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Response of Van Der Pol's Oscillator to Random Excitation

This paper considers the response of Van der Pol's oscillator to random excitation. It is shown that the output of the oscillator consists of a periodic term, plus a narrow band noise term centered at the natural frequency of the oscillator. The root-mean-square amplitude of this noise term is shown to be proportional to the square root of the spectral density of the excitation, and inversely proportional to the amplitude of self-oscillation.

In recent years considerable interest has been aroused in the response of nonlinear systems to various types of excitation. Most of the recent work has been in the field of servomechanisms and feedback systems, where the equivalent linearization technique of nonlinear mechanics has been used in various guises and under various names. With the exception of Garstens' paper,[1] the analytical studies have been on the response of systems to either periodic excitation alone or to random excitation alone. In this paper an analysis will be presented on a self-excited oscillator acted upon by an external random excitation. Because of the self-excitation, the solution will contain periodic terms and, owing to the random excitation, the solution also will contain random terms. The results of this study show that the noise component of the solution decreases with increasing amplitude of self-excited oscillation; however, the band width of the noise increases with increasing amplitude of self-oscillation.

To verify the results of the analysis, the problem was simulated on an analog computer. The results of this study are in good agreement with the theory.

Equation of Oscillator With Noise as Driving Force

The equation of a Van der Pol oscillator acted upon by noise is

$$\frac{d^2V}{dt^2} - (\alpha - r)\frac{dV}{dt} - \omega_0^2 V + \gamma \frac{d}{dt}(V^3) = N(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_N(\omega) e^{i\omega t} d\omega \quad (1)$$

where $N(t)$ is a small Gaussian noise force, assumed to have a white power spectrum of density $4D$/cycle. Let

$$V(t) = V_p(t) + V_N(t)$$

where $V_p(t) = A \sin \omega_0 t$ is the periodic part of the solution and $V_N(t)$ is the randomly varying part of the solution.

If equation (1) was linear, the noise component $V_N(t)$ would be Gaussian since $N(t)$ is Gaussian; further, if the damping is small the noise component $V_N(t)$ is confined to a narrow band of frequencies close to ω_0. In a nonlinear system the noise component in general will not be Gaussian; however, if the nonlinearity is small the assumption of a Gaussian distribution is a fairly good one.

If the assumption is made that the noise component V_N is narrow band Gaussian noise, then the power spectrum $W(\omega)$ of V^3 can be calculated easily by the method of Rice[2] and Bennett.[3]

Power Spectrum of $[V_p(t) + V_N(t)]^3$

The power spectrum $W(\omega)$ of a function $V_1(t)$ representing a random voltage or force given by

$$W(\omega) = 4\int_0^{\infty} \Psi(\tau)\cos(\omega\tau)d\tau \quad (2)$$

where the auto correlation function $\Psi(\tau)$ is defined as the time average of $V_1(t)V_1(t+\tau)$. It is convenient to express $V_1(\tau)$ as a contour integral of an exponential since the averaging required to obtain $\Psi(\tau)$ is relatively easily done. Thus, following the method of Rice,[2] set

$$V_1(t) = V^3 = (V_p + V_N)^3 = \frac{3!}{2\pi}\int_c \frac{e^{iVu}}{u^4} du \quad (3)$$

where the path of integration is a positive loop around $u = 0$. Thus

$$\begin{aligned}\Psi(\tau) &= \langle V_1(t)V_1(t+\tau)\rangle_{av} \\ &= \lim_{T\to\infty} \frac{1}{T}\int_0^T V_1(t)V_1(t+\tau)dt \\ &= \frac{1}{4\pi^2}\int_c \frac{6}{u^4} du \int_c \frac{6}{v^4} dv \lim \frac{1}{T} \\ &\quad \left\{\int_0^T \exp[iuV(t) + ivV(t+\tau)]dt\right\} \quad (4)\end{aligned}$$

In the present case, that of a sine wave plus noise, equation (4) can be reduced to an expression of the form (see Rice[2] section 4.9):

$$\Psi = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{k!}\psi_\tau^k h_{nk}^2 \epsilon_n \cos n\omega_0\tau \quad (5)$$

where

$$h_{nk} = \frac{i^{n+k}}{2\pi}\int_c \frac{6}{u^4} u^k J_n(Au)\exp\left(\frac{-\psi_0}{2}u^2\right)du \quad (6)$$

$$\epsilon_0 = 1, \quad \epsilon_n = 2 \quad \text{for} \quad n \geqslant 1$$

J_n is a Bessel function of the first kind order n, and $\psi_\tau = \psi(\tau)$

[1] M. A. Garstens, "Noise in Non-Linear Oscillators," *Journal of Applied Physics*, vol. 28, 1957, pp. 353–356.

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Note: Statements and opinions advanced in papers are to be understood as individual expressions of their authors and not those of the Society. Manuscript received by ASME Applied Mechanics Division, November 18, 1957. Paper No. 59—APMW-5.

[2] S. O. Rice, *Bell System Technical Journal*, vol. 23, 1944, p. 283, and vol. 24, 1945, p. 46; also "Noise and Stochastic Processes," edited by N. Wax, Dover Publications, Inc., New York, N. Y., 1954, pp. 133–293, sections 4.8 and 4.9.

[3] W. R. Bennett, "Response of a Linear Rectifier to Signal and Noise," *Journal of The Acoustical Society of America*, vol. 15, 1944, pp. 164–172

is the correlation function of $V_N(t)$. In order to determine the spectral distribution of V^3, h_{nk} must be evaluated: $\Psi(\tau)$ can then be calculated and finally by means of equation (2), $W(\omega)$ can be evaluated. The reduction of the auto correlation function in equation (4) to the form given by Rice in equation (5) allows a selection in the modulation products of the solution, of those components whose frequencies fall near ω_0 and are largest in magnitude.

Examination of equation (6) shows that:

$$h_{nk} = 0 \text{ if } n + k \geqslant 4$$

A table of h_{nk} for $n + k \leq 3$ follows:

$$\left.\begin{array}{lll} h_{00} = 0 & h_{10} = \dfrac{A}{2}\left(3\psi_0 + \dfrac{3A^2}{4}\right); & h_{21} = \dfrac{3}{2}A^2 \\ h_{01} = 3\left(\dfrac{A^2}{2} + \psi_0\right) & h_{11} = 0 & h_{30} = \dfrac{A^3}{8} \\ h_{02} = 0 & h_{12} = 3A & \\ h_{03} = 6 & & \end{array}\right\} \quad (7)$$

Substituting (7) into (5) one finds that:

(i) h_{21} and h_{30} have no contribution at ω_0.

(ii) If $A^2 > A > \psi_0$ the largest contribution to Ψ comes from h_{01} and h_{10}.

Substituting equation (5) into equation (2)

$$W(\omega) \simeq 4h_{01}^2 \int_0^\infty \psi_\tau \cos \omega\tau d\tau + 2h_{10}^2 \int_0^\infty \cos \omega\tau \cos \omega_0\tau d\tau \quad (8)$$

Therefore the power spectrum of V_1^3 is given by

$$W(\omega) \simeq h_{01}^2 W_n(\omega) + 2h_{10}^2\delta(\omega_0) \quad (9)$$

where $Wn(\omega)$ is the power spectrum of $V_N(\tau)$. Substituting into (9) the values of h_{01} and h_{10} given in (7)

$$W(\omega) = (3\psi_0 + {}^3/_2A^2)^2 W_n(\omega) + \frac{A^2}{2}(3\psi_0 + {}^3/_4A^2)^2\delta(\omega_0) \quad (10)$$

It is to be noted that

$$\left.\begin{array}{l} Wn(\omega) \text{ is the power spectrum of } V_N(t) \\ \dfrac{A^2}{2}\delta(\omega_0) \text{ is the power spectrum of } A \sin \omega_0 t \end{array}\right\} \quad (11)$$

Thus the power spectrum of V^3 as given by equation (10) is the same as would have been obtained by passing the noise component through a linear device with a gain of $(3\psi_0 + {}^3/_2A^2)$ and passing the periodic part of the solution through a linear device with a gain of $(3\psi_0 + {}^3/_4A^2)$. This result is in complete agreement with that obtained by a straightforward application of Van der Pol's method neglecting higher order cross-modulation terms.

Equivalent Linear Form of Equation (1)

Equation (1) may now be replaced by an equivalent linear equation having the same first-order power spectrum. Equation (1) becomes

$$\left[\frac{d^2V_p}{dt^2} - \left\{\alpha - r - \gamma\,(3\psi_0 + 3A^2/4)\right\}\frac{dV_p}{dt} + \omega_0^2V_p\right] + \left[\frac{d^2V_n}{dt^2} - \{\alpha - r - \gamma(3\psi_0 + {}^3/_2A^2)\}\frac{dV_n}{dt} + \omega_0^2V_n\right] = N(t) \quad (12)$$

The terms in the first bracket are all periodic, none of the other terms are periodic, hence the terms in the first bracket must be zero. Equation (12) therefore splits into two separate linear equations:

$$\frac{d^2V_p}{dt^2} - \left\{\alpha - r - \gamma\,(3\psi_0 + 3A^2/4)\right\}\frac{dV_p}{dt} + \omega_0^2V_p = 0 \quad (13)$$

and

$$\frac{d^2V_N}{dt^2} - \{\alpha - r - \gamma(3\psi_0 + {}^3/_2A^2)\}\frac{dV_N}{dt} + \omega_0^2V_N = N(t) \quad (14)$$

Solution of Equations (13) and (14)

Consider equation (13); two stable solutions are possible:
(a) $V_p = A \sin \omega_0 t \quad A \neq 0$. In this case

$$\alpha - r - \gamma\,(3\psi_0 + 3A^2/4) = 0.$$

Hence

$$A^2 = \frac{\alpha - r}{{}^3/_4\gamma} - 4\psi_0 \quad (15)$$

But $(\alpha - r)/{}^3/_4 = A_0^2$, where A_0 is the amplitude of oscillation in the absence of noise

$$\therefore \quad A^2 = A_0^2 - 4\psi_0 \quad (16)$$

Equation (16) requires that $A_0^2 > 4\psi_0$; this condition will naturally be satisfied if $N(t)$ is kept sufficiently small.

(b) Another possibility is that $V_p = 0$. If this is to be a stable solution

$$\alpha - r - \gamma(3\psi_0) < 0 \quad (17)$$

i.e.,

$$\psi_0 > {}^1/_4\,\frac{\alpha - r}{{}^3/_4} = {}^1/_4A_0^2 \quad (18)$$

In deriving equation (9) it was assumed that $A^2 > \psi_0$; since this condition is violated in case (b) the analysis breaks down. For this reason, and also for the reason that the effect of small noise is of more practical interest, further analysis will be restricted to case (a).

Using the results of (a), substitute equation (15) into equation (12):

$$\frac{d^2V_N}{dt^2} - \left\{\alpha - r - \gamma\left(3\psi_0 + 2\,\frac{(\alpha - r)}{\gamma} - 6\psi_0\right)\right\}\frac{dV_N}{dt} + \omega_0^2V_N = N(t) \quad (19)$$

i.e.,

$$\frac{d^2V_N}{dt^2} + \{\alpha - r - 3\gamma\psi_0\}\frac{dV_N}{dt} + \omega_0^2V_N = N(t) \quad (20)$$

Let

$$\beta = \alpha - r - 3\gamma\psi_0 \quad (21)$$

by equation (15) $\beta \geqslant 0$. Equation (20) is therefore identical with the Langevin equation for the Brownian motion of a simple harmonic oscillator.

Since $N(t)$ is Gaussian, it is clear that $V_N(t)$ will also be a Gaussian random process with a spectrum:

$$W_N(\omega) = \frac{4D}{|\omega_0^2 - \omega^2 + i\omega\beta|^2} \tag{22}$$

from which it follows that

$$\psi(\tau) = \langle V_N(t)V_N(t+\tau)\rangle av = \frac{2D}{\pi}\int_0^\infty \frac{\cos\omega\tau}{(\omega^2-\omega_0^2)^2+\beta^2\omega^2}d\omega$$

$$\therefore \psi(\tau) = \frac{D}{\beta\omega_0^2}\exp\left(-\frac{\beta\tau}{2}\right)\left(\cos\omega_1\tau + \frac{\beta}{2\omega_1}\sin\omega_1\tau\right) \tag{23}$$

where $\omega_1^2 = \omega_0^2 - \beta^2/4$.

In particular, when $\tau = 0$

$$\psi_0 = \sigma^2 = \langle V_N(t)^2\rangle_{av} = \frac{D}{\beta\omega_0^2} \tag{24}$$

Thus, substituting for β, a quadratic equation in ψ_0 is obtained; viz.,

$$3\gamma\psi_0^2 - (\alpha - r)\psi_0 + \frac{D}{\omega_0^2} = 0 \tag{25}$$

$$\therefore \psi_0 = {}^1/_8A_0^2 \pm \left[({}^1/_8A_0^2)^2 - \frac{D}{3\gamma\omega_0^2}\right]^{1/2} \tag{26}$$

where

$$A_0 = \left(\frac{\alpha - r}{{}^3/_4\gamma}\right)^{1/2}$$

is the amplitude of self-oscillation in the absence of noise.

The positive root in equation (26) can be eliminated on physical grounds, since ψ_0 must tend to zero as D tends to zero.

$$\therefore \psi_0 = {}^1/_8A_0^2 - \left[({}^1/_8A_0^2)^2 - \frac{D}{3\gamma\omega_0^2}\right]^{1/2} \tag{27}$$

In particular if

$$\frac{D}{3\gamma\omega_0^2} \ll ({}^1/_8A_0^2)^2$$

equation (27) reduces to

$$\psi_0 \simeq \frac{4}{3\gamma}\frac{D}{A_0^2\omega_0^2} \tag{28}$$

Hence

$$\sqrt{\psi_0} = \sigma = \frac{1}{A_0}\left(\frac{4D}{3\gamma\omega_0^2}\right)^{1/2} \tag{29}$$

Thus for small noise, the rms amplitude of the noise component of $V(t)$ is proportional to the square root of the spectral density and inversely proportional to the amplitude of self-oscillator in the absence of noise. The effect of the noise on the amplitude of self-oscillator can be seen by substituting (27) into (16). Thus

$$A^2 = {}^1/_2A_0^2 + {}^1/_2A_0^2\left\{\left(1 - \frac{64D}{3\gamma A_0^4\omega_0^2}\right)^{1/2}\right\} \tag{30}$$

For a given A_0, A will decrease as the spectral density D of the noise is increased.

Bandwidth of $V_N(t)$

The power spectrum of the noise component $V_N(t)$ of the solution is given by (22)

$$W_n(\omega) = \frac{4D}{(\omega_0^2 - \omega^2) + (\omega\beta)^2}$$

Thus the bandwidth to the half-power points is given by

$$\Delta\omega = \beta \tag{31}$$

But by (21) $\beta = \alpha - r - 3\gamma\psi_0$, and if D is small

$$\psi_0 \simeq \frac{4}{3\gamma}\frac{D}{A_0^2\omega_0^2}$$

by (28).

Thus

$$\beta = {}^3/_4\gamma\frac{\alpha - r}{{}^3/_4\gamma} - \frac{4D}{A_0^2\omega_0^2}$$

$$\therefore \beta = {}^3/_4\gamma A_0^2 - \frac{4D}{A_0^2\omega_0^2} \tag{32}$$

Thus, if A_0 increases, β increases, and hence the bandwidth $\Delta\omega$ also increases.

Summary of Results of Analysis

The results of the analysis may be summarized as follows: If a Van der Pol oscillator is excited by a small Gaussian noise force, then:

(i) The amplitude of self-oscillation decreases with increasing $N(t)$.

(ii) The largest part of the noise component of the signal is narrow band Gaussian, centered at the frequency of self-oscillation, with a bandwidth increasing with increasing amplitude of self-oscillation.

(iii) The mean squared amplitude of the noise component is proportional to the input noise power, but inversely proportional to the square of the amplitude of self-oscillation.

The result (iii) is in disagreement with the results obtained by Garstens[1] who finds that the noise component increases rapidly with the amplitude of self-oscillation. To resolve this problem, equation (1) was simulated on an electronic differential analyzer, and a study was made of the effect of the amplitude of self-oscillation on the magnitude of the noise component of the output.

Summary of Results of Computer Studies

The following results were obtained from a computer study of equation (1) for small $N(t)$:

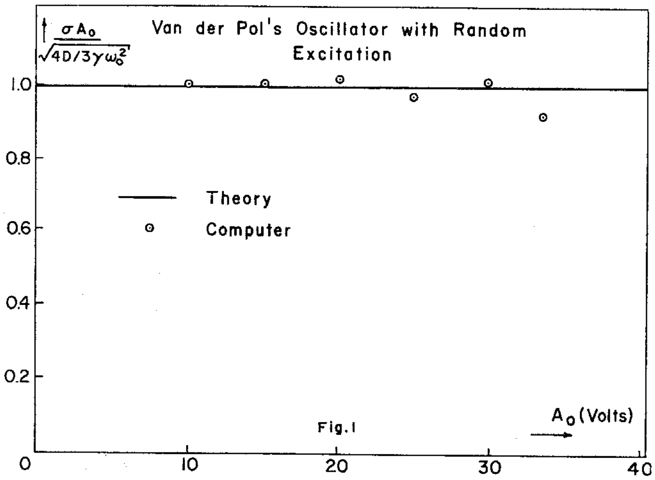

Fig. 1

(*a*) The amplitude of self-oscillation decreases only slightly with increasing noise power.

(*b*) The noise component of the output consists of narrow band noise centered at ω_0, the frequency of self-oscillation, together with a much smaller noise component at $3\omega_0$. The bandwidth of the noise increased as the amplitude of self-oscillation was increased. (In these experiments, A_0 was varied by varying $\alpha - r$ keeping γ constant.)

(*c*) The root-mean-squared amplitude of the noise component was found to vary proportionally with the root mean square of the noise input, but inversely with the amplitude of self-oscillation. Fig. 1 shows the results of one such test, in which the input was maintained constant while the amplitude of self-oscillation was varied. The results show that $A_0\sigma$ is constant, within ± 5 per cent, thus verifying the results of the analysis.

Conclusions

The main findings of the theory were confirmed by the computer studies, thereby proving Garstens' conclusions incorrect. As Garstens points out, however, actual oscillators do exhibit increasing noise with increasing amplitude of self-oscillation, from which one is forced to conclude that the model of the oscillator described by equation (1), in which the noise $N(t)$ remains constant, is incorrect. A more realistic model would require that the noise increase with increasing self-oscillation.

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