# ON THE PROBLEM OF THE ENTROPY OF THE UNIVERSE AS A WHOLE

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Abstract

The well-known problem of the entropy of the universe as a whole arises from the difficulties encountered by classical thermodynamics—first in failing to account for the presumed fact that the entropy of the universe has always been increasing at an enormous rate and nevertheless has not yet reached its maximum value—and second in failing to allow an emotionally satisfactory feeling towards our universe whose ultimate fate would be the stagnation of "heat-death." The purpose of the present article is to examine this problem from the point of view of the extension of thermodynamics to general relativity which has previously been made by the author.

A number of earlier contributions to the solution of the problem, which have been made from the standpoint of classical thermodynamics or statistical mechanics, are first briefly described in order to emphasize the very different character of the contribution to the problem made in the present article. It is then pointed out that the problem of the entropy of the universe arises in the classical thermodynamics because of the presumption that thermodynamic processes cannot take place both reversibly and at a finite rate, and that the general nature of the contribution to the problem offered by relativistic thermodynamics consists in showing the possibility of thermodynamic changes which could take place at a finite rate and at the same time reversibly without increase in entropy.

The principles of relativistic thermodynamics are then reviewed, and this difference between the classical and relativistic thermodynamics is shown by considering the possibilities of carrying out reversible changes at a finite rate in the properties of a thermodynamic fluid. In the classical thermodynamics it is found that no change in the thermodynamic properties could be allowed to take place at a finite rate, the entropy density of the fluid necessarily remaining constant in accordance with the equation

$$\frac{d\phi_0}{dt} = 0.$$

On the other hand, in relativistic thermodynamics it is found possible to allow changes to take place at a finite rate in the proper volume of the fluid, due to changes in the gravitational potentials $g_{\mu\nu}$, and still maintain reversibility provided the changes satisfy the relation

$$\frac{\partial}{\partial x_4}\left(\phi_0\sqrt{-g}\,\frac{dx_4}{ds}\right) = 0.$$

To exhibit the nature of the reversible changes at a finite rate thus permitted in relativistic thermodynamics, consideration is given to the highly idealized model of a non-static universe filled with black-body radiation as a thermodynamic fluid, and it is shown that the radius, total proper volume, and entropy density of such a universe could be changing at a finite rate and yet reversibly without increase in entropy. Furthermore, in the case of an expanding model of the kind considered, it is shown that an ordinary observer, who marks out with rigid meter sticks a small region of this universe in his immediate vicinity for study, would find the energy density, energy content, and the temperature of this region decreasing with the time, and would find the number of quanta leaving the region per second greater than the number entering, and the average frequency of the quanta that leave greater than that of those that return. These phenomena would be interpreted by the observer, from a classical point of view, as due to radiation from his neighborhood into the colder surroundings of space and hence as leading to an increase in entropy, in spite of the fact that all the processes taking place in such a model would actually be reversible from the point of view of the relativistic thermodynamics which should be applied to such a problem.

In conclusion remarks are made concerning the disparity between the above model, which was chosen for purposes of illustration because of its mathematical simplicity, and models which would be more suitable to serve as representations of the actual universe. And some indication is given of the further developments that should be undertaken.

## Part I. Introduction

### §1. Purpose of present article.

IN A number of previous articles I have endeavored to present the principles for an extension of thermodynamics to general relativity[1] and to consider some of the applications of the new system of relativistic thermodynamics based on these principles.[2] The purpose of the present article is to examine the bearings of relativistic thermodynamics on the well-known problem of the entropy of the universe as a whole. It will be found that this extended thermodynamics provides new possibilities for thermodynamic processes to take place at a finite rate without increase in entropy. And it will be shown that the recognition of these new possibilities not only appears to be essential for a true understanding of the problem of the entropy of the universe, but may even provide to a greater or lesser extent the basis for its solution.

### §2. The nature of the problem of the entropy of the universe.

In accordance with the views of the classical thermodynamics all thermodynamic processes, actually taking place in the universe at a finite rate, were regarded as accompanied by an increase in entropy. Among these processes appeared a wide variety of immediately appreciated terrestial occurrences of a meteorological, biological or technological nature in which the increase of entropy depended for the most part on the degradation of energy originally received as radiation from the sun,—in addition, various tidal actions in which the increase of entropy resulted from the degradation into heat of mechanical energy of astronomical motions;—and quantitatively most important of all, the continuous flow of radiation from the stars with a great increase in entropy due to the presumable drop in temperature in passing from the hot interior of the stars to the cold depths of intergalactic space.

In general the view was held that entropy was everywhere increasing at an enormous rate and that this would continue until the entropy of the universe

[1] Tolman, Proc. Nat. Acad. **14,** 268 (1928); ibid. **14,** 701 (1928); Phys. Rev. **35,** 875 (1930); ibid. **35,** 896 (1930).

[2] Tolman, Proc. Nat. Acad. **14,** 348 (1928); ibid. **14,** 353 (1928); ibid. **17,** 153 (1931); Phys. Rev. **35,** 904 (1930); Tolman and Ehrenfest, Phys. Rev. **36,** 1791 (1930).

had reached its maximum,—the sun and stars cold, all of creation dead and unchanging.

Such a view, however, carries with it two difficulties. The first difficulty has genuine intellectual validity and can be expressed by the question: Why has not the entropy of the universe already reached its maximum value in the infinite past time which has presumably been available? The second difficulty has perhaps only emotional validity and can be expressed by the question: What significance can we ascribe to a universe whose ultimate fate is merely the "heat-death" of maximum entropy? These are the difficulties which constitute the problem of the entropy of the universe.

### §3. Nature of earlier contributions to the solution of the problem.

Various suggestions have been made with regard to the solution of the problem. It will be profitable to consider some of them briefly in order to emphasize the very different nature of the suggestion which will be made in this article.

*a. Finite time since creation.* The most obvious treatment of the problem is to assume that the universe was indeed created at a finite time in the past with sufficient available energy so that the entropy has not yet reached its maximum value. In the future this maximum would be reached and all significant changes would cease. A modification of the treatment could be made by assuming an infinite past during which the universe was in a quiescent metastable state of large available energy, and a disturbance at a finite time in the past which initiated the process of degradation.

These suggestions depend too greatly on special *ad hoc* assumptions to be scientifically satisfying.

*b. Continuous regeneration.* A second type of suggestion depends on the assumption of the existence of regenerative processes of such a nature as to maintain the universe in an approximately steady condition. Thus Millikan[3] has suggested the four-step cycle: (1) Matter in the stars is transformed into radiation which flows out into intergalactic space; (2) the radiation in intergalactic space is transformed into electrons and protons; (3) the electrons and protons combine to form helium and other elements, giving rise to the production of the observed cosmic rays; (4) the matter thus formed drifts back into the stars, thus completing the cycle. The evidence for step (2) is completely lacking at present; steps (2) and (3) assume the occurrence of processes of synthesis under the theoretically unfavorable conditions of extremely low concentration; and the cycle contradicts the principle of microscopic reversibility.[4] The evidence for step (1), however, is very strong, the evidence for step (3) cannot be dismissed as trivial, and there is no inherent improbability in step (4).

*c. Continuous approach to maximum entropy.* A third type of suggestion

[3] Millikan and Cameron, Proc. Nat. Acad. **14,** 637 (1928).

[4] For a partial historical account of this principle, see Tolman, Proc. Nat. Acad. **11,** 436 (1925). For a discussion of the principle, see for example Tolman, "Statistical Mechanics" Chap. 15, Chemical Catalog Co., New York, 1927.

would be to assume an infinite past for the universe, coupled with ever decreasing values for the entropy of the universe as we examine backwards in the past and a continuous asymptotic approach to the maximum of entropy in the future.[5] If such an assumption were allowable, it would avoid the difficulty of a limited past time for the existence of the universe, but the difficulties of a practical exhaustion of the energy available for human needs within a finite time in the future would not appear to be avoided. The assumption would perhaps not be a possible one for a finite universe of finite energy content, for which we have some evidence.

*d. Fluctuations in entropy in accordance with its statistical-mechanical interpretation.* A fourth of type of contribution to the problem depends on the statistical mechanical interpretation of entropy, as given most clearly by Boltzmann's famous $H$-theorem. In accordance with this theorem it is found that, although there is a great probability for the entropy of a system to increase when it has less than its maximum value, it is not certain that this will take place and fluctuations away from the maximum of entropy will occur. This furnishes the possibility that the universe has existed for an infinite time in the past and that we are now experiencing a return of the universe or of that portion which is within our range of observation towards a condition of maximum entropy after a major fluctuation away from that value.

This important possibility was clearly presented by Boltzmann[6] over thirty years ago. The enormous improbability of a major fluctuation of the kind assumed does not necessarily furnish a valid argument against the explanation, since, as pointed out to me in conversation by Mrs. Ehrenfest-Afanassjewa, the existence of sentient beings to observe the rare phenomenon could presumably only occur at the time of decay of such a fluctuation. From the point of view of human wishes, however, the explanation is not entirely satisfying, since it implies that man himself is a transitory and improbable phenomenon, that our surroundings are now headed with almost complete certainty towards a condition at least close to that of maximum entropy, and that the conditions under which life, as we know it, is possible are almost never present. Nevertheless, these objections have emotional rather than intellectual validity and the part played by the theory of statistical fluctuations in a relatively complete solution of the problem of the entropy of the universe may prove to be no mean one.

### §4. Nature of the present contribution to the solution of the problem.

The present contribution to the problem of the entropy of the universe is based on the system of relativistic thermodynamics which I have developed. The general nature of the contribution depends on an extension given by this relativistic thermodynamics in our ideas as to the kind of processes which can occur at a finite rate without producing any increase in entropy.

As an illustration of this extension in our ideas, it would be impossible

[5] This possibility was suggested to me by some remarks of the late Professor William James which were told me in conversation by Professor Gilbert N. Lewis.

[6] Boltzmann, Wied. Ann. **60**, 392 (1897).

from the point of view of classical thermodynamics to carry out an actual expansion of a thermodynamic fluid reversibly and at a finite rate, since the friction of moving parts and the deficiency between the actual pressure exerted by the fluid and that which could be exerted with an infinitely slow rate of expansion would lead to an increase in entropy. Nevertheless, in relativistic thermodynamics we shall find that the *proper* volume associated with a thermodynamic fluid could increase at a finite rate, owing to a finite rate of change in the gravitational potentials $g_{\mu\nu}$, without involving any increase in entropy.

In further illustration, it appeared impossible in the classical thermodynamics for a flow of heat to take place reversibly and at a finite rate, owing to the increase in entropy connected with the finite temperature drop necessary to maintain the finite rate of flow. Nevertheless, in relativistic thermodynamics, we shall find that the reversible increase in proper volume, mentioned in the paragraph above, would make it possible for heat radiation to be regarded by an ordinary observer as flowing out of a given region of interest at a finite rate, without any increase in the entropy of the system as a whole.

It is evident from these examples, that relativistic thermodynamics increases in an important manner the variety of changes which could be taking place in a universe which is actually in a state of maximum entropy, and makes it necessary to re-examine processes which we have formerly taken as evidence that the entropy of our own universe is actually increasing at an enormous rate.

In Part II, we shall first consider the general bearing of relativistic thermodynamics on changes in thermodynamic condition without increase in entropy. In Part III, we shall then apply relativistic thermodynamics to the very special model of a non-static universe filled solely with radiation. Such a model ignores very characteristic features of the actual universe, but mathematically is relatively simple to handle and will present some features which appear analogous to phenomena in the actual universe. Finally in Part IV, we shall try to give some criticism of the role that the new ideas might play in the general solution of the problem of the entropy of the universe.

## Part II. Relativistic Thermodynamics

### §5. The first and second laws of relativistic thermodynamics.

The extension of thermodynamics to general relativity can be based on two principles which may be regarded as the relativistic generalization of the first and second laws of classical thermodynamics.

In accordance with the first of these principles, any thermodynamic process occurring in the universe must take place in such a way as to agree with the principles of relativistic mechanics as given by the tensor density equation

$$\frac{\partial \mathfrak{T}_{\mu}^{\nu}}{\partial x_{\nu}} - \frac{1}{2} \mathfrak{T}^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_{\mu}} = 0 \tag{1}$$

or the equivalent non-tensorial yet nevertheless covariant equation

$$\frac{\partial(\mathfrak{T}_\mu^\nu + \mathfrak{t}_\mu^\nu)}{\partial x_\nu} = 0 \tag{2}$$

where $\mathfrak{T}_\mu^\nu$ is the tensor density of material energy and momentum, $\mathfrak{t}_\mu^\nu$ the pseudo tensor density of potential energy and the $g_{\alpha\beta}$ the gravitational potentials.

Since these equations reduce to the ordinary energy-momentum principle in flat space-time where the gravitational field is negligible, the analogy of this first principle to the ordinary first law of thermodynamics is evident. In applying the principle to thermodynamic considerations, the system involved will of course be treated from a macroscopic point of view, and this is an advantage since the applicability of these equations to microscopic phenomena would certainly not be in accord with the development of quantum mechanics, which has taken place since their formulation.

The second principle of relativistic thermodynamics may be stated in the form

$$\frac{\partial}{\partial x_\mu}\left(\phi_0\sqrt{-g}\,\frac{dx_\mu}{ds}\right)dx_1dx_2dx_3dx_4 \geqq \frac{dQ_0}{T_0} \tag{3}$$

where $\phi_0$ is the proper density of entropy as measured by a local observer, using Galilean coordinates which are at rest with respect to the mass motion of the thermodynamic fluid at the point of interest, the quantities $dx_\mu/ds$ are the macroscopic "velocities" of the fluid at the point of interest as measured in the coordinate system $x_1$, $x_2$, $x_3$, $x_4$, the quantity $dQ_0$ is the heat flowing through the boundary into the infinitesimal region *and* during the infinitesimal time, denoted by $dx_1dx_2dx_3dx_4$, as measured in proper coordinates, and $T_0$ the temperature of the boundary also measured in proper coordinates.

The justification for the principle lies in the fact that it has been shown to be a natural covariant generalization of the ordinary second law of thermodynamics valid in flat space-time, Eq. (3) being a tensor equation of rank zero which reduces in flat space-time and Galilean coordinates to

$$\left[\frac{\partial\phi}{\partial t} + \frac{\partial}{\partial x}(\phi u) + \frac{\partial}{\partial y}(\phi v) + \frac{\partial}{\partial z}(\phi w)\right]dxdydzdt \geqq \frac{dQ}{T} \tag{4}$$

where $\phi$ is the density of entropy, $u$, $v$ and $w$ are the component velocities of the fluid, $dQ$ is the heat flowing into the region $dxdydz$ in the time $dt$, and $T$ is the temperature, all these quantities now being measured in the particular set of Galilean coordinates $x$, $y$, $z$, $t$ which is being used.

### §6. Application of the relativistic second law to a finite adiabatic system.

Let us now apply our new form of the second law as given by Eq. (3) to a finite thermodynamic system, by taking $x_1$, $x_2$, $x_3$ as being the space-like coordinates and carrying out an integration over the spatial region of interest. If we carry out such an integration, using coordinates such that the limits of integration necessary to include the system fall on the boundary which separates the system from its surroundings, it is evident that the summation of $dQ_0/T_0$ over the interior of the system will cancel out, since any heat entering a given element of volume is abstracted from neighboring elements. Hence dividing Eq. (3) by $dx_4$, writing out the separate terms corresponding to the different values of $\mu$, and performing the integration we obtain, with some rearrangement in order,

$$\begin{aligned}\iiint \frac{\partial}{\partial x_4}\left(\phi_0\sqrt{-g}\,\frac{dx_4}{ds}\right)dx_1dx_2dx_3 \\ \geqq -\iiint\left[\frac{\partial}{\partial x_1}\left(\phi_0\sqrt{-g}\,\frac{dx_1}{ds}\right)+\frac{\partial}{\partial x_2}\left(\phi_0\sqrt{-g}\,\frac{dx_2}{ds}\right)\right. \\ \left.+\frac{\partial}{\partial x_3}\left(\phi_0\sqrt{-g}\,\frac{dx_3}{ds}\right)\right]dx_1dx_2dx_3 + \sum\left(\frac{1}{T_0}\,\frac{dQ_0}{dx_4}\right)_{\text{BOUNDARY}}\end{aligned} \tag{5}$$

The last term on the right hand side of the above inequality is the total value of the quantity $(1/T_0)(dQ_0/dx_4)$ taken over the boundary which separates the system from its surroundings, and by performing the indicated integrations the other terms on the right hand side of the expression can also be seen to depend solely on conditions at the boundary. We obtain

$$\begin{aligned}\iiint \frac{\partial}{\partial x_4}\left(\phi_0\sqrt{-g}\,\frac{dx_4}{ds}\right)dx_1dx_2dx_3 \\ \geqq -\iint\left|\phi_0\sqrt{-g}\,\frac{dx_1}{ds}\right|_{x_1}^{x_1'}dx_2dx_3 - \iint\left|\phi_0\sqrt{-g}\,\frac{dx_2}{ds}\right|_{x_2}^{x_2'}dx_1dx_3 \\ -\iint\left|\phi_0\sqrt{-g}\,\frac{dx_3}{ds}\right|_{x_3}^{x_3'}dx_1\,dx_2 + \sum\left(\frac{1}{T_0}\,\frac{dQ_0}{dx_4}\right)_{\text{BOUNDARY}}\end{aligned} \tag{6}$$

where the limits of integration at the boundary are denoted by $x_1$, $x_1'$ etc.

This expression (6) may be regarded as a general statement of the relativistic second law of thermodynamics as applied to finite systems. Defining the entropy of the system as

$$S = \iiint\left(\phi_0\sqrt{-g}\,\frac{dx_4}{ds}\right)dx_1dx_2dx_5 \tag{7}$$

it gives the relation which must hold between the rate at which the entropy of a finite system is changing with the time $x_4$ and those conditions at the boundary which determine the flux of matter and the flow of heat between the system and its surroundings.

For an adiabatic system with no flux of matter or flow of heat between the system and its surroundings we shall have the quantities $dx_1/ds$, $dx_2/ds$, $dx_3/ds$, and $dQ_0/dx_4$ equal to zero at the boundary and the expression will then reduce to

$$\frac{dS}{dx_4} = \iiint \frac{\partial}{\partial x_4}\left(\phi_0\sqrt{-g}\,\frac{\partial x_4}{ds}\right) dx_1 dx_2 dx_3 \geqq 0. \tag{8}$$

In accordance with this expression the entropy of an adiabatic system cannot decrease with the time but can only increase or remain constant. As in the classical thermodynamics, adiabatic processes in which the entropy increases with the time may be called irreversible, since neglecting improbable fluctuations the system after such a process could not of itself return to the original state of lower entropy; while processes in which the entropy remains constant may be called reversible.

### §7. Increased possibility for reversible processes in relativistic thermodynamics.

With the help of the foregoing considerations we may now compare the conditions which would be imposed by classical and by relativistic thermodynamics on the occurrence of reversible processes. In the present section we shall show that the new thermodynamics offers the possibility for a kind of thermodynamic change which was not contemplated in the classical thermodynamics, and which might take place at a finite rate without increase in entropy, in contrast to the conclusion of classical thermodynamics that reversible thermodynamic processes could not take place at a finite rate. And in later sections we shall show by a simple specific example that such reversible processes taking place at a finite rate might actually be realized, and play a possible part in cosmological happenings.

*a. Classical treatment of entropy changes in a thermodynamic fluid.* Let us first illustrate the kinds of classical considerations which have formerly lead to the conclusion that thermodynamic processes could not take place both reversibly and at a finite rate.[7] To do this we may consider the conditions which would be imposed by classical thermodynamics on reversible changes in the condition of a thermodynamic fluid.

In the classical thermodynamics we could evidently write for the entropy of a finite portion of thermodynamic fluid enclosed in a suitable container the expression

$$S = \iiint \phi \, dxdydz \tag{9}$$

where $\phi$ is the density of entropy as measured in the particular set of (Galilean) coordinates $x$, $y$, $z$, $t$ which the observer uses, and the integration is to be taken over the whole volume of the container.

If now we consider the possible reversible changes which could take place in this thermodynamic fluid, it is evident in the first place that we could permit no relative motion between different portions of the fluid, since the decay of this motion would lead to an increase in entropy, and we may hence

[7] Of course the classical thermodynamics permitted ideal *mechanical* processes to take place at a finite rate without increase in entropy, but the distinction between mechanical and thermodynamic processes was clear enough so that this did not prove to be a source of confusion.

use a set of coordinates in which the thermodynamic fluid as a whole would be at rest and rewrite our expression for the entropy in the form

$$S = \iiint \phi_0 \, dxdydz \tag{10}$$

where $\phi_0$ is the proper density of entropy. In the second place, it is evident that we could only permit adiabatic changes, since if we allowed heat flow in our system at a finite rate we should have increases in entropy arising from the finite temperature gradient which would be necessary to maintain this flow. Hence in accordance with the classical thermodynamics the condition for our contemplated process to be reversible would be that of constant entropy as given by the equation

$$\frac{d}{dt}\iiint \phi_0 \, dxdydz = 0. \tag{11}$$

In the third place, it is evident that our process could not involve a change in volume at a finite rate, for example by the withdrawal of a piston, since this would involve mass flow of portions of the fluid which would lead to an increase in entropy that could be calculated from the difference between the pressure actually exerted by the fluid on the moving piston and that which would be exerted at an infinitesimally slow rate of expansion. Hence the condition given by Eq. (11) for our contemplated reversible process might now be rewritten with the differentiation inside the integral sign in the form

$$\iiint \frac{d\phi_0}{dt} \, dxdydz = 0. \tag{12}$$

This final condition, moreover, could evidently be satisfied in our case only by taking

$$\frac{d\phi_0}{dt} = 0 \tag{13}$$

at all points of the fluid, since for a stationary fluid with no flow of heat there would be no possibility at any point for negative values of the quantity $d\phi_0/dt$.

This, however, completes the considerations necessary for the classical conclusion that there could be no thermodynamic change at all in our fluid which takes place both reversibly and at a finite rate. Indeed we see that no changes could take place in the system as a whole through interaction with its surroundings, since we have found that its volume could not be allowed to change at a finite rate and heat could not be allowed to flow through its boundary at a finite rate, and no changes could take place in the interior condition of the fluid since we have found that it could have no macroscopic internal motions, no internal flow of heat, and no changes in local entropy density which take place at a finite rate.

*b. Relativistic treatment of entropy changes in a thermodynamic fluid.* We must now compare this conclusion with that which we would obtain by applying relativistic thermodynamics to the same system, namely a finite portion of thermodynamic fluid. In this case in accordance with expression (8) in the preceding section §6, the condition for a reversible adiabatic change in the condition of the fluid would be

$$\iiint \frac{\partial}{\partial x_4}\left(\phi_0\sqrt{-g}\,\frac{dx_4}{ds}\right)dx_1dx_2dx_3 = 0 \tag{14}$$

where $x_4$ is the time like coordinate and the integration is to be taken over the whole range of spatial coordinates $x_1$, $x_2$, $x_3$ necessary to include the fluid. And this condition can evidently be satisfied if we have the equality holding at each point in the fluid

$$\frac{\partial}{\partial x_4}\left(\phi_0\sqrt{-g}\,\frac{dx_4}{ds}\right) = 0 \tag{15a}$$

or

$$\frac{1}{\phi_0}\frac{\partial\phi_0}{\partial x_4} = -\frac{1}{\sqrt{-g}\,\dfrac{dx_4}{ds}}\frac{\partial}{\partial x_4}\left(\sqrt{-g}\,\frac{dx_4}{ds}\right). \tag{15b}$$

This expression gives a relation between the percentage rate at which the proper entropy density $\phi_0$ is changing with the time $x_4$ at a given point and the percentage rate at which the quantity $(\sqrt{-g}\,dx_4/ds)$ is changing with the time at that same point. The value of this latter quantity, however, is determined by the gravitational field at the point in question, and by the kind of coordinate system $x_1 \cdots x_4$ which is being used.

If now we assumed the gravitational field negligible, as is tacitly done in the classical thermodynamics, and had a fluid with no relative motion between its parts, we could choose a system of Galilean coordinates $x$, $y$, $z$, $t$ in which the fluid as a whole would be at rest. In this system of coordinates the quantities $\sqrt{-g}$ and $dx_4/ds$ would have the constant value unity and the condition given by Eqs. (15) would reduce to the result

$$\frac{d\phi_0}{dt} = 0 \tag{16}$$

which we have already found to be characteristic of the classical thermodynamics. It is thus by a neglect of the gravitational field and its possible change with time that the classical thermodynamics has been led to the conclusion that no reversible processes can occur at a finite rate.

On the other hand in relativistic thermodynamics we must not assume that the gravitational field is necessarily negligible but must specifically consider the part which it plays in thermodynamic processes. Hence in relativistic thermodynamics we must consider the condition for reversibility given by Eq. (15) in its full form, and retain the possibility of mutual changes in gravitational field and entropy density taking place together at a finite rate in such a way as to satisfy this condition for reversibility.

In Part III we shall consider a definite model which exhibits such a mutual change in gravitational field and entropy density taking place at a finite rate and satisfying the condition for reversibility. To conclude the present section, however, we may first investigate somewhat further the general nature of the reversibility requirement given by Eq. (15).

Consider the case of a thermodynamic fluid which is at rest with respect to the spatial coordinates $x_1$, $x_2$, $x_3$ which are being used. Since the macroscopic velocities $dx_1/ds$, $dx_2/ds$, and $dx_3/ds$ are everywhere zero by hypothesis, it is evident that the amount of fluid in any given coordinate range $dx_1dx_2dx_3$ would not be changing with the time since there is no flow across the boundary. In accordance with the principles of relativity, however, we can then write for the proper volume $dV_0$ of the small element of fluid in such a coordinate range the well-known equation

$$dV_0 = \sqrt{-g}\,dx_1dx_2dx_3\,\frac{dx_4}{ds} \tag{17}$$

and substituting this expression into the condition for reversibility, as given by Eq. (15a), we can rewrite this condition in the new form

$$\frac{\partial}{\partial x_4}\left(\phi_0\sqrt{-g}\,dx_1dx_2dx_3\,\frac{dx_4}{ds}\right) = \frac{\partial}{\partial x_4}(\phi_0\,dV_0) = 0 \tag{18}$$

since the coordinates $x_1$, $x_2$ and $x_3$ are independent of the coordinate $x_4$.

This equation, however, states that the total entropy for each given small element of fluid shall be constant as measured by a local observer, and this is merely the condition for a change in the proper volume of the element with no flow of heat and with balance between internal and external pressures. Hence if we had a fluid with no flow of heat and constant proper pressure throughout, a finite rate of alteration in the gravitational field which produced no flow of heat and changed the proper pressure at the same rate throughout the fluid would satisfy the condition of reversibility. This alteration in gravitational field, however, would lead to an alteration in proper volume and thus to alteration in the entropy density, so that the thermodynamic state of the fluid would be changing reversibly and at a finite rate. It is this dependence of proper volume on gravitational field, which was quite outside of the considerations of the classical thermodynamics, which leads in relativistic thermodynamics to the possibility of reversible processes which take place at a finite rate.

## Part III. Application to a Specific Model

### §8. The general nature of the model.

We may now apply the foregoing considerations to a specific model. For this purpose we shall take a non-static universe[8] filled with a uniform density of black-body radiation. The choice of this model is not made because it is

[8] For an account of various treatments which have been given to the non-static line element for the universe, see Tolman, Proc. Nat. Acad. **16,** 582 (1930).

thought to give a close approximation to the actual state of the universe but because the mathematical treatment will be relatively simple. The model neglects the presence of matter and its agglomeration into stellar systems which are very characteristic features of the actual universe. Nevertheless, we shall find that the behaviour of the radiation in such a universe furnishes a surprizing possibility of insight into the flow of radiation from the stars which is such a puzzling feature of the actual universe.

### §9. The line element for the non-static universe.

The line element for a non-static universe filled with a uniform distribution of matter and energy can be derived[9] by treating the contents of the universe for the purposes of large-scale considerations as though filled with a perfect fluid, on the basis of the two requirements, (a) that the fluid shall at all times be uniformly distributed spatially, and (b) that particles (nebulae) which are stationary in the coordinate system used shall fulfill the stability requirement of not being subject to acceleration.

The line element so obtained can be written in a variety of forms depending on the choice of coordinates, and for the purpose of the discussions in the present article it will be most convenient to write it in the form[10]

$$ds^2 = -e^{g(t)}\left(\frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) + dt^2 \tag{19}$$

where $r$, $\theta$ and $\phi$ are the spatial coordinates, $t$ is the time coordinate, $R$ is a constant, and the dependence of the line element on the time is given by the exponent $g(t)$.

### §10. Certain general properties of the non-static universe.

Before proceeding to our special model, it will be desirable to recall certain properties which are implied in general for the non-static universe by the form of the line element and which will be needed in our later discussion.

In accordance with the requirement (a) on which the line element was derived, the proper macroscopic density $\rho_{00}$ and the proper pressure $p_0$ of the fluid which fills the universe will be independent of the position $r$, $\theta$, $\phi$, but may be changing with the time $t$. And indeed working out the components of the energy-momentum tensor $T_\mu^\nu$ which correspond to the line element (19) and equating to those for a perfect fluid we obtain as the only non-vanishing components[11]

$$8\pi T_1^1 = 8\pi T_2^2 = 8\pi T_3^3 = -8\pi p_0 = \frac{1}{R^2}e^{-g} + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \tag{20}$$

$$8\pi T_4^4 = 8\pi \rho_{00} = \frac{3}{R^2}e^{-g} + \frac{3}{4}\dot{g}^2 - \Lambda \tag{21}$$

[9] Tolman, Proc. Nat. Acad. **16**, 320 (1930). See also Ibid. **16**, 409 (1930), and note that the five assumptions mentioned in §2 of that article can be included under the heading of the two requirements (a) and (b) given above.

[10] Tolman, Proc. Nat. Acad. **16**, 511 (1930). Eq. (5). Note that the $\bar{r}$ of that article is our present $r$.

[11] Tolman, Proc. Nat. Acad. **16**, 409 (1930). Eq. (2).

where $\Lambda$ is the cosmological constant; and these equations give the dependence of pressure and density on the exponent $g$ and its time derivatives $\dot{g}$ and $\ddot{g}$, and thus on the time itself.

With the help of these expressions for the components of the energy-momentum tensor we can now easily apply the principles of relativistic mechanics in the well-known form

$$\frac{\partial \mathfrak{T}_\mu^\nu}{\partial x_\nu} - \frac{1}{2} \mathfrak{T}^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_\mu} = 0. \tag{22}$$

With $\mu=1, 2, 3$ we merely obtain identities, but substituting into this equation for the case $\mu=4$ we can easily obtain after dividing through by a constant factor[12]

$$\frac{d}{dt}(\rho_{00}\, e^{3g/2}) + p_0 \frac{d}{dt} e^{3g/2} = 0.. \tag{23}$$

This important result can evidently also be obtained directly by combining Eqs. (20) and (21).

In accordance with the requirement (b) on which the line element was derived, particles which are at rest with respect to the coordinate system $r$, $\theta$, $\phi$ will not be subject to acceleration but will remain at rest. And this can be directly verified by calculating the Christoffel three-index symbols which correspond to the line element (19) and substituting in the geodesic equation which governs the motions of particles in general relativity.

As a result of the foregoing, observers who are at rest with respect to the coordinate system will remain permanently so. And in accordance with the form of the line element (19), for such observers, the proper time as measured by local clocks will evidently agree with the coordinate time $t$. On the other hand, for the proper distance $dl_0$ as measured with rigid meter sticks we shall evidently have

$$dl_0 = \frac{e^{g/2} dr}{\sqrt{1 - r^2/R^2}} \tag{24}$$

for points at the coordinate distance $dr$ in the radial direction, and

$$dl_0 = re^{g/2} d\theta \quad \text{and} \quad dl_0 = r \sin\theta e^{g/2} d\phi \tag{25}$$

for the $\theta$ and $\phi$ directions. For the proper volume $dV_0$ associated with a given small range of coordinates we shall have

$$dV_0 = \frac{r^2 \sin\theta\, e^{3g/2}}{\sqrt{1 - r^2/R^2}} dr d\theta d\phi. \tag{26}$$

Although particles which are at rest in the coordinate system $r$, $\theta$, $\phi$ will remain so, nevertheless it is evident from Eqs. (24) and (25), that the proper distance between such particles as measured with rigid meter sticks will in general be changing with the time, since the exponent $g$ is itself a function of

[12] See reference 11, Eq. (4).

the time. Thus for the proper distance between a particle located at the origin $r=0$ and a particle permanently located at the coordinate distance $r=r$, we shall have

$$l_0 = \int_0^r \frac{e^{g/2}dr}{\sqrt{1 - r^2/R^2}} = e^{g/2}R \sin^{-1}\frac{r}{R} \tag{27}$$

and this will be increasing or decreasing with the time in accordance with the dependence of $g$ on the time. Also in accordance with Eq. (26), the proper volume associated with a given coordinate range will be a function of the time, and for the proper volume of the universe as a whole we shall have

$$V_0 = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{r^2 \sin\theta\, e^{3g/2}}{\sqrt{1 - r^2/R^2}} dr d\theta d\phi = \pi^2 R^3 e^{3g/2}. \tag{28}$$

In accordance with this result it is natural to consider $Re^{g/2}$ as the radius of the universe and to speak of an expanding universe if $g$ is increasing with the time and of a contracting universe if $g$ is decreasing with the time.

The change with time in the proper distance between objects in the universe leads to a shift in the observed wave-length of light coming from distant objects, a shift towards the red in an expanding universe and a shift towards the violet in a contracting universe. The magnitude of this shift is given by the formula[13]

$$\frac{\lambda_0 + \delta\lambda}{\lambda_0} = e^{(g-g_0)/2} \tag{29}$$

where $g_0$ is the value of the exponent occurring in the general expression for the line element at the time when the light was emitted with the original wave-length $\lambda_0$, and $g$ is its value at the time the light is received and observed to have the wave-length $\lambda_0+\delta\lambda$.

In accordance with this formula we may regard the wave-length, which would be found by a local observer for any given quantum of light, as a quantity which is changing with the time in accordance with the change of $g$ with the time. And, indeed, differentiating Eq. (29) with respect to the time we can evidently write

$$\frac{1}{\lambda}\frac{d\lambda}{dt} = \frac{1}{\lambda_0 + \delta\lambda}\frac{d}{dt}(\lambda_0 + \delta\lambda) = \frac{1}{2}\frac{dg}{dt} \tag{30}$$

as an expression for the fractional change in the wave-length of any given quantum with the time. Or in terms of frequency we can write

$$\frac{1}{\nu}\frac{d\nu}{dt} = -\frac{1}{2}\frac{dg}{dt} \tag{31}$$

as an expression for the change in the frequency of radiation with the time, $\nu$ being, of course, the frequency as measured by proper observers who are at rest with respect to the fluid in the universe and hence also at rest with respect to the coordinates $r, \theta, \phi$.

[13] See reference 11, Eq. (21).

This completes the statement of general properties of the non-static universe which we shall need in discussing our special model.

**§11. Properties of non-static universe filled with black-body radiation.**

We may now turn to the discussion of the special model in which we take the thermodynamic fluid filling the universe to be a uniform distribution of black-body radiation of the same proper density throughout, a proper observer being one who finds no net flow of radiation.

Under these circumstances we can obtain a great simplification in treatment since it is evident that the proper macroscopic density of the fluid $\rho_{00}$ and its proper pressure $p_0$ will be related by the well-known expression connecting the density and pressure of radiation

$$\rho_{00} = 3p_0. \tag{32}$$

And this permits us to obtain an immediate relation between the pressure of radiation in such a universe and the time variable $g$, since by substituting in the general Eq. (23) we have

$$3\frac{d}{dt}(p_0 e^{3g/2}) + p_0 \frac{d}{dt} e^{3g/2} = 0 \tag{33}$$

and this can at once be integrated to give

$$p_0 = Ae^{-2g} \quad \text{and} \quad \rho_{00} = 3Ae^{-2g} \tag{34}$$

where $A$ is the constant of integration, the pressure and density of radiation thus being quantities which decrease as the radius of the universe $Re^{g/2}$ increases.

As an important consequence of this result it now becomes possible to obtain a solution for $g$ as a function of $t$. Substituting the expression for proper density given by Eq. (34) into Eq. (21) we obtain after some rearrangement

$$\frac{d}{dt}(e^g) = \pm \sqrt{32\pi A - \frac{4}{R^2}e^g + \frac{4}{3}\Lambda e^{2g}} \tag{35}$$

as a differential equation for the dependence of $g$ on $t$, where the plus sign corresponds to an expanding universe and the negative sign to a contracting universe. Eq. (35) can itself then easily be integrated to give an explicit solution for $g$ as a function of $t$. The form of the solution will depend on the sign of the cosmological constant $\Lambda$, and it is merely of interest for our present purposes to remark that for the case of a universe containing nothing but radiation there appears to be no solution, having physical reality, which would make $g$ a periodic function of $t$.

As the most important consequence, however, of the expression for density given by Eq. (34), we can now show that the changes taking place in such a universe on account of the changing value of $g$ are thermodynamically reversible. In accordance with Eq. (34) and the well known relation of Boltzmann connecting the density of black-body radiation with its temperature, we can write

$$\rho_{00} = 3Ae^{-2g} = aT_0^4 \tag{36}$$

where $a$ is the Stefan-Boltzmann constant and $T_0$ is the proper temperature. Solving this for the temperature and substituting in the known expression for the entropy density of black-body radiation, we obtain

$$\phi_0 = \frac{4}{3} aT_0^3 = 4\left(\frac{a}{3}\right)^{1/4} A^{3/4} e^{-3g/2} \tag{37}$$

as an expression for the proper entropy density of our fluid.[14] On the other hand in accordance with Eq. (26) we have

$$dV_0 = \frac{r^2 \sin\theta\, e^{3g/2}}{\sqrt{1 - n^2/R^2}} dr d\theta d\phi \tag{38}$$

as an expression for the proper volume associated with the coordinate range $drd\theta d\phi$. Hence combining the two expressions (37) and (38), we can evidently write

$$\frac{\partial}{dt}(\phi_0 dV_0) = 0 \tag{39}$$

since $g$ is the only quantity involved which depends on the time and this is seen to cancel out from the product.

The final result, however, is the very expression which we obtained in Part II (§7, Eq. (18)) as a general condition for a reversible process in relativistic thermodynamics. And since $g$ and hence $\phi_0$ will in general be changing in such a universe at a finite rate, we have thus actually illustrated by a specific example the possibility provided by relativistic thermodynamics for reversible processes to take place at a finite rate.

### §12. Interpretation by an ordinary observer of phenomena in an expanding universe filled with radiation.

Turning our attention now in particular to the case of expansion, with the radius $Re^{g/2}$ increasing with the time, we can show that the special model of a universe, filled with black-body radiation and expanding reversibly without increase in entropy, would nevertheless exhibit important phenomena which

[14] The relations connecting energy density and entropy density with temperature, used in (36) and (37), presuppose that the frequency distribution of the radiation remains that for black-body radiation for all values of $g$. This introduces no difficulty, however, since even if there were a tendency for the frequency distribution to change away from that for black-body radiation, as the size of the universe changes, this could be prevented by the introduction of a small amount of material to act as a catalyst; and in actuality there is of course no such tendency since it would involve a decrease in entropy. In addition it can be shown in detail that the dependencies of frequency and energy density on $g$ given by Eqs. (31) and (34) are such as to preserve the black-body distribution of frequency for all values of $g$, if we have such a distribution for one value of $g$.

would be interpreted by an ordinary observer as similar to phenomena in the actual universe which have been regarded in the past as important evidence for an increasing entropy of the universe. To obtain a description of these phenomena, we shall consider that the observer in question marks out a small region of the universe in his immediate vicinity, using rigid meter sticks, and then studies the changes taking place in this region. We shall then show that the observer will find the density of energy in this region and the total energy content of the region continually decreasing with the time, its temperature dropping, the number of light quanta leaving the region always greater than the number entering, and the average frequency of the quanta which leave greater than the average frequency of those that enter. Evidently our ordinary unsophisticated observer would interpret these findings as evidence that his immediate neighborhood was cooling off by radiation into the colder depths of space, and with a knowledge only of the classical thermodynamics he would conclude that the entropy of the universe was increasing at an enormous rate, in spite of the fact that the relativistic thermodynamics, which must be used under the circumstances, actually shows that there would be no increase in entropy in such a universe. The analogy between the phenomena interpreted by this unsophisticated observer as leading to an increase of entropy and phenomena in the actual universe which have hitherto been interpreted in a similar manner is close enough so that we must certainly be cautious lest we draw too hasty conclusions as to increases in entropy in our actual universe.

To proceed now to the detailed exposition, let us consider that the observer in our idealized model of the universe is located for convenience at the origin of the $r$, $\theta$, $\phi$ system of coordinates and is provided with a rigid scale of proper length $dl_0$. With the help of this scale he marks out a small sphere around the origin of constant proper radius $l_0$, which gives him a small region of the universe in his immediate vicinity to serve as the subject of his studies.

For the relation between the constant proper radius of this sphere and the coordinate $r$ of its boundary we may evidently write in accordance with Eq. (24)

$$l_0 = \int_0^r \frac{e^{g/2}dr}{\sqrt{1 - r^2/R^2}} = e^{g/2}R \sin^{-1}\frac{r}{R} \tag{40}$$

and for the case in hand where the sphere considered is very small compared with the whole universe, so that $r$ is small compared with $R$, we obtain from this the approximate relation

$$r \approx l_0\, e^{-g/2}. \tag{41}$$

Since the proper radius of the sphere $l_0$ is constant by hypothesis, we note that the coordinate $r$ of its boundary is a quantity which is decreasing with the time in an expanding universe owing to the increase in $g$ with time.

For the proper volume of this sphere contained within the radius $l_0$ we can evidently write in accordance with Eq. (26)

$$V_0 = \int_0^r \frac{4\pi r^2 e^{3g/2} dr}{\sqrt{1 - r^2/R^2}}$$
$$= 4\pi e^{3g/2} R \left| -\frac{r}{2}\sqrt{R^2 - r^2} + \frac{R^2}{2}\sin^{-1}\frac{r}{R} \right|_0^r . \tag{42}$$

Developing this in the form of a series in $r/R$ and neglecting higher powers, we obtain

$$V_0 = 4\pi e^{3g/2} R^3 \left( -\frac{1}{2}\frac{r}{R} + \frac{1}{4}\frac{r^3}{R^3} + \frac{1}{16}\frac{r^5}{R^5} + \cdots \right.$$
$$\left. + \frac{1}{2}\frac{r}{R} + \frac{1}{12}\frac{r^3}{R^3} + \frac{3}{80}\frac{r^5}{R^5} + \cdots \right) \approx \frac{4}{3}\pi r^3 e^{3g/2}. \tag{43}$$

And substituting the value of $r$ given by Eq. (41), we obtain for the proper volume of the sphere in terms of its proper radius $l_0$, as a close approximation, the result which might be expected

$$V_0 = \frac{4}{3}\pi l_0^3 \tag{44}$$

which is a constant independent of the time.

We may now consider the nature of the observations which our observer would find in studying this sphere of constant measured radius which he has marked off.

As a result of Eq. (34), the proper energy density at every point in our special model of the universe would be changing with the time $t$ in accordance with the expression

$$\frac{1}{\rho_{00}}\frac{d\rho_{00}}{dt} = -2\frac{dg}{dt}. \tag{45}$$

Moreover, the measurements of energy density which our observer would make in his immediate neighborhood would actually be measurements of proper energy density, and from the form of the line element (19) the proper time which he uses would agree with the coordinate time $t$. Hence it is evident that our observer would find the energy density in his vicinity to be decreasing with the time in accordance with Eq. (45).

Furthermore, in accordance with Eq. (44), the proper volume of his sphere of constant measured radius is itself independent of the time. Hence it is evident that our observer would find the total energy content $E_0$ of his sphere decreasing at the rate

$$\frac{1}{E_0}\frac{dE_0}{dt} = -2\frac{dg}{dt}. \tag{46}$$

In addition, owing to the relation between energy density and temperature for black-body radiation given by Eq. (36), it is evident that our observer would find the temperature in his vicinity to be dropping at the rate

$$\frac{1}{T_0}\frac{dT_0}{dt} = -\frac{1}{2}\frac{dg}{dt}. \tag{47}$$

Still further, since the total number of light quanta in the universe would be independent of the time, and the proper volume of the universe as a whole would be increasing with the time while the proper volume of the observer's sphere remained constant, it is evident that the observer would find a larger number of light quanta leaving his sphere per second than entering. To calculate this excess we may evidently write for the number of quanta $n$ inside the sphere in terms of the total number of quanta $N$ in the universe

$$n = \frac{\frac{4}{3}\pi l_0^3}{\pi^2 R^3 e^{3g/2}} N \tag{48}$$

where the numerator of the fraction is the proper volume of the sphere as given by Eq. (44) and the denominator is the total proper volume of the universe as given by Eq. (28). And carrying out a logarithmic differentiation of this with respect to the time we obtain

$$\frac{1}{n}\frac{dn}{dt} = -\frac{3}{2}\frac{dg}{dt} \tag{49}$$

which gives the net loss per unit time in the number of quanta within the observer's sphere. The result so obtained, when combined with the rate at which the frequencies of the quanta are decreasing with the time as given by Eq. (31), is just sufficient to account for the rate of decrease in the proper energy of the observer's sphere as given by Eq. (46).

Finally, we may point out a curious circumstance which would reinforce our unsophisticated observer in his interpretation of the above phenomena as radiation into surroundings of lower temperature. Let us suppose that our observer, ever active in his scientific investigations, stations one of his assistants on the boundary of his sphere at the fixed distance $l_0$ from the origin as measured with rigid meter sticks, and instructs him to observe the average frequency of the light entering and leaving the sphere through its surface. This assistant will not be at rest in the coordinate system $r, \theta, \phi$, but in accordance with Eq. (41) will have the coordinate velocity

$$\frac{dr}{dt} = -\frac{1}{2} l_0 e^{-g/2}\frac{dg}{dt} = -\frac{1}{2} r \frac{dg}{dt}\,. \tag{50}$$

Hence, since it is evident that the average frequency of the radiation would be independent of direction for an observer at rest in the coordinate system, this assistant will find the average frequency of the radiation entering the sphere less than that of the radiation leaving the sphere, as a result of the Doppler effect corresponding to the velocity given by Eq. (50).

This completes a considerable chain of evidence which would lead an ordinary observer, unfamiliar with the expansion of the universe, to conclude that the region in his immediate neighborhood was cooling off by radiation into surroundings of lower temperature, in spite of the fact that the changes taking place in the model actually involve no increase in entropy. The analogy between the findings of this observer in the hypothetical model and those of the classical thermodynamist in the actual universe is very striking.

## PART IV. CONCLUSION

### §13. Summary.

In the foregoing article an attempt has been made to show the bearing of relativistic thermodynamics on the well-known problem of the entropy of the universe as a whole. The origin of this problem lies in the difficulties encountered by the classical thermodynamics,—first in failing to account for the presumed fact that the entropy of the universe has always been increasing at an enormous rate and nevertheless has not yet reached its maximum value,—and second in failing to allow an emotionally satisfactory feeling towards our universe whose ultimate fate would be the stagnation of "heat-death."

In the present article a brief description was first given of various older contributions to the solution of this problem, which have been based on the standpoints of the classical thermodynamics and statistical mechanics. This was done in order to show the very different character of the new contribution proposed in this article. A summarized account of the nature of this contribution may now be given.

The problem of the entropy of the universe arises because of the commonly accepted conclusion that the entropy of the universe is actually increasing at an enormous rate, and this conclusion is in turn based on the presumption, familiar in classical thermodynamics, that thermodynamic processes cannot be taking place at a finite rate, as observed, and at the same time reversibly without increase in entropy. The general nature of the contribution to the problem offered by relativistic thermodynamics lies in showing that there can be thermodynamic processes which take place both reversibly and at a finite rate.

To illustrate this difference between the classical and relativistic thermodynamics, we may consider the possibility of carrying out a reversible change in the thermodynamic properties of a finite portion of thermodynamic fluid. In the classical mechanics it is found that no internal motions of the fluid, no flow of heat, and no change in volume can be allowed to take place at a finite rate and hence that no change at all in the thermodynamic properties of the fluid can take place at a finite rate, the entropy density of the fluid remaining constant in accordance with the equation

$$\frac{d\phi_0}{dt} = 0.$$

On the other hand, in relativistic thermodynamics it appears possible to allow changes in the proper volume of the fluid, due to changes in the gravitational potentials, to take place at a finite rate and still maintain reversibility. Indeed, the condition for reversibility is found to be satisfied if we have at each point in the fluid the relation

$$\frac{\partial}{\partial x_4}\left(\phi_0\sqrt{-g}\,\frac{dx_4}{ds}\,dx_1dx_2dx_3\right) = \frac{\partial}{\partial x_4}(\phi_0 dV_0) = 0$$

and this permits a finite rate of change in the proper entropy density $\phi_0$ with the time $x_4$, provided it satisfies the equation

$$\left(\sqrt{-g}\,\frac{dx_4}{ds}\right)\frac{\partial\phi_0}{\partial x_4} + \phi_0\frac{\partial}{\partial x_4}\left(\sqrt{-g}\,\frac{dx_4}{ds}\right) = 0.$$

To exhibit the nature of the reversible changes at a finite rate thus permitted in relativistic thermodynamics, we may consider the highly idealized model of a non-static universe filled with black-body radiation as our thermodynamic fluid. It is found that the radius and total proper volume of such a universe could be changing at a finite rate with the time and yet reversibly without increase in entropy.

Furthermore, if we take the case of an expanding universe and consider an observer who marks out with rigid meter sticks a small region of the universe in his vicinity, it can be shown that he would find the energy density, the energy content, and the temperature of this region decreasing with the time, and in addition would find the number of light quanta leaving the region per second greater than the number returning and the average frequency of those passing outward through the boundary greater than that of those returning. He would thus be led to interpret the phenomena taking place in such a universe as a flow of radiation from his immediate neighborhood out into the colder regions of space, in spite of the fact that the changes in the universe would in reality be taking place without any increase in entropy.

The general nature of the contribution to the problem of entropy made in this article has thus been to show that phenomena which have hitherto been regarded from the point of view of classical thermodynamics as furnishing unmistakable evidence for an increasing entropy in the universe are not necessarily leading to any increase in entropy at all, and to emphasize the necessity for analyzing the phenomena of the universe from the more acceptable point of view of relativistic thermodynamics before conclusions are drawn as to what extent the entropy of the universe is increasing if at all.

## §14. Critique.

Finally a few words of criticism will not be out of place. The foregoing statement as to the nature of the contribution made in this article carries with it at least the possible implication that an analysis of the phenomena of the actual universe from the standpoint of relativistic thermodynamics would show that there are in reality no important changes at all taking place in the entropy of the universe.

I feel, however, that although the article has clearly demonstrated the necessity of using relativistic thermodynamics in analyzing the entropy changes of the universe as a whole, it would be premature to assert too precisely what is to be expected as the result of such an analysis until it has been applied to a model of the universe which is not so over-simplified as the one employed in this article. This model of a universe, containing nothing but radiation, neglects two of the most characteristic features of the actual universe, namely the presence of matter and its high degree of concentration into stellar systems. At a later time I hope to give more consideration to these properties of the universe. They appear, however, greatly to increase the mathematical complexity of the problem and for this reason I have contented myself for the present with a very simple model, which can nevertheless give us considerable insight into the problem of the entropy of the universe.

One feature of the model which was mentioned in §11 should perhaps be emphasized, namely that there appear to be no periodic solutions of Eq. (35) for $g$ as a function of $t$ which would have physical interest. Hence if we had an expanding model of the kind considered it would continue to expand, reversibly to be sure, but without actual return to its original condition. Our model is so over-simplified, however, that we must no conclude therefrom that periodic solutions would not be of interest for the actual universe.

Further it should perhaps also be emphasized again that the theory of fluctuations may play an important part in a relatively complete treatment of the entropy of the universe. At the present time, however, we cannot say just how large this part may be.

In conclusion then, it has apparently been definitely demonstrated that the problem of the entropy of the universe as a whole must be treated with the help of relativistic rather than classical thermodynamics, and it has been shown that the application of relativistic thermodynamics to a highly over-simplified model of the universe gives results of great interest. It remains for the future, however, to consider the application of relativistic thermodynamics to more complicated models which would give a better approximation to the actual universe.