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WHIRLING OF A HEAVY CHAIN

Approximate solutions are obtained for the non-linear problem of the forced whirling of a heavy chain. It is shown that for a given mode of whirling, triple valued solutions are obtained for speeds of rotation above the linear critical speed for that mode. It is also shown that two of these solutions are stable while the third is unstable.

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This problem is of some antiquity, the linearized solutions having been derived by D. Bernoulli (1700–1782) and L. Euler (1707–1783). A simple experiment with a key chain is sufficient to show that the linearized solution completely fails to explain the physics of the problem. Almost two hundred years passed before a more suitable solution was obtained which included the non-linear effects. In 1955 Kolodner [1] at New York University made a detailed mathematical study of the free whirling of a heavy chain and showed that if a given mode is considered, then above the linear critical speed, the deflection is a continuous function of the speed of rotation.

In this paper approximate non-linear solutions will be obtained for the forced whirling of a heavy chain. It will be shown that for a given mode, triple valued solutions are obtained for speeds of rotation above the linear critical speed for that mode. It will further be shown that two of these solutions are stable while the third is unstable.

Formulation

Consider the problem of a heavy chain being rotated about a vertical axis with an angular velocity ω, the upper end of the chain being a fixed distance ϵ from the axis of rotation, the lower end being free.

The problem is to determine the possible configurations such a chain will take in steady rotation. By considering a heavy chain, the effect of air damping may be neglected in comparison with the other forces acting on the chain.

Equations of Motion

Denote by:

S = the arclength measured from the free end of the chain.
$\bar{x} = \{x(S,t),\ y(S,t),\ z(S,t)\}$ – the position vector of a point in the chain.
$T(S,t)$ = the tension
$\bar{g} = (o,o,\ -g)$ – the acceleration of gravity vector.

Consider the motion of the chain in a set of rectangular coordinate axes rotating about the z axis with an angular velocity ω.

The equations of motion are then

$$(\rho\bar{x}_t)_t + \rho\bar{g} = (T\bar{x}_S)_S \tag{1}$$

$$\bar{x}_S{}^2 = 1 \tag{2}$$

With the end conditions

$$T(o,t) = 0;\ \bar{x}(L,t) = (\epsilon,o,o) \tag{3}$$

Writing out the three components of (1)

$$\frac{\partial}{\partial S}\left(T(S,t)\frac{\partial x}{\partial S}\right) = \rho(\ddot{x} - \omega^2 x - 2\omega\dot{y}) \tag{4}$$

$$\frac{\partial}{\partial S}\left(T(S,t)\frac{\partial y}{\partial S}\right) = \rho(\ddot{y} - \omega^2 y + 2\omega\dot{x} \tag{5}$$

$$\frac{\partial}{\partial S}\left(T(S,t)\frac{\partial z}{\partial S}\right) = \rho g + \rho\ddot{z} \tag{6}$$

The geometric condition (2) becomes

$$\left(\frac{\partial x}{\partial S}\right)^2 + \left(\frac{\partial y}{\partial S}\right)^2 + \left(\frac{\partial z}{\partial S}\right)^2 = 1 \tag{7}$$

Steady State Solutions

For a steady state of rotation, the velocities and accelerations in the rotating coordinate system must be zero.

Hence

$$\dot{x} = \dot{y} = \dot{z} = 0$$

$$\ddot{x} = \ddot{y} = \ddot{z} = 0$$

Equations (4), (5), and (6) therefore reduce to:

$$\frac{d}{dS}\left(T(S)\,\frac{dx}{dS}\right) = -\rho\omega^2 x \tag{8}$$

$$\frac{d}{dS}\left(T(S)\,\frac{dy}{dS}\right) = -\rho\omega^2 y \tag{9}$$

$$\frac{d}{dS}\left(T(S)\,\frac{dz}{dS}\right) = \rho g \tag{10}$$

Integrating (10) and using the boundary condition at $S = 0$

$$\therefore T(S) = \frac{\rho g S}{\left(\frac{dz}{dS}\right)} \tag{11}$$

Substituting (11) into (8) and (9) and using (7)

$$\frac{d}{dS}\left\{S\,\frac{dx}{dS}\Big/\left[1-\left(\frac{dx}{dS}\right)^2-\left(\frac{dy}{dS}\right)^2\right]^{\frac{1}{2}}\right\} + \frac{\omega^2}{g}x = 0 \tag{12}$$

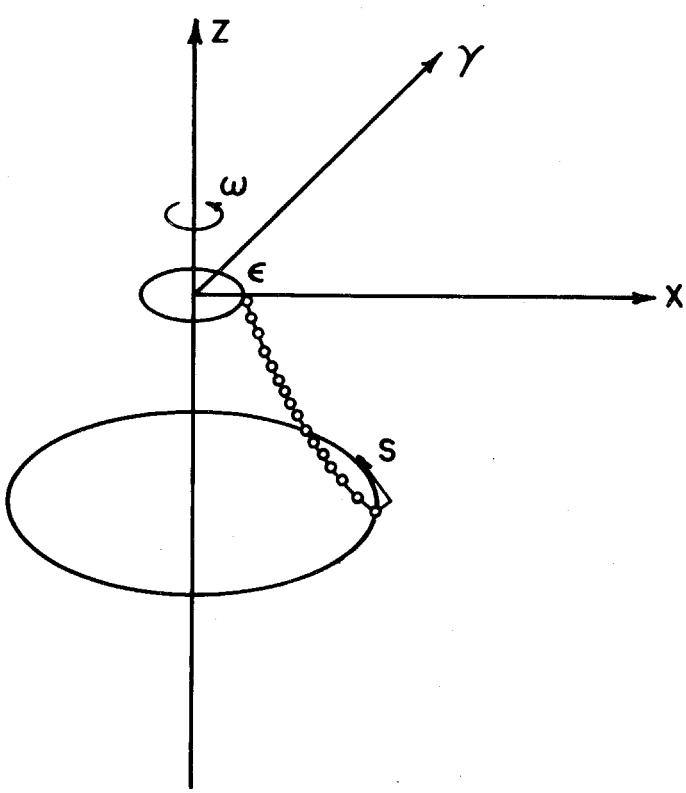

FIGURE 1. FORCED WHIRLING OF HEAVY CHAIN.

$$\frac{d}{dS}\left\{S\,\frac{dy}{dS}\Big/\left[1-\left(\frac{dx}{dS}\right)^2-\left(\frac{dy}{dS}\right)^2\right]^{\frac{1}{2}}\right\} + \frac{\omega^2}{g}y = 0 \tag{13}$$

Before proceeding to the solution of (12) and (13), consider the linearized problem given by assuming x and y small. In this case it can easily be shown that the solution of (12) and (13) is:

$$\left.\begin{aligned} x &= \epsilon + \sum_{i=1}^{\infty} A_i J_o(\alpha_i (S/L)^{\frac{1}{2}}) \\ y &= 0 \end{aligned}\right\} \text{provided } \omega_i \neq \omega \tag{14}$$

where

$$A_i = \frac{\epsilon \int_o^L J_o(\alpha_i (S/L)^{\frac{1}{2}})\,dS}{\left(\frac{\omega_i^2}{\omega^2}-1\right)\int_o^L J_o^2(\alpha_i (S/L)^{\frac{1}{2}})\,dS} \tag{15}$$

$$\alpha_i \text{ are the roots of } J_o(\alpha_i) = 0 \tag{16a}$$

and

$$\omega_i = \frac{\alpha_i}{2}(g/L)^{\frac{1}{2}} \tag{16b}$$

is the critical speed for the ith mode in free whirling.

From (15) it will be observed that as $\omega \longrightarrow \omega_i$, $A_i \longrightarrow \infty$; thus, in the vicinity of the critical speed a single term represents the summation with considerable accuracy. In the case of the non-linear problem it seems reasonable to express the solution in the form:

$$\left.\begin{aligned} x &= \epsilon + \sum_{i=1}^{\infty} A_i J_o(\alpha_i (S/L)^{\frac{1}{2}}) \\ y &= 0 \end{aligned}\right\} \tag{17}$$

where the A_i's are to be chosen to satisfy the differential equation

$$\frac{d}{dS}\left\{S\,\frac{dx}{dS}\Big/\left[1-\left(\frac{dx}{dS}\right)^2\right]^{\frac{1}{2}}\right\} + \frac{\omega^2}{g}x = 0 \tag{18}$$

Thus the problem of solving the non-linear differential equation is reduced to the problem of solving an infinite set of non-linear algebraic equations. In general this

problem is scarcely less difficult than the solution of the differential equation; however, the linearized solution suggests that in the vicinity of a critical speed a single term in the summation will give a reasonable approximation to the solution.

Thus, take

$$x \simeq \epsilon + A_i J_o(\alpha_i (S/L)^{\frac{1}{2}}) \qquad (19)$$
$$i = 1, 2, 3, \ldots.$$

From the geometry of the chain it is obvious that

$$\left(\frac{dx}{dS}\right)^2 \leq 1$$

This condition imposes an upper bound to A_i in (19), namely

$$\left[A_i \frac{\alpha_i}{2} \frac{1}{\sqrt{SL}} J_1(\alpha_i (S/L)^{\frac{1}{2}})\right]^2 \leq 1 \qquad (20)$$

Since the maximum value of $J_1(x)/x$ occurs at $x = 0$.

$$\left| A_i \right| \leq \frac{4L}{\alpha_i^2} = A_i^* \qquad (21)$$

This restriction on A_i is a consequence of the approximate nature of (19) and has nothing to do with the differential equation (18) per se. Bearing this restriction in mind, substitute (19) into (18). Thus

$$\frac{d}{dS}\left[\frac{\left\{SA_i \frac{d}{dS} J_o(\alpha_i\sqrt{S/L})\right\}}{\sqrt{1 - A_i^2\left[\frac{d}{dS} J_o(\alpha_i\sqrt{S/L})\right]^2}}\right] + \frac{\omega^2}{g}[A_i J_o(\alpha_i (S/L)^{1/2}) + \epsilon] = 0 \qquad (22)$$

To evaluate A_i, multiply (22) by $J_o(\alpha_i\sqrt{S/L})$ and integrate from 0 to L. Thus

$$A_i \int_o^L \left\{J_o(\alpha_i\sqrt{S/L}) \frac{d}{dS} \times \left[\frac{S\frac{d}{dS} J_o(\alpha_i\sqrt{S/L})}{\sqrt{1 - A_i^2\left[\frac{d}{dS} J_o(\alpha_i\sqrt{S/L})\right]^2}}\right]\right\} dS + \frac{\omega^2}{g}\left\{A_i \int_o^L J_o^2(\alpha_i\sqrt{S/L})\, dS + \epsilon \int_o^L J_o(\alpha_i\sqrt{S/L})\, dS\right\} = 0 \qquad (23)$$

Integrating the first term by parts:

$$A_i\left\{J_o(\alpha_i\sqrt{S/L}) \left[\frac{S\frac{d}{dS}[J_o(\alpha_i\sqrt{S/L})]}{1 - A_i^2\left[\frac{d}{dS} J_o(\alpha_i\sqrt{S/L})\right]^2}\right]\right\}\Bigg|_0^L - \int_o^L \left[\frac{S\left[\frac{d}{dS} J_o(\alpha_i\sqrt{S/L})\right]^2}{\sqrt{1 - A_i^2\left[\frac{d}{dS} J_o(\alpha_i\sqrt{S/L})\right]^2}}\right] dS + \frac{\omega^2}{g}\left\{A_i \int_o^L J_o^2(\alpha_i\sqrt{S/L})\, dS + \epsilon \int_o^L J_o(\alpha_i\sqrt{S/L})\, dS\right. = 0 \qquad (24)$$

The first term vanishes since $J_o(\alpha_i) = 0$, by the choice of α_i. Thus (24 can be written in the form

$$\frac{\omega^2}{g/L}(A_i + \epsilon_i) = L\, F_i(A_i) \qquad (25)$$

where

$$\epsilon_i = \frac{\epsilon \int_o^L J_o(\alpha_i\sqrt{S/L})\, dS}{\int_o^L J_o^2(\alpha_i\sqrt{S/L})\, dS} = \frac{2\epsilon}{\alpha_i J_1(\alpha_i)} \qquad (26)$$

and

$$F_i(A_i) = \frac{A_i \int_o^L \left\{\frac{S\left[\frac{d}{dS} J_o(\alpha_i\sqrt{S/L})\right]^2}{\sqrt{1 - A_i^2\left[\frac{d}{dS} J_o(\alpha_i\sqrt{S/L})\right]^2}}\, dS\right\}}{\int_o^L J_o^2(\alpha_i\sqrt{S/L})\, dS}$$

$$F_i(A_i) = A_i \frac{\omega_i^2}{g} \times$$

$$\left\{ \int_o^1 \frac{z J_1^2(\alpha_i z)}{\sqrt{1 - \frac{\alpha_i^2}{4L^2} A_i^2 \left[\frac{J_1(\alpha_i z)}{z}\right]^2}} dz \right\} \frac{2}{J_1^2(\alpha_i)} \quad (27)$$

Consider $F_i(A_i)$ defined by (27):

1) $F_i(A_i)$ is an odd function of A_i
i.e., $F_i(A_i) = -F_i(-A_i)$
2) $F_i(A_i)$ is a monotonically increasing function of $|A_i|$ provided $A_i < A_i^*$
3) The slope of the function $F_i(A_i)$ is:

$$\frac{\partial F_i(A_i)}{\partial A_i} = \frac{\omega_i^2}{g} \left\{ \int_o^1 \frac{z J_1^2(\alpha_i z)}{\left[\sqrt{1 - \frac{\alpha_i^2}{4L^2} A_i^2 \left[\frac{J_1(\alpha_i z)}{z}\right]^2}\right]^3} dz \right\} \times \frac{2}{J_1^2(\alpha_i)} \quad (28a)$$

is a monotonically increasing function of $|A_i|$. The function $F_i(A_i)$ has its smallest slope at $A_i = 0$

$$\left. \frac{\partial F_i(A_i)}{\partial A_i} \right|_{A_i=0} = \frac{\omega_i^2}{g} \quad (28b)$$

Graphical Solution of (25)

Equation (25) lends itself to simple graphical interpretation. If

$$\lambda_1 = [\omega^2/(g/L)]\,(A_i + \epsilon)$$

and

$$\lambda_2 = L\,F_i(A_i)$$

are plotted against A_i, the points of intersection of λ_1 and λ_2 will determine the solutions A_i.

It will be noted from Fig. 2 that:

1) If $\frac{\omega^2}{g/L} < \frac{\omega_i^2}{g/L}$, then only one intersection can occur and this will be for $A_i > 0$.
2) If $\frac{\omega^2}{g/L} > \frac{\omega_i^2}{g/L}$, then three intersections are possible, one is positive while two are negative.

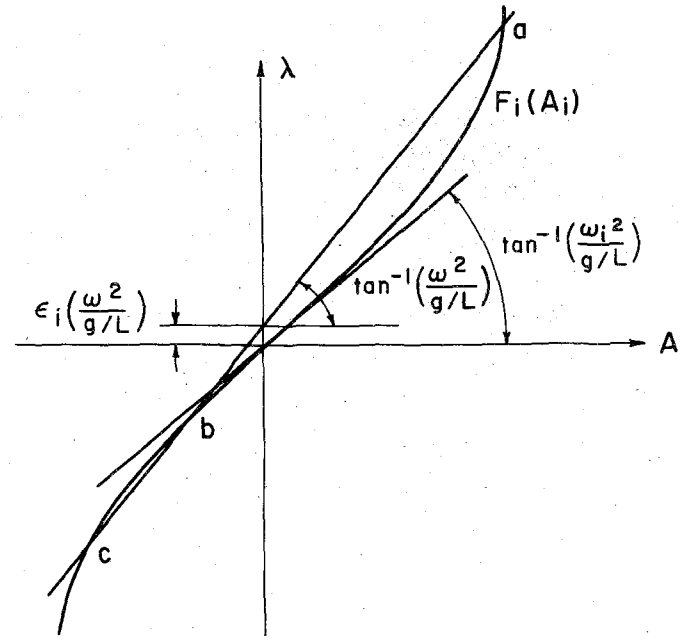

FIGURE 2. GRAPHICAL SOLUTION OF EQUATION (25).

From Fig. 2 it is possible to construct an amplitude/rotational velocity curve such as that in Fig. 3. The analysis breaks down beyond an amplitude A^* given by (21) since $F_i(A_i)$ is complex beyond this point.

Stability of Steady State Solutions

In order to establish the range of ω within which the approximate solution (19) is valid and also to answer the question of which of the three solutions a, b, or c will occur in steady rotation, it is necessary to study the stability of the steady state solutions.

Perturbation Equations

Give small perturbations ξ, η, ζ to x, y, z in (4), (5), (6) and (7); these will cause a small perturbation τ in the tension T.

Substituting

$$\left.\begin{aligned} x &= x_0 + \xi \\ y &= 0 + \eta \\ z &= z_0 + \zeta \\ T &= T_0 + \tau \end{aligned}\right\}$$

into (4), (5), (6) and (7) the following equations are obtained:

$$\frac{\partial}{\partial S}\left(T_0 \frac{\partial \xi}{\partial S} + \tau \frac{\partial x_0}{\partial S}\right) = \rho(\ddot{\xi} - \omega^2 \xi - 2\omega\dot{\eta}) \quad (29)$$

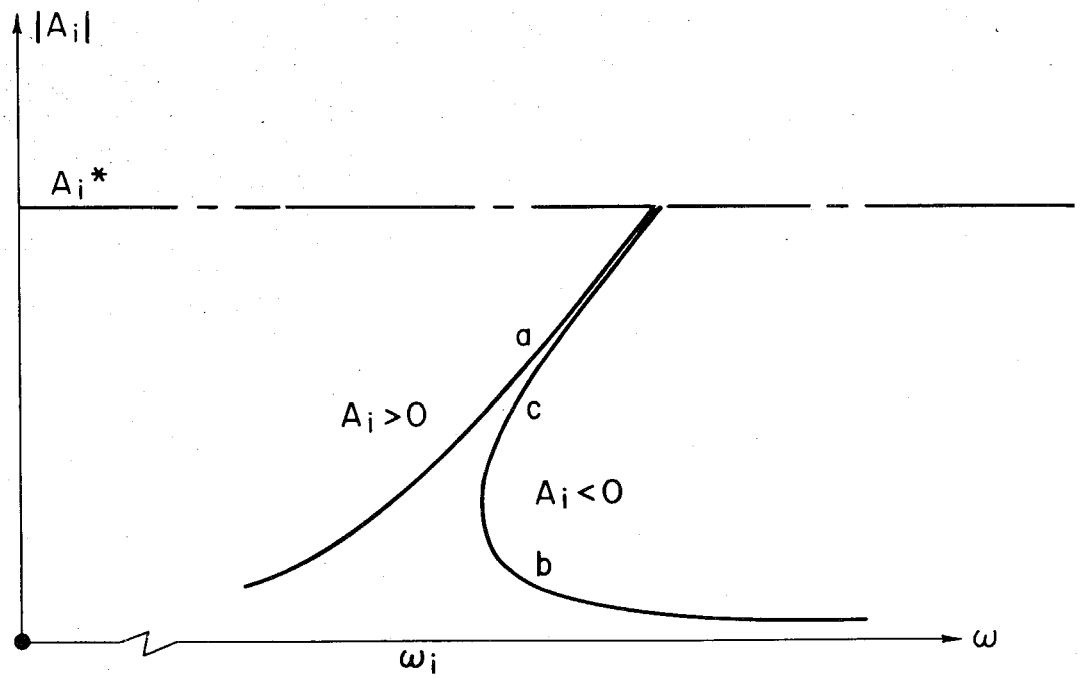

FIGURE 3. AMPLITUDE OF ith MODE.

$$\frac{\partial}{\partial S}\left(T_0 \frac{\partial \eta}{\partial S}\right) = \rho(\ddot{\eta} - \omega^2 \eta + 2\omega\dot{\xi}) \tag{30}$$

$$\frac{\partial}{\partial S}\left(T_0 \frac{\partial \rho}{\partial S} + \tau \frac{\partial z_0}{\partial S}\right) = \rho\ddot{\xi} \tag{31}$$

$$\frac{\partial x_0}{\partial S}\frac{\partial \xi}{\partial S} + \frac{\partial z_0}{\partial S}\frac{\partial \rho}{\partial S} = 0 \tag{32}$$

where the subscript 0 indicates steady state solutions. Since the perturbations are assumed small, the equations are linear in ξ, η, ζ and τ and the time dependence may be assumed to be exponential. Thus

$$\left.\begin{aligned} \xi &= \xi e^{\lambda t} \\ \eta &= \eta e^{\lambda t} \\ \zeta &= \zeta e^{\lambda t} \\ \tau &= \tau e^{\lambda t} \end{aligned}\right\} \tag{33}$$

Integrating (31) and using (32) and the boundary conditions (2)

$$\tau \frac{\partial x_0}{\partial S} = T_0 \frac{\left(\frac{\partial x_0}{\partial S}\right)^2}{\left(\frac{\partial z_0}{\partial S}\right)^2} + \lambda^2 \zeta \frac{\left(\frac{\partial x_0}{\partial S}\right)}{\left(\frac{\partial z_0}{\partial S}\right)} \int_0^S d\beta \int_\beta^L \frac{\left(\frac{\partial x_0}{\partial \gamma}\right)}{\left(\frac{\partial z_0}{\partial \gamma}\right)} \frac{\partial \xi}{\partial \gamma} d\gamma \tag{34}$$

Substituting into (29) and (30)

$$\frac{\partial}{\partial S}\left(T_0 \frac{1}{\left(\frac{\partial z_0}{\partial S}\right)^2} + \lambda^2 \zeta \frac{\left(\frac{\partial x_0}{\partial S}\right)}{\left(\frac{\partial z_0}{\partial S}\right)} \int_0^S d\beta \int_\beta^L \frac{\left(\frac{\partial x_0}{\partial \gamma}\right)}{\left(\frac{\partial z_0}{\partial \gamma}\right)} \frac{\partial \xi}{\partial \gamma} d\gamma\right) = \rho((\lambda^2 - \omega^2)\xi - 2\omega\lambda\eta) \tag{35}$$

$$\frac{\partial}{\partial S}\left(T_0 \frac{\partial \eta}{\partial S}\right) = \rho((\lambda^2 - \omega^2)\eta + 2\omega\gamma\xi) \tag{36}$$

Substituting for T_0 from (11) and using (7), (35) and (36) become

$$\frac{d}{dS}\left\{\frac{S\frac{d\xi}{dS}}{\left[1 - \left(\frac{dx_0}{dS}\right)^2\right]^{3/2}} + \lambda^2 \frac{\frac{dx_0}{dS}}{\sqrt{1 - \left(\frac{dx_0}{dS}\right)^2}} \int_0^S d\beta \int_\beta^L \frac{\frac{dx_0}{d\gamma}\frac{d\xi}{d\gamma}}{\sqrt{1 - \left(\frac{dz_0}{d\gamma}\right)^2}} d\gamma\right\} = \frac{1}{g}\{(\lambda^2 - \omega^2)\xi - 2\omega\lambda\eta\} \tag{37}$$

$$\frac{d}{dS}\left\{\frac{S\dfrac{d\eta}{dS}}{\left[1-\left(\dfrac{dx_0}{dS}\right)^2\right]^{1/2}}\right\}=\frac{1}{g}\left\{(\lambda^2-\omega^2)\eta+2\omega\lambda\xi\right\} \quad (38)$$

Subject to the boundary condition that

$$\eta(L)=\xi(L)=0 \quad (39)$$

Solution of (37) and (38)

The method of solution is to assume $\xi(S)$ and $\eta(S)$ to have the form

$$\left.\begin{aligned}\xi(S)&=\sum_{i=1}^{\infty}a_iJ_0(\alpha_i\sqrt{S/L})\\ \eta(S)&=\sum_{i=1}^{\infty}b_iJ_0(\alpha_i\sqrt{S/L})\end{aligned}\right\} \quad (40)$$

where the α_i's are the roots of $J_0(\alpha_i)=0$.

Substitution of (40) into (37) and (38) and the use of the orthogonality properties of the Bessel functions will lead to a Hill's determinant for the determination of λ. In general the evaluation of such a determinant is difficult, and this particular case, almost impossible. Since the solutions x_0 and T_0 are only approximate it is hardly worth the additional labor to try to compute any but the simplest approximation to the Hill's determinant.

Thus, to the same order of approximation as the steady state solution take

$$\left.\begin{aligned}\xi(S)&\simeq a_jJ_0(\alpha_j\sqrt{S/L})\\ \eta(S)&\simeq b_jJ_0(\alpha_j\sqrt{S/L})\end{aligned}\right\} \quad (41)$$

$$j=1,2,3,\ldots i,\quad i+1,\ldots$$

Substituting (41) into (37) and (38), two simultaneous equations in a_j and b_j may be obtained by expanding the left hand side of (37) and (38) in a series of Bessel functions and equating the coefficients of the $J_0(\alpha_j\sqrt{S/L})$ terms. Thus

$$\left[\frac{\lambda^2-\omega^2}{g}+F_{ij}^{(3)}(A_i)+\frac{\lambda^2}{g}G_{ij}(A_i)\right]a_j-\frac{2\omega x}{g}b_j=0 \quad (42)$$

$$\frac{2\omega\lambda}{g}a_j+\left[\frac{\lambda^2-\omega^2}{g}+F_{ij}^{(1)}(A_i)\right]b_j=0 \quad (43)$$

where

$$F_{ij}^{(3)}(A_i)=\frac{\displaystyle\int_0^L\frac{\left(\dfrac{dJ_0(\alpha_j\sqrt{S/L})}{dS}\right)^2}{\left(\sqrt{1-A_i^2\left[\dfrac{dJ_0(\alpha_i\sqrt{S/L})}{dS}\right]^2}\right)^3}dS}{\displaystyle\int_0^L J_0^2(\alpha_j\sqrt{S/L})\,dS}$$

i.e.,

$$F_{ij}^{(3)}=\frac{\omega_j^2}{g}\left\{\int_0^1\frac{zJ_1^2(\alpha_jz)\,dz}{\left(\sqrt{1-\dfrac{A_i^2\alpha_i^2}{4L^2}\dfrac{J_1^2(\alpha_iz)}{z^2}}\right)^3}\right\}\frac{2}{J_1^2(\alpha_j)} \quad (44)$$

$$\geqslant\frac{\omega_j^2}{g}$$

$$F_{ij}^{(1)}=\frac{\omega_j^2}{g}\left\{\int_0^1\frac{zJ_1^2(\alpha_jz)\,dz}{\sqrt{1-\dfrac{A_i^2\alpha_i^2}{4L^2}\dfrac{J_1^2(\alpha_iz)}{z^2}}}\right\}\frac{2}{J_1^2(\alpha_j)} \quad (45)$$

$$\geqslant\omega_j^2/g$$

$$G_{ij}=A_i^2\left\{\int_0^L\frac{\dfrac{1}{S}\left(\dfrac{\alpha_i\alpha_j}{4L}\right)^2J_1(\alpha_i\sqrt{S/L})\,J_1(\alpha_j\sqrt{S/L})}{\sqrt{1-\dfrac{A_i^2\alpha_i^2}{4L}\dfrac{J_1^2(\alpha_i\sqrt{S/L})}{S}}}\times\left[\int_0^S d\beta\int_\beta^L\frac{\dfrac{1}{\gamma}J_1(\alpha_i\sqrt{\gamma/L})\,J_1(\alpha_j\sqrt{\gamma/L})}{\sqrt{1-\dfrac{A_i^2\alpha_i^2}{4L}\dfrac{J_1^2(\alpha_i\sqrt{\gamma/L})}{\gamma}}}d\gamma\right]dS\right\}\Bigg/\int_0^L J_0^2(\alpha_i\sqrt{S/L})\,dS$$

Integrating by parts, G_{ij} may be put into the form

$$G_{ij}=A_i^2\frac{\alpha_i^2\alpha_j^2}{4L^2}\left\{\int_0^1\left[\int_z^1\frac{1}{\beta}\frac{J_1(\alpha_i\beta)J_1(\alpha_j\beta)\,d\beta}{\sqrt{1-\dfrac{A_i^2\alpha_i^2}{4L^2}J_1^2(\alpha_i\beta)/\beta^2}}\right]^2dz\right\}\frac{1}{J_1^2(\alpha_j)}$$

$$\geqslant 0 \quad (46)$$

Frequency Equation

From (42) and (43) either $a_j = b_j = 0$ or

$$\left[\frac{\lambda^2 - \omega^2}{g} + F_{ij}^{(3)} + \frac{\lambda^2}{g} G_{ij}\right]\left[\frac{\lambda^2 - \omega^2}{g} + F_{ij}^{(1)}\right] + \frac{4\omega^2\lambda^2}{g^2} = 0 \quad (47)$$

Expanding:

$$(1 + G_{ij})\left(\frac{\lambda^2}{g}\right)^2 + \left(\frac{\lambda^2}{g}\right)\left\{(1 + G_{ij}) F_{ij}^{(1)} + F_{ij}^{(3)} + \frac{\omega^2}{g}(2 - G_{ij})\right\} + \left(F_{ij}^{(3)} - \frac{\omega^2}{g}\right)\left(F_{ij}^{(1)} - \frac{\omega^2}{g}\right) = 0 \quad (48)$$

For stability the following conditions must be satisfied:

$$\left.\begin{aligned}
&1)\ (1 + G_{ij}) F_{ij}^{(1)} + F_{ij}^{(3)} + \frac{\omega^2}{g}(2 - G_{ij}) > 0 \\
&2)\ \left(F_{ij}^{(1)} - \frac{\omega^2}{g}\right)\left(F_{ij}^{(3)} - \frac{\omega^2}{g}\right) > 0 \\
&3)\ \left[(1 + G_{ij}) F_{ijk}^{(1)} + F_{ij}^{(2)} + \frac{\omega^2}{g}(2 - G_{ij})\right]^2 - \\
&\qquad 4(1 + G_{ij})\left(F_{ij}^{(1)} - \frac{\omega^2}{g}\right)\left(F_{ij}^{(3)} - \frac{\omega^2}{g}\right) > 0
\end{aligned}\right\} \quad (49)$$

Expanding (3), the condition becomes

$$\left[(1 + G_{ij})\left(F_{ij}^{(1)} - \frac{\omega^2}{g}\right) - \left(F_{ij}^{(3)} - \frac{\omega^2}{g}\right)\right]^2 + \frac{8\omega^2}{g}\left[(1 + G_{ij})\left(F_{ij}^{(1)} - \frac{\omega^2}{g}\right) + \left(F_{ij}^{(3)} - \frac{\omega^2}{g}\right) + \frac{2\omega^2}{g}\right] > 0 \quad (50)$$

If ω is restricted to the range $\omega_{i-1} \leqslant \omega \leqslant \omega_{i+1}$ and $\epsilon/L \ll 1$, conditions (1) and (3) are automatically satisfied. The stability is then governed entirely by condition (2).

$\therefore \left(F_{ij}^{(1)} - \frac{\omega^2}{g}\right)\left(F_{ij}^{(3)} - \frac{\omega^2}{g}\right) > 0$ is the condition for stability for $\epsilon/L \ll 1$ and ω in the range $\omega_{i-1} \leqslant \omega \leqslant \omega_{i+1}$.

In particular when $j = i$ it will be observed from (27) and (45) that:

$$F_{ii}^{(1)} = \frac{F_i(A_i)}{A_i} \quad (51)$$

From (28a) and (44) it will be observed that:

$$F_{ii}^{(3)} = \frac{\partial}{\partial A_i}(F_i(A_i)) \quad (52)$$

Using (51) and (52) it can easily be shown that condition (2) for $j = i$ is simply the requirement that solutions lie outside the region enclosed by the locus of vertical tangency. Application of this criterion to (25) shows

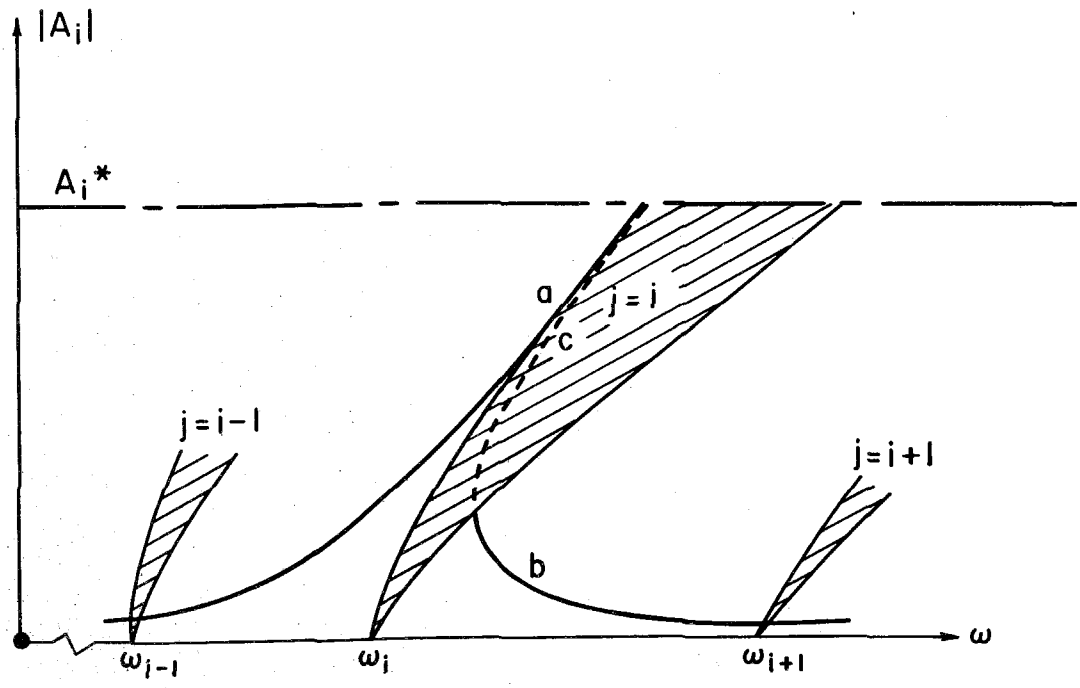

FIGURE 4. STABILITY OF ith MODE.

that solutions a and b are stable while solution c is unstable.

Fig. 4 shows a plot of the unstable regions for the i th mode, for ϵ small. It will be observed that the approximate solution $x = A_i J_0(\alpha_i \sqrt{S/L})$ becomes unstable at the end points of the interval $\omega_{i-1} < \omega < \omega_{i+1}$. Within this interval there are two stable solutions, a large amplitude solution a and a small amplitude solution b, and an unstable solution c.

Experimental Observations

Experiments with a chain rotated by a variable speed motor show that:

1) For a given mode, there are two stable solutions for rotational speeds above the linear critical speed for that mode.
2) The small amplitude solutions become unstable in the vicinity of the linear critical speeds, giving rise to mode conversion.
3) For $i > 1$, the large amplitude solutions become unstable for high enough rotational speeds.
4) In the lower modes particularly, amplitudes are observed which are in excess of A_i^*, the maximum for the approximate theory.
5) Subharmonic modes of rotation are observed in which the fixed end of the chain performs an integral number of rotations in the time that the free end takes to make one complete revolution.

Observations 1) and 2) are in agreement with the approximate theory of this paper and cannot be explained by the linearized theory. Observations 3), 4) and 5) cannot be explained by the approximate theory and will require a much refined analysis.

Bibliography

1. Heavy Rotating String—A Nonlinear Eigenvalue Problem, by Ignace I. Kolodner, Communications on Pure & Applied Mathematics, Vol. VIII, pp. 395-408 (1955).