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Response of a Nonlinear String to Random Loading

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This paper considers the response of a nonlinear string to random excitation. It is shown that, owing to the additional stress induced by the stretching of the string, the mean squared deflection at every point is smaller than that for the equivalent linear string.

The response of a nonlinear string to deterministic loading has been analyzed by several authors.[1–4] G. F. Carrier[1,2] has analyzed the free oscillations of a nonlinear string, while Lee[3] and the author[4] have analyzed the response of a nonlinear string to deterministic loading. This paper is devoted to the study of a nonlinear string subjected to a random excitation. It is shown that, as in the case of deterministic loading, the additional stress induced by the stretching of the string reduces the mean squared deflection at every point compared with that for the equivalent linear string.

Formulation

Consider an elastic string, clamped at its ends subjected to a random loading. The nomenclature used is as follows:

- u = transverse deflection of string
- T_0 = initial tension in string
- ρ = linear density of string
- $\rho\beta$ = viscous damping coefficient
- A = cross-sectional area of string
- E = elastic modulus of string
- L = length of string
- $f(x, t)$ = loading on string

If the deflections and slopes are considered to be moderately small (consistent with the string remaining elastic) the equation of motion of the string is

$$\rho\left(\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t}\right) = \left[T_0 + \frac{AE}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2 dx\right]\frac{\partial^2 u}{\partial x^2} + f(x, t) \quad (1)$$

subject to the boundary conditions that

$$u(0, t) = u(L, t) = 0 \quad (2)$$

In the analysis that follows it will be assumed that $f(x, t)$ is a randomly varying force with a clipped Gaussian white spectrum and has the following cross correlation:

$$\overline{f(x, t_1)f(\eta, t_2)} = 4D\delta(x - \eta)\frac{\sin\omega_c\tau}{\tau} \quad (3)$$

where

$$\tau = t_1 - t_2$$

Since the eigenmodes for equation (1) are sine waves, let

$$f(x, t) = \sum_{i=1}^{\infty} a_i(t)\sin\frac{i\pi x}{L}$$
$$u(x, t) = \sum_{i=1}^{N} b_i(t)\sin\frac{i\pi x}{L} \quad (4)$$

where N is chosen such that $\omega_N < \omega_c < \omega_{N+1}$.

As a consequence of the assumed form of the cross correlation (3), the a_i in (4) are uncorrelated and the power spectrum $g_i(\omega)$ of a_i is given by

$$g_i(\omega) = \frac{2}{L}\frac{4D}{2\pi} \qquad 0 < \omega < \omega_c$$
$$= 0 \qquad \omega > \omega_c \quad (5)$$

Substituting (4) into (1)

$$\frac{d^2b_n}{dt^2} + \beta\frac{db_n}{dt} + \left(\frac{n\pi}{L}\right)^2\frac{T_0}{\rho}\left[1 + \frac{AE}{2LT_0}\times \int_0^L\left(\sum_{i=1}^{N} b_i\frac{i\pi}{L}\cos\frac{i\pi x}{L}\right)^2 dx\right]b_n = a_n(t)/\rho \quad (6)$$

Using the orthogonality of the circular functions, equation (6) becomes

$$\frac{d^2b_n}{dt^2} + \beta\frac{db_n}{dt} + \left(\frac{n\pi}{L}\right)^2\frac{T_0}{\rho}\left(1 + \alpha\sum_{i=1}^{N} i^2b_i^2\right)b_n = a_n/\rho \quad (7)$$

where

$$\alpha = \frac{AE}{4T_0L}\left(\frac{\pi}{L}\right)^2 \quad (8)$$

Equivalent Linear Form of Equation (7)

The method of equivalent linearization has been applied successfully in the past to nonlinear differential equations with harmonic excitation. It will now be shown that, by a slightly different approach to the problem, this method may be applied to equation (7).

Rewriting equation (7) in the form

[1] G. F. Carrier, "On the Nonlinear Vibration Problem of the Elastic String," *Quarterly of Applied Mathematics*, vol. 3, 1945, pp. 157–165.

[2] G. F. Carrier, "A Note on the Vibrating String," *Quarterly of Applied Mathematics*, vol. 7, 1949, pp. 97–101.

[3] E. W. Lee, "Nonlinear Forced Vibrations of a Stretched String," *British Journal of Applied Physics*, vol. 8, 1957, pp. 411–413.

[4] T. K. Caughey, unpublished CalTech Report No. 85, 1956.

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$$\frac{d^2b_n}{dt^2} + \beta\frac{db_n}{dt} + \left(\frac{n\pi}{L}\right)^2 \frac{T_0}{\rho} K_n b_n + \epsilon(b_1, b_2, \ldots b_N) = \frac{a_n}{\rho} \quad (9)$$

If in (9), $\epsilon(b_1, b_2, \ldots b_N)$ is neglected, (9) is linear and can be solved readily. The smaller $\epsilon(b_1, b_2, \ldots b_N)$ is, the smaller the error in neglecting it. The logical choice of K_n is, therefore, that value of K_n which minimizes $\epsilon(b_1, b_2, \ldots b_N)$.

The choice of minimization procedure is somewhat arbitrary, but by analogy with Galerkin's method, and for mathematical expediency, it is desirable to use the minimization of the mean squared error

$$\overline{\epsilon^2} = \frac{1}{2T}\int_{-T}^{T}\left[\left(\frac{n\pi}{L}\right)^2\frac{T_0}{\rho}\right]^2 \left\{\left[1 + \alpha\sum_{i=1}^{N} i^2b_i^2\right]b_n - K_nb_n\right\}^2 dt \quad (10)$$

If the process is an ergodic one, then equation (10) may be replaced by the stochastic average. Further, since the a_i are uncorrelated

$$\therefore \overline{\epsilon^2} = \int_{-\infty}^{\infty}{}_{N\text{-fold}}\int_{-\infty}^{\infty}\left[\left(\frac{n\pi}{L}\right)^2\frac{T_0}{\rho}\right]^2 \left\{\left[1 + \alpha\sum_{i=1}^{N} i^2b_i^2\right] - K_n\right\}^2 b_n^2 \prod_{i=1}^{N}\left\{p(b_i)db_i\right\} \quad (11)$$

where $p(b_i)$ is the first probability density function of b_i.

If in (9), $\epsilon(b_1, b_2, \ldots b_N)$ is neglected, b_n is Gaussian since a_n is Gaussian.

Thus

$$p(b_i) = \frac{1}{(2\pi)^{1/2}\sigma_i}\exp(-b_i^2/2\sigma_i^2)$$

where

$$\sigma_i^2 = \overline{b_i^2} \quad (12)$$

$$\therefore \overline{\epsilon^2} = \left[\prod_{i=1}^{N}(2\pi)^{1/2}\sigma_i\right]^{-1}\int_{-\infty}^{\infty}{}_{N\text{-fold}}\int_{-\infty}^{\infty}\left[\left(\frac{n\pi}{L}\right)^2\frac{T_0}{\rho}\right]^2\left[1 + \alpha\sum_{i=1}^{N} i^2b_i^2 - K_n\right]^2 b_n^2 \prod_{i=1}^{N}\left[\exp(-b_i^2/2\sigma_i^2)db_i\right] \quad (13)$$

Minimizing $\overline{\epsilon^2}$ with respect to K_n:

$$K_n = \left[\int_{-\infty}^{\infty}{}_{N\text{-fold}}\int_{-\infty}^{\infty} b_n^2\prod_{i=1}^{N}\{\exp(-b_i^2/2\sigma_i^2)db_i\}\right]^{-1} \int_{-\infty}^{\infty}{}_{N\text{-fold}}\int_{-\infty}^{\infty}\left[1 + \alpha\sum_{i=1}^{N} i^2b_i^2\right]b_n^2 \prod_{i=1}^{N}[\exp(-b_i^2/2\sigma_i^2)db_i] \quad (14)$$

Evaluating the integrals:

$$K_n = 1 + \alpha\left[\sum_{i=1}^{N} i^2\sigma_i^2 + 2n^2\right]\sigma_n^2 \quad (15)$$

If in equation (9) $\epsilon(\ )$ is neglected, then

$$\frac{d^2b_n}{dt^2} + \beta\frac{db_n}{dt} + \left(\frac{n\pi}{L}\right)^2\frac{T_0}{\rho}K_nb_n = \frac{a_n(t)}{\rho} \quad (16)$$

The power spectrum of b_n is therefore given by

$$B_n(\omega) = g_n(\omega)\rho^{-1}\left\{\left[\left(\frac{n\pi}{L}\right)^2\frac{T_0}{\rho}K_n - \omega^2\right]^2 + [\omega\beta]^2\right\}^{-1} \quad (17)$$

where $g_n(\omega)$ is given by equation (5).

The mean squared value of b_n is readily computed from (17). Thus

$$\sigma_n^2 = \overline{b_n^2} = \int_0^\infty B_n(\omega)dw = \int_0^\infty \frac{2^2}{\rho L}\frac{D}{\pi}\left\{\left[\left(\frac{n\pi}{L}\right)^2\frac{T_0}{\rho}K_n - \omega^2\right]^2 + [w\beta]^2\right\}^{-1} dw \quad (18)$$

If β is small and

$$\left(\frac{n\pi}{L}\right)^2\frac{T_0}{\rho}K_n < \omega_c^2$$

$$\sigma_n^2 \doteq \frac{2^2}{2\rho L}D\left\{\left(\frac{n\pi}{L}\right)^2\frac{T_0}{\rho}K_n\beta\right\}^{-1} \quad (19)[5]$$

The main contribution to the integral (18) coming from a small region close to

$$\omega^2 = \left(\frac{n\pi}{L}\right)^2\frac{T_0}{\rho}K_n$$

By the same token it will be seen that if

$$\left(\frac{i\pi}{L}\right)^2\frac{T_0}{\rho}K_i > \omega_c^2$$

the integral (18) will be very small, thus justifying the approximate representation of u in (4).

Substituting K_n from (15) into (19)

$$\sigma_n^2 = \frac{2}{\rho L}D\left\{n^2\left(\frac{\pi}{L}\right)^2\frac{T_0}{\rho}\beta \left(1 + \alpha\left[\sum_{i=1}^{N} i^2\sigma_i^2 + 2n^2\sigma_n^2\right]\right)\right\}^{-1} \quad (20)$$

Noting that

$$\frac{2}{\rho L}D\left[\left(\frac{\pi}{L}\right)^2\frac{T_0}{\rho}\beta\right]^{-1} = \sigma_{1,0}^2 \quad (21)$$

the mean squared deflection in the first mode of the linearized problem obtained by setting $\alpha = 0$ in (7). Thus

$$n^2\frac{\sigma_n^2}{\sigma_{1,0}^2} = \left\{1 + \alpha\sigma_{1,0}^2\left[\sum_{i=1}^{N} i^2\frac{\sigma_i^2}{\sigma_{1,0}^2} + 2n^2\frac{\sigma_n^2}{\sigma_{1,0}^2}\right]\right\}^{-1} \quad (22)$$

Denoting

$$i^2\frac{\sigma_i^2}{\sigma_{1,0}^2} = y_i$$

$$\alpha\sigma_{1,0}^2 = z \quad (23)$$

$$\sum_{i=1}^{N} i^2\frac{\sigma_i^2}{\sigma_{1,0}^2} = \sum_{i=1}^{N} y_i = S$$

Equation (22) becomes

$$2zy_n^2 + [1 + zS]y_n - 1 = 0 \quad (24)$$

Solving:

$$y_n = \{-(1 + zS) + [(1 + zS)^2 + 8z]^{1/2}\}[4z]^{-1} \quad (25)$$

[5] See Appendix.

The positive root being taken in (25) since y_n by definition is positive. Multiplying both sides of (25) by $4z$, and summing on n

$$4zS = -N(1 + zS) + N[(1 + zS)^2 + 8z]^{1/2} \qquad (26)$$

Rearranging terms and squaring

$$\left[1 + z\left(1 + \frac{4}{N}\right)S\right]^2 = (1 + zS)^2 + 8z \qquad (27)$$

Solving for zS:

$$zS = \{-1 + [1 + 4(N + 2)z]^{1/2}\}\left[2\left(1 + \frac{2}{N}\right)\right]^{-1} \qquad (28)$$

Since zS is positive, the positive radical is taken.

Substituting (28) into (25), y_n may be calculated. It will be observed from (25) that y_n is independent of n, thus there is equipartition of energy among the modes.

Calculation of Mean Squared Deflection $\overline{u^2}$

Using equation (4)

$$\overline{u^2} = \overline{\left[\sum_{i=1}^{N} b_i \sin\frac{i\pi x}{L}\right]^2} \qquad (29)$$

$$\overline{U^2} = \sum_{i,j}^{N} \overline{b_i b_j} \sin\frac{i\pi x}{L} \sin\frac{j x\pi}{L} \qquad (30)$$

Owing to the particular cross correlation function (3), the b_i are uncorrelated: thus

$$\overline{b_i b_j} = 0 \quad i \neq j \qquad (31)$$

Equation (30) therefore becomes

$$\overline{U^2} = \sum_{i=1}^{N} \overline{b_i^2} \sin^2\frac{i\pi x}{L} \qquad (32)$$

From equations (23) and (25)

$$\overline{b_i^2} = \sigma_i^2 = y_i \frac{\sigma_{1,0}^2}{i^2}$$

$$= \frac{\sigma_{1,0}^2}{i^2}\{-(1 + zS) + [(1 + zS)^2 + 8z]^{1/2}\}[4z]^{-1} \qquad (33)$$

Substituting (33) into (32)

$$\overline{U^2} = \sigma_{1,0}^2\{-(1 + zS) + [(1 + zS)^2 + 8z]^{1/2}\}[4z]^{-1} \sum_{i=1}^{N} \frac{1}{i^2}\sin^2\frac{i\pi x}{L} \qquad (34)$$

In the linearized problem given by setting $\alpha = 0$ in (7), the mean squared deflection $\overline{U_0^2}$ is given by

$$\overline{U_0^2} = \sigma_{1,0}^2 \sum_{i=1}^{N} \frac{1}{i^2}\sin^2\frac{i\pi x}{L} \qquad (35)$$

Thus

$$\frac{\overline{U^2}}{\overline{U_0^2}} = \{-(1 + zS) + [(1 + zS)^2 + 8z]^{1/2}\}[4z]^{-1} \qquad (36)$$

where zS is given by equation (28).

If in (36) $z \ll 1$, (36) reduces to

$$\frac{\overline{U^2}}{\overline{U_0^2}} \doteq [1 + zS]^{-1} \leq 1 \qquad (37)$$

Further, if in (28) N is large,

$$zS \doteq {}^1/_2[-1 + (1 + 4Nz)^{1/2}] \qquad (38)$$

Using (38), equation (37) becomes

$$\frac{\overline{U^2}}{\overline{U_0^2}} \doteq 2[1 + (1 + 4Nz)^{1/2}]^{-1} \leq 1 \qquad (39)$$

Thus the mean squared deflection in the nonlinear case is always less than the mean squared deflection in the equivalent linear string.

Conclusions

From the foregoing analysis it is clearly seen that the effect of the stress induced by the stretching of the string is to reduce the mean squared deflection at every point in the string compared with the corresponding mean squared deflection in the equivalent linear string.

APPENDIX

(a) Evaluation of integral (18). Let

$$\left(\frac{n\pi}{L}\right)^2 \frac{T_0}{\rho} K_n = \tilde{\omega}_n^2$$

Thus

$$\sigma_n^2 = 2^2D/\rho L\pi \int_0^{\omega c} \{[\tilde{\omega}_n^2 - \omega^2]^2 + [\beta w]^2\}^{-1} dw \qquad (40)$$

Using integral 211.16c of "Integraltaffel, Ester Teil, Unbestimmte Integral," by W. Grobner and N. Hofreiter (Springer, 1949), equation (40) becomes

$$\begin{aligned}\sigma_n^2 = 2D/\rho L\pi\tilde{\omega}_n\{&\sin\phi_0 \ln[(\tilde{\omega}_n^2 + \omega_c^2 + 2\tilde{\omega}_n\omega_c\cos\phi_0)/\\ &(\tilde{\omega}_n^2 + \omega_c^2 - 2\tilde{\omega}_n\omega_c\cos\phi_0)]\\ &+ \cos\phi_0 \tan^{-1}[2\tilde{\omega}_n\omega_e\sin\phi_0/(\tilde{\omega}_n^2 - \omega_c^2)]\}\\ &\{\beta^2(\tilde{\omega}_n^2 - \beta^2/4)\}^{-1/2}\end{aligned} \qquad (41)$$

where

$$\tan 2\phi_0 = \{\beta^2(\tilde{\omega}_n^2 - \beta^2/4)\}^{1/2}\{\tilde{\omega}_n^2 - \beta^2/2\}^{-1} \qquad (42)$$

If β is small compared to $\tilde{\omega}_n$, equation (41) reduces to

$$\sigma_n^2 \simeq 2D/\rho L\pi\tilde{\omega}_n^2\beta\{\tan^{-1}[\omega_c\beta/(\tilde{\omega}_n^2 - \omega_c^2)] + 0[\beta/2\tilde{\omega}_n \ln(2\tilde{\omega}_n/\beta)]\} \qquad (43)$$

For $\beta/\tilde{\omega}_n$ very small, the second term may be neglected entirely. Thus

$$\sigma_n^2 \doteq 2D/\rho L\pi\tilde{\omega}_n^2\beta\{\tan^{-1}[\omega_c\beta/(\tilde{\omega}_n^2 - \omega_c^2)]\} \qquad (44)$$

If $\omega_c > \tilde{\omega}_n$

$$\sigma_n^2 \doteq 2D/\rho L\tilde{\omega}_n^2\beta \qquad (45)$$

If $\tilde{\omega}_n > \omega_c$

$$\begin{aligned}\sigma_n^2 &\doteq 2D/\rho L\pi\tilde{\omega}_n^2\beta\{\omega_c\beta/(\tilde{\omega}_n^2 - \omega_c^2)\}\\ &= 2\beta D\omega_c/\rho L\pi\beta\tilde{\omega}_n^2(\tilde{\omega}_n^2 - \omega_c^2)\\ &\ll 2D/\rho L\pi\beta\tilde{\omega}_n^2 \quad \text{for} \quad \beta/\tilde{\omega}_n \ll 1\end{aligned} \qquad (46)$$

This justifies the approximate representation of u in equation (4)

(b) Justification for neglecting term $z\sum_{N+1}^{\infty} i^2(\sigma_n^2/\sigma_{1,0}^2)$. As is shown in (46), for $\tilde{\omega}_i^2 > \omega_c^2$

$$\sigma_i^2 \doteq 2D\omega_c[\rho L\pi\tilde{\omega}_i^2(\tilde{\omega}_i^2 - \omega_c^2)]^{-1} \qquad (47)$$

i.e.,

$$i^2 \frac{\sigma_i^2}{\sigma_{1,0}^2} \doteq \omega_c\beta[\pi(\omega_i^2 K_i - \omega_c^2)]^{-1} \tag{48}$$

$$\therefore \sum_{N+1}^{\infty} i^2 \frac{\sigma_i^2}{\sigma_{1,0}^2} \simeq \frac{\omega_c\beta}{\pi\omega_0^2 K_{N+1}} \sum_{N+1}^{\infty} \left(i^2 - \frac{\omega_c^2}{\omega_0^2 K_{N+1}}\right)^{-1} \tag{49}$$

$$\leq \frac{\beta}{2\pi\omega_0 K_{N+1}^{1/2}} \int_{N+1}^{\infty} \left[\left(i - \frac{\omega_c}{\omega_0 K_{N+1}^{1/2}}\right)^{-1} - \left(i + \frac{\omega_c}{\omega_0 K_{N+1}^{1/2}}\right)^{-1}\right] di \tag{50}$$

$$\leq \frac{\beta}{2\pi\omega_0 K_{N+1}^{1/2}} \ln\ [(N + 1 + \omega_c/\omega_0 K_{N+1}^{1/2})/(N + 1 - \omega_c/\omega_0 K_{N+1}^{1/2})] \tag{51}$$

$$\doteq \frac{\beta}{2\pi\omega_0 K_{N+1}^{1/2}} \ln\ [(N + 2)/N] \tag{52}$$

Comparing

$$\sum_{N+1}^{\infty} i^2 \frac{\sigma_i^2}{\sigma_{1,0}^2} \quad \text{with} \quad \sum_{i=1}^{N} i^2 \frac{\sigma_i^2}{\sigma_{1,0}^2}$$

$$\sum_{N+1}^{\infty} i^2 \frac{\sigma_i^2}{\sigma_{1,0}^2} \Big/ \sum_{1}^{N} i^2 \frac{\sigma_i^2}{\sigma_{1,0}^2} \doteq \frac{2\beta\left(1 + \dfrac{2}{N}\right) z \ln\ [(N + 2)/N]}{2\pi\omega_0 K_{N+1}^{1/2}\{-1 + [1 + 4(N + 2)z]^{1/2}} \tag{53}$$

For z small

$$\sum_{N+1}^{\infty} i^2 \frac{\sigma_i^2}{\sigma_{1,0}^2} \Big/ \sum_{1}^{N} i^2 \frac{\sigma_i^2}{\sigma_{1,0}^2} \leq \frac{\beta z^{1/2}}{\omega_0}\, 0 \left[\frac{\ln\ (1 + 2/N)}{N}\right] \tag{54}$$

If β is small and N is large, this ratio becomes very small and may be neglected, thus justifying the treatment given in the paper.

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