) = \delta^3(\vec{x}) \times \begin{cases} \infty & \text{if } 3+m-c<0, \\ 1 & \text{if } 3+m-c=0, \; m \text{ odd} \\ \epsilon(x_0) & \text{if } 3+m-c=0, \; m \text{ even} \\ 0 & \text{if } 3+m-c>0, \end{cases}$$

if $c-m<4$. If $c-m \geq 4$, there would be additional terms proportional to derivatives of $\delta^3(\vec{x})$. Since only terms with $c-m<4$ appear in Eq. (3.6) if $d<4$, there is no diagonal matrix element of an operator Schwinger term in the equal-time commutator of J_0 and $\partial^\mu J_\mu^\dagger$. In order that the equal-time limit of the matrix element of Eq. (2.10) is $\delta^3(\vec{x})$ times a finite constant, the terms (3.6) for which $3+m-c<0$ or $3+m-c=0$ (m even) should have zero coefficients. This implies

$$g_0^{(\alpha)}=0 \;\text{ if }\; c_\alpha>0,$$
$$g_1^{(\alpha)}=0 \;\text{ if }\; c_\alpha \geq 1, \tag{3.7}$$
$$h_0^{(\beta)}=0 \;\text{ if }\; c_\beta \geq 1.$$

We see further that there must be a term with $c_\alpha=0$ in the expansion (2.11) for $\mathcal{O}_4$, if the matrix element of $S(0)$ is not to vanish. In fact, we have

$$\lim_{x_0\to 0}\langle N(p)|[J_0(x), \partial^\mu J_\mu^\dagger(0)]|N(p)\rangle = \delta^3(\vec{x}) g_0^{(\alpha)}/m^2, \quad c_\alpha=0 \tag{3.8}$$

so that

$$g_0^{(\alpha)}/m^2 = -i\langle N(p)|S(0)|N(p)\rangle, \quad c_\alpha=0. \tag{3.9}$$

IV. SUM RULES

We shall now discuss the observable consequences of Eqs. (3.7) and (3.8). To this end, we shall first consider the scaling behavior of W_4 and W_5 in the Bjorken limit. In view of the preceding discussions, we may write Eq. (2.17) as

$$W_4^{(\nu,\bar\nu)}(\nu, q^2) \sim \pm \frac{1}{2\pi}\int d^4x\, e^{\pm iq\cdot x}\sum_\alpha E(x; c_\alpha) G^{(\alpha)}(p\cdot x),$$
$$W_5^{(\nu,\bar\nu)}(\nu, q^2) \sim \frac{1}{2\pi}\int d^4x\, e^{\pm iq\cdot x}\sum_\beta E(x; c_\beta) H^{(\beta)}(p\cdot x). \tag{4.1}$$

We may express $G^{(\alpha)}$ and $H^{(\beta)}$ in terms of Fourier transforms

$$G^{(\alpha)}(p\cdot x) = \int d\lambda\, e^{i\lambda p\cdot x}\tilde{G}^{(\alpha)}(\lambda),$$
$$H^{(\beta)}(p\cdot x) = \int d\lambda\, e^{i\lambda p\cdot x}\tilde{H}^{(\beta)}(\lambda). \tag{4.2}$$

When Eq. (4.2) is substituted in Eq. (4.1) and the limit $-q^2, \nu \to \infty$, with $\alpha = -q^2/2m\nu$ held fixed, is taken, we obtain

$$W_4^{(\nu,\bar\nu)}(\nu, q^2) \sim \sum_\alpha \frac{1}{(2m\nu)^{\zeta_\alpha}} F_4^{(\nu,\bar\nu)}(\alpha; [\zeta_\alpha]),$$
$$W_5^{(\nu,\bar\nu)}(\nu, q^2) \sim \sum_\beta \frac{1}{(2m\nu)^{\zeta_\beta}} F_5^{(\nu,\bar\nu)}(\alpha; [\zeta_\beta]), \tag{4.3}$$

where

$$\zeta_\alpha = 2 - \tfrac{1}{2}c_\alpha, \quad \zeta_\beta = 2 - \tfrac{1}{2}c_\beta.$$

The scaling functions F_4 and F_5 are integrals over $\tilde{G}^{(\alpha)}$ and $\tilde{H}^{(\beta)}$:

$$F_4^{(\nu)}(\alpha; [\zeta_\alpha]) = \frac{1}{2\pi}\int_{-1}^{1} d\lambda\, \tilde{G}^{(\alpha)}(\lambda)[(\alpha-\lambda-i\epsilon)^{-\zeta_\alpha} - (\alpha-\lambda+i\epsilon)^{-\zeta_\alpha}],$$
$$F_5^{(\nu)}(\alpha; [\zeta_\alpha]) = \frac{1}{2\pi}\int_{-1}^{1} d\lambda\, \tilde{H}^{(\alpha)}(\lambda)[(\alpha-\lambda-i\epsilon)^{-\zeta_\alpha} - (\alpha-\lambda+i\epsilon)^{-\zeta_\alpha}] \tag{4.4}$$

and

$$F_4^{(\bar\nu)}(\alpha; [\zeta_\alpha]) = \frac{1}{2\pi}\int_{-1}^{1} d\lambda\, \tilde{G}^{(\alpha)}(-\lambda)[(\alpha-\lambda-i\epsilon)^{-\zeta_\alpha} - (\alpha-\lambda+i\epsilon)^{-\zeta_\alpha}],$$
$$F_5^{(\bar\nu)}(\alpha; [\zeta_\alpha]) = -\frac{1}{2\pi}\int_{-1}^{1} d\lambda\, \tilde{H}^{(\alpha)}(-\lambda)[(\alpha-\lambda-i\epsilon)^{-\zeta_\alpha} - (\alpha-\lambda+i\epsilon)^{-\zeta_\alpha}], \tag{4.5}$$

where we have used the spectral condition

$$W_4^{(\nu,\bar\nu)} = W_5^{(\nu,\bar\nu)} = 0 \;\text{ for }\; q^2 < -|2m\nu|$$

to deduce that

$$\tilde{G}^{(\alpha)}(\lambda)=\tilde{H}^{(\beta)}(\lambda)=0 \quad \text{for} \quad |\lambda|>1\,.$$

The conditions (3.7) and (3.8) may be stated in terms of $\tilde{G}$ and $\tilde{H}$. Thus,

$$\int_0^1 d\lambda[\tilde{G}^{(\alpha)}(\lambda)+\tilde{G}^{(\alpha)}(-\lambda)]=0 \quad \text{if} \quad \zeta_\alpha<2\,,$$
$$\int_0^1 d\lambda\,\lambda[\tilde{G}^{(\alpha)}(\lambda)-\tilde{G}^{(\alpha)}(-\lambda)]=0 \quad \text{if} \quad \zeta_\alpha\le\tfrac{3}{2}\,, \qquad (4.6)$$
$$\int_0^1 d\lambda[\tilde{H}^{(\beta)}(\lambda)+\tilde{H}^{(\beta)}(-\lambda)]=0 \quad \text{if} \quad \zeta_\beta\le\tfrac{3}{2}\,,$$

and, for $\zeta_\alpha=2$,

$$-i\langle N(p)|S|N(p)\rangle=\frac{1}{m^2}\int_0^1 d\lambda[\tilde{G}^{(\alpha)}(\lambda)+\tilde{G}^{(\alpha)}(-\lambda)]\,. \qquad (4.7)$$

Our task now is to express the conditions (4.6) and (4.7) in terms of F_4 and F_5, which, in principle, can be measured experimentally. For this purpose, we must invert Eqs. (4.4) and (4.5). Consider the equation

$$F(\alpha)=\frac{1}{2\pi}\int_{-1}^1 d\lambda\,\tilde{G}(\lambda)[(\alpha-\lambda-i\epsilon)^{-\zeta}-(\alpha-\lambda+i\epsilon)^{-\zeta}]$$
$$=i\,\frac{\sin\pi\zeta}{\pi}\int_\alpha^1 d\lambda\,\tilde{G}(\lambda)(\lambda-\alpha)^{-\zeta}\,. \qquad (4.8)$$

This equation must be understood in the sense of generalized functions for $\zeta\ge1$. Thus, for $1\le\zeta<2$, for example, Eq. (4.8) means

$$F(\alpha)=i\,\frac{\sin\zeta\pi}{\pi}(\zeta-1)^{-1}\frac{d}{d\alpha}\int_\alpha^1 d\lambda\,\tilde{G}(\lambda)(\lambda-\alpha)^{1-\zeta}\,.$$

The inversion problem associated with Eq. (4.8) is known as Abel's integral equation.[13] The solution is

$$i\tilde{G}(\lambda)=(\zeta-1)\int_\lambda^1 d\alpha\,F(\alpha)(\alpha-\lambda)^{\zeta-2}\,. \qquad (4.9)$$

For $\zeta\le1$, Eq. (4.9) must be understood in the sense of generalized functions. Similarly, Eqs. (4.4) and (4.5) can be solved for $\tilde{G}(\lambda)$ and $\tilde{H}(\lambda)$ in terms of $F_{4,5}^{(\nu)}$ and $F_{4,5}^{(\bar\nu)}$. When the expressions for $\tilde{G}(\lambda)$ and $\tilde{H}(\lambda)$ are substituted in Eqs. (4.6) and (4.7), we obtain the desired results:

$$\int_0^1 d\alpha\,\alpha^{\zeta-1}[F_4^{(\nu)}(\alpha;[\zeta])+F_4^{(\bar\nu)}(\alpha;[\zeta])]=0, \quad \zeta<2$$
$$\int_0^1 d\alpha\,\alpha^{\zeta}[F_4^{(\nu)}(\alpha;[\zeta])-F_4^{(\bar\nu)}(\alpha;[\zeta])]=0, \quad \zeta\le\tfrac{3}{2}$$
$$\int_0^1 d\alpha\,\alpha^{\zeta-1}[F_5^{(\nu)}(\alpha;[\zeta])-F_5^{(\bar\nu)}(\alpha;[\zeta])]=0, \quad \zeta\le\tfrac{3}{2} \qquad (4.10)$$

and

$$\langle N(p)|S(0)|N(p)\rangle=\frac{1}{m^2}\int_0^1 d\alpha\,\alpha[F_4^{(\nu)}(\alpha;[2])+F_4^{(\bar\nu)}(\alpha;[2])]\,. \qquad (4.11)$$

V. COMMENTS AND CONCLUSIONS

Our central results are the sum rules (4.10) and (4.11). The sum rules (4.10) provide a test of the broken scale-invariance hypothesis for the strong interactions, which forms the framework for this entire analysis. They depend only on this hypothesis, plus the rather plausible assumptions that the equal-time singularity is the smooth limit of the light-cone singularity, and that the one-nucleon matrix elements of the $[Q,[Q^\dagger,\mathcal{H}]]$ double commutator are finite. If $\zeta_{\min}<2$, these sum rules (for $\zeta=\zeta_{\min}$) will be much easier to evaluate experimentally than those which measure the double commutators.

The sum rule (4.11) expresses the nucleon matrix element of $[Q,[Q^\dagger,\mathcal{H}]]$ in terms of an integral over scaling functions. More information about the chiral $SU(3)\otimes SU(3)$ transformation properties of $\mathcal{H}$ can be extracted from measurements if one separates total inelastic neutrino scattering into two subprocesses – events in which the net strangeness of the final hadrons is 0 and events in which it is 1. For these partial processes, one finds

$$\int_0^1 \alpha\,d\alpha[F_{\text{nonstrange}}^{(N\nu)}(\alpha;[2])+F_{\text{nonstrange}}^{(N\bar\nu)}(\alpha;[2])]=m^2\cos^2\theta_C\langle N(p)|[Q^1-iQ^2,[Q^1+iQ^2,\mathcal{H}]]|N(p)\rangle\,,$$
$$\int_0^1 \alpha\,d\alpha[F_{\text{strange}}^{(N\nu)}(\alpha;[2])+F_{\text{strange}}^{(N\bar\nu)}(\alpha;[2])]=m^2\sin^2\theta_C\langle N(p)|[Q^4-iQ^5,[Q^4+iQ^5,\mathcal{H}]]|N(p)\rangle\,, \qquad (5.1)$$

where θ_C is the Cabibbo angle.

These double commutators reflect the representation of $\mathcal{H}_{\text{break}}$. If, for example, we choose the symmetry-breaking Hamiltonian to contain terms u_i which transform like $(3,\bar{3})+(3,\bar{3})$ and terms s_i which transform like $(8,1)+(1,8)$,

$$\mathcal{H}_{\text{break}}=u_0+cu_8+s_8\,, \qquad (5.2)$$

then the one-nucleon matrix elements of the double commutator are

$$\langle N(p)|[Q,[Q^\dagger,\mathcal{H}]]|N(p)\rangle = \cos^2\theta_C\langle N(p)|[Q^1-iQ^2,[Q^1+Q^2,\mathcal{H}]]|N(p)\rangle + \sin^2\theta_C\langle N(p)|[Q^4-iQ^5,[Q^4+iQ^5,\mathcal{H}]]|N(p)\rangle$$

$$= \cos^2\theta_C\langle N(p)|\frac{2\sqrt{2}+2c}{3}(\sqrt{2}\,u_0+u_8)|N(p)\rangle + \sin^2\theta_C\langle N(p)|\frac{4-\sqrt{2}\,c}{3}u_0+\frac{-\sqrt{2}+5c}{3}u_8+\frac{\sqrt{2}+c}{\sqrt{3}}u_3+3s_8+\sqrt{3}\,s_3|N(p)\rangle\,. \quad (5.3)$$

Note that the nonstrange part of the weak current gives an equal contribution to the proton and neutron matrix elements. This is a general theorem, not simply a peculiarity of this example.

The difficulty of evaluating the sum rule (4.11) depends critically on $\zeta_{\min}$, the smallest power of $1/\nu$ in the asymptotic expansion of W_4. If $\zeta_{\min}<2$, then the evaluation of the integrals requires the extraction of the coefficient, as a function of α, of a *subsidiary* power of ν, surely a formidable task. But if $\zeta_{\min}=2$, then the required scaling functions F will be the coefficients of the *leading* power of ν, and would be far more accessible. Which alternative nature has chosen we will know when $\zeta_{\min}$ is measured. This basic strong-interaction parameter is related to the scale dimension of $\mathcal{H}_{\text{break}}$, and its measurement will be much simpler than, and preliminary to, those suggested here.

---

*Permanent address: Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, L. I., New York 11790.

†Work supported in part by the U. S. Atomic Energy Commission. Prepared under Contract No. AT(11-1)-68 for the San Francisco Operations Office, U. S. Atomic Energy Commission.

[1]M. Gell-Mann, Physics 1, 63 (1964).

[2]J. D. Bjorken, Phys. Rev. 179, 1547 (1969).

[3]T. P. Cheng and R. F. Dashen, Phys. Rev. Letters 26, 594 (1971).

[4]This is a generalization of the short-distance expansion of K. Wilson, Phys. Rev. 179, 1499 (1969).

[5]R. A. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971).

[6]Y. Frishman, Phys. Rev. Letters 25, 966 (1970).

[7]H. Fritzsch and M. Gell-Mann, in *Coral Gables Conference on Fundamental Interactions at High Energy II*, edited by A. Perlmutter, G. J. Iverson, and R. M. Williams (Gordon and Breach, New York, 1970).

[8]M. Gell-Mann, in *Proceedings of the Third Hawaii Conference on Particle Physics*, edited by S. F. Tuan (Western Periodicals, Los Angeles, California, 1969).

[9]J. Mandula, A. Schwimmer, J. Weyers, and G. Zweig, Caltech Report No. CALT-68-302, 1971 (unpublished).

[10]A. Pais, in Proceedings of the International Conference on Expectations for Particle Reactions at the New Accelerators, University of Wisconsin, 1970 (unpublished).

[11]C. H. Llewellyn Smith, CERN Report No. CERN TH-1188, 1970 (unpublished).

[12]There are more terms on the right-hand side of Eq. (3.5) involving one or more $g^{\rho\sigma}$ $(\rho,\sigma=\mu_1,\ldots,\mu_n)$. However, such terms do not contribute to the light-cone singularities and may be ignored in the ensuing discussion.

[13]See, for example, I. M. Gelfand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. I., pp. 117–118; E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge Univ. Press, New York, 1943), p. 229.