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IMBALANCED STRONG MHD TURBULENCE

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ABSTRACT

We present a phenomenological model of imbalanced MHD turbulence in an incompressible magnetofluid. The steady–state cascades, of waves traveling in opposite directions along the mean magnetic field, carry unequal energy fluxes to small length scales, where they decay due to viscous and resistive dissipation. The inertial–range scalings are well–understood when both cascades are weak. We study the case when both cascades are, in a sense, *strong*. The inertial–range of this imbalanced cascade has the following properties: (i) the ratio of the r.m.s. Elsasser amplitudes is independent of scale, and is equal to the ratio of the corresponding energy fluxes; (ii) in common with the balanced strong cascade, the energy spectra of both Elsasser waves are of the anisotropic Kolmogorov form, with their parallel correlation lengths equal to each other on all scales, and proportional to the two–thirds power of the transverse correlation length; (iii) the equality of cascade time and waveperiod (critical balance) that characterizes the strong balanced cascade does not apply to the Elsasser field with the larger amplitude. Instead, the more general criterion that always applies to both Elsasser fields is that the cascade time is equal to the correlation time of the straining imposed by oppositely-directed waves. (iv) in the limit that the energy fluxes are equal, the turbulence corresponds to the balanced strong cascade. Our results are particularly relevant for turbulence in the solar wind. Spacecraft measurements have established that, in the inertial range of solar wind turbulence, waves travelling away from the sun have higher amplitudes than those travelling towards it. Result (i) allows us to infer the turbulent flux ratios from the amplitude ratios, thus providing insight into the origin of the turbulence.

Subject headings: MHD — turbulence

1. INTRODUCTION

Magnetohydrodynamic (MHD) turbulence is present in many astronomical settings, such as the solar wind, the interstellar medium, molecular clouds, accretion disks, and clusters of galaxies (Biskamp 2003; Kulsrud 2005; Schekochihin & Cowley 2005). Its theory has attracted a sizable literature (Iroshnikov 1963; Kraichnan 1965; Shebalin et al. 1983; Goldreich & Sridhar 1995, 1997; Ng & Bhattacharjee 1996; Cho & Vishniac 2000; Biskamp & Müller 2000; Maron & Goldreich 2001; Cho et al. 2002; Galtier et al. 2000, 2002; Müller et al. 2003; Galtier et al. 2005; Boldyrev 2005; Müller & Grappin 2005; Beresnyak & Lazarian 2005). The simplest of cases concerns the small–scale dynamics of the excitations of an incompressible fluid with a mean magnetic field. The turbulent cascade of energy to small scales is the result of non linear interactions between Alfvén waves traveling in opposite directions along the local, mean magnetic field (Iroshnikov 1963; Kraichnan 1965). Whereas this broad picture of Iroshnikov and Kraichnan still endures, our appreciation of MHD turbulence has undergone significant revisions due, mainly, to the recognition of the importance of anisotropy and the consequent strengthening of non linear interactions. The inertial–range, which includes length scales between the stirring and dissipation scales, is best understood in those cases in which the oppositely directed waves are excited with equal power: these *balanced* cascades can be *weak* (Ng & Bhattacharjee 1996; Goldreich & Sridhar 1997), as well as *strong* (Goldreich & Sridhar 1995). *Imbalanced* cascades are understood only in the case when the turbulence is *weak* (Galtier et al. 2000; Lithwick & Goldreich 2003). In this Letter we consider the general case of imbalanced cascades.

The solar wind is the best laboratory that we have to investigate MHD turbulence. In-situ measurements by spacecraft yield spectra for velocity and magnetic field fluctuations over many decades of lengthscale (e.g., Horbury 1999). On the largest scales, the spectrum is flat, presumably reflecting the spectrum with which fluctuations are injected into the solar wind by shocks or dynamical instabilities. On smaller scales, the spectrum is Kolmogorov, and fluctuations are thought to be undergoing an active turbulent cascade. On these scales, the amplitudes of the two Elsasser fields are not equal: waves travelling away from the Sun have higher amplitudes than those travelling towards it.[1] Because of this imbalance, the theory of MHD turbulence has been inadequate for application to the solar wind. Our solution for the strong imbalanced cascade removes this inadequacy.

In §2, we summarize the properties of MHD cascades that were previously understood. Our solution for the strong imbalanced cascade is given in §3.

2. BALANCED CASCADES AND THE IMBALANCED WEAK CASCADE

The system we consider is an incompressible magnetofluid of mass density ρ and mean magnetic field $B_0\hat{z}$. Let $\mathbf{v}(\mathbf{r}, t)$ be the fluid velocity, and $\mathbf{b}(\mathbf{r}, t)$ the magnetic field fluctuation. The MHD equations determining their

[1] This imbalance is more pronounced closer to the Sun.

time evolution are expressed conveniently in terms of the Elsasser fields, $\mathbf{w}^{\pm} = \mathbf{v} \mp \mathbf{b}/\sqrt{4\pi\rho}$: [2]

$$(\partial_t \pm V_A \partial_z)\mathbf{w}^{\pm} \;+\; (\mathbf{w}^{\mp} \cdot \boldsymbol{\nabla})\,\mathbf{w}^{\pm} \;=\; -\boldsymbol{\nabla}\,(p/\rho) \qquad (1)$$

where $V_A = B_0/\sqrt{4\pi\rho}$ is the Alfvén speed, and p is the total pressure, determined by requiring $\boldsymbol{\nabla} \cdot \mathbf{w}^{\pm} = 0$. We have neglected viscous and resistive dissipative terms, which are important on small scales; we have also neglected forcing terms, which are important on large scales. When either $\mathbf{w}^+$ or $\mathbf{w}^-$ is initially zero everywhere, the nonlinear term vanishes for all time. Then, either $\{\mathbf{w}^- = \mathbf{w}^-(x, y, z + V_A t),\ \mathbf{w}^+ = \mathbf{0}\}$, or $\{\mathbf{w}^+ = \mathbf{w}^+(x, y, z - V_A t),\ \mathbf{w}^- = \mathbf{0}\}$, is a nonlinear solution of arbitrary form, that propagates in the $-\hat{z}$ or $+\hat{z}$ direction with speed V_A. Kraichnan recognised that the existence of these nonlinear solutions implies that MHD turbulence can be described as interactions between oppositely directed wavepackets. Equations (1) conserve the Elsasser energies, $E^{\pm} = \frac{1}{2}\int |\mathbf{w}^{\pm}|^2\, d^3x$, hence collisions between wavepackets do not lead to exchanges between E^+ and E^-, but only to a redistribution of the energies over different length scales.

We label the r.m.s. amplitudes of the Elsasser fields in the inertial range of the turbulence by $w_\lambda^{\pm}$, where λ is the lengthscale transverse to the mean B-field. MHD turbulence is known to be anisotropic: wavepackets of each type of Elsasser field with transverse scale λ have an extent $\Lambda_\lambda^{\pm}$ parallel to the mean B-field.[3] We note that $\Lambda_\lambda^{\pm}$ is scale-dependent, and is typically much larger than λ. We define the strength of a wavepacket as the ratio of the straining rate that it imposes on oppositely directed waves, $w_\lambda^{\pm}/\lambda$, to its linear wave frequency:

$$\chi_\lambda^{\pm} \;\equiv\; \frac{\Lambda_\lambda^{\pm} w_\lambda^{\pm}}{\lambda V_A}\,. \qquad (2)$$

If a wavepacket of +ve waves has $\chi_\lambda^+ = 1$, then −ve waves with scale λ that encounter it are cascaded in the time that they cross it, assuming that the backreaction of the −ve waves on the +ve ones is negligible.We wish to understand how $\chi_\lambda^{\pm}$ influence the inertial–range scalings of physical quantities. Below we provide a brief review of the cases that are understood to some extent; this enables us to pose our question more sharply.[4]

1. *Balanced weak turbulence*, $\chi_\lambda \ll 1$: The turbulent cascade is due to resonant 3–wave interactions (Ng & Bhattacharjee 1996; Goldreich & Sridhar 1997; Galtier et al. 2000; Lithwick & Goldreich 2003). The r.m.s. amplitude across a transverse scale λ is $w_\lambda \propto \lambda^{1/2}$. Frequency resonance conditions prevent a parallel cascade, so the parallel scale is independent of λ; its value is set by conditions at the stirring scale. Perturbation theory is valid when $\chi_\lambda \ll 1$, and can be used to derive kinetic equations, describing the inertial–range. However, the cascade strengthens on small scales, because $\chi_\lambda \propto 1/\lambda^{1/2}$ increases as λ decreases; this limits the validity of perturbation theory, and the inertial range of weak turbulence.

2. *Balanced strong turbulence*, $\chi_\lambda \sim 1$: Perturbation theory is inapplicable. According to the phenomenological theory of Goldreich & Sridhar (1995), the strength of the interactions remains of order unity: $\chi_\lambda \sim 1$ ("critical balance"). The cascade is of anisotropic Kolmogorov form, with $w_\lambda \propto \lambda^{1/3}$ and $\Lambda_\lambda \propto \lambda^{2/3}$. Turbulence that is excited with $\chi_\lambda > 1$, weakens to $\chi_\lambda \sim 1$ in less than the Alfvén crossing time.

3. *Imbalanced weak turbulence*, $\chi_\lambda^{\pm} \ll 1$: In common with the balanced case, the process can be described in detail, using the weak turbulence theory of resonant 3–wave interactions (Galtier et al. 2000; Lithwick & Goldreich 2003). As before, the product $w_\lambda^+ w_\lambda^- \propto \lambda$, and frequency resonance conditions forbid a parallel cascade, so the two parallel scales $\Lambda_\lambda^{\pm}$ are scale-independent. However, in contrast to the balanced case, (i) kinetic equations are required to relate the spectral indices of the oppositely directed waves to the ratio of their fluxes; (ii) the ratio of w_λ^+ to w_λ^- in the inertial range depends on the ratio of λ to the dissipation scale, $\lambda_{\rm diss}$, since the two amplitudes are forced to equal one another at $\lambda_{\rm diss}$.

As discussed in Lithwick & Goldreich (2003), when the dissipation scale is large enough (while still remaining smaller than the stirring scale), imbalanced turbulence can be weak throughout the inertial–range. However, both χ_λ^+ and χ_λ^- increase with decreasing λ, thereby limiting the inertial range. Therefore, in the physically important limit of very small dissipation, it is inevitable that at least one of the +ve or −ve cascades becomes strong.

3. THE IMBALANCED STRONG CASCADE

In the present section, the heart of this paper, we derive the spectrum of imbalanced strong turbulence. We assume that $w_\lambda^+ \gg w_\lambda^-$.[5]

Let $\chi_\lambda^+ \geq 1$. Otherwise, the turbulence would be weak. The +ve waves impose a strain on −ve ones, with straining rate $\sim w_\lambda^+/\lambda$. Since $\chi_\lambda^+ \geq 1$, this strain is nearly constant over the cascade time, τ_λ^-, of the −ve waves. Thus $\tau_\lambda^- \sim \lambda/w_\lambda^+$ and the energy flux in the −ve cascade is

$$\varepsilon^- \;\sim\; \frac{(w_\lambda^-)^2}{\tau_\lambda^-} \;\sim\; \frac{(w_\lambda^-)^2 w_\lambda^+}{\lambda}\,. \qquad (3)$$

Waves of w_λ^- travel a parallel distance $V_A \tau_\lambda^-$ before cascading to smaller λ. It *follows* that a wavepacket with given transverse size λ has parallel size

$$\Lambda_\lambda^- \;\sim\; V_A \tau_\lambda^- \;\sim\; \left(\frac{V_A}{w_\lambda^+}\right)\lambda\,. \qquad (4)$$

Since this relation mixes Λ_λ^- and w_λ^+, it does not amount to critical balance for either the +ve or −ve waves.

Equation (4), together with our starting point $\chi_\lambda^+ \geq 1$, imply that $\Lambda_\lambda^+ \gtrsim \Lambda_\lambda^-$. But, since −ve waves backreact on the +ve ones, they impose uncorrelated strains on parts of a + wavepacket separated in z by $\gtrsim \Lambda_\lambda^-$. Thus they imprint their parallel scales on the +ve waves, and

$$\Lambda_\lambda^+ \;\sim\; \Lambda_\lambda^-\,. \qquad (5)$$

[2] The superscripts on $\mathbf{w}^+$, $\mathbf{w}^-$ refer to the direction of wave propagation.

[3] A wavepacket can be defined more precisely in terms of correlation lengths perpendicular and parallel to the mean field (Lithwick & Goldreich 2001).

[4] In the balanced cases, we set $w_\lambda^+ = w_\lambda^- = w_\lambda$ and $\Lambda_\lambda^+ = \Lambda_\lambda^- = \Lambda_\lambda$ which implies that $\chi_\lambda^+ = \chi_\lambda^- = \chi_\lambda = (\Lambda_\lambda w_\lambda/\lambda V_A)$.

[5] This assumption is made only for clarity of presenation. Our discussion is more generally valid, for all $w_\lambda^+ \geq w_\lambda^-$.

Henceforth we denote the common parallel scales by Λ_λ. This *discovery* that the parallel scales are similar in an imbalanced cascade is a *nontrivial* result. Equations (4) and (5) yield

$$\chi_\lambda^+ = \frac{w_\lambda^+ \Lambda_\lambda}{V_A \lambda} \sim 1 , \tag{6}$$

proving that the cascade of −ve waves is critically balanced.

It remains to calculate the +ve waves' cascade time. All of the following material up to equation (9), as well as the material in the Appendix, is devoted to *deriving* the result: $\tau_\lambda^+ \sim \lambda/w_\lambda^-$. This result is remarkable. It shows that the straining rate imposed by the −ve waves on +ve ones, w_λ^-/λ, is imposed coherently over a time λ/w_λ^-. Yet the waveperiod of −ve waves is much shorter than this, by the factor

$$\chi_\lambda^- = \frac{w_\lambda^- \Lambda_\lambda}{V_A \lambda} \sim \frac{w_\lambda^-}{w_\lambda^+} \ll 1 . \tag{7}$$

Since $\chi_\lambda^- \ll 1$, one might be tempted to conclude, erroneously, that +ve waves undergo a weak cascade, i.e., that −ve waves impose on them small, short uncorrelated strains of amplitude χ_λ^- over time intervals $\sim \Lambda_\lambda/V_A$ resulting in $\tau_\lambda^+ \sim (\chi_\lambda^-)^{-2}(\Lambda_\lambda/V_A) \sim (\chi_\lambda^-)^{-1}(\lambda/w_\lambda^-)$. Instead, the correct conclusion is that a coherent strain is imposed over time interval λ/w_λ^-.

How can the coherence time exceed Λ_λ/V_A? The key point is that the straining of the +ve waves is due to the $\mathbf{w}^-$ field as seen from the +ve waves' rest frame (which has $x' = x,\ y' = y,\ z' = z - V_A t,\ t' = t$). In this frame, the MHD equations (1) transform to

$$\begin{aligned} \partial_{t'} \mathbf{w}^+ + (\mathbf{w}^- \cdot \boldsymbol{\nabla}')\, \mathbf{w}^+ &= -\boldsymbol{\nabla}'\,(p/\rho) \\ (\partial_{t'} - 2V_A \partial_{z'})\mathbf{w}^- + (\mathbf{w}^+ \cdot \boldsymbol{\nabla}')\, \mathbf{w}^- &= -\boldsymbol{\nabla}'\,(p/\rho) \end{aligned} \tag{8}$$

To appreciate that the correlation time of $\mathbf{w}^-$ in the primed frame can exceed Λ_λ/V_A, consider the limiting case in which $\mathbf{w}^-$ is so small that backreaction onto the +ve waves can be neglected. Then $\mathbf{w}^+$ is independent of t', and is a function only of $\mathbf{r}'$, so $\mathbf{w}^-$ satisfies a linear (integro–differential) equation, whose coefficients are independent of t'. If −ve waves are injected on a lengthscale much larger than the scale of interest, with a long coherence time T, then as they cascade down to smaller scales their coherence time remains fixed, $\tau^-_{\mathrm{corr},\lambda} = T$, where $\tau^-_{\mathrm{corr},\lambda}$ is defined as the correlation time of the −ve waves in the frame of the +ve waves. In the limiting case that $\mathbf{w}^-$ is held fixed at the injection scale ($T = \infty$), then on smaller scales $\mathbf{w}^-$ is independent of t' ($\tau^-_{\mathrm{corr},\lambda} = \infty$), even though it is undergoing an active cascade to small scales.

To estimate $\tau^-_{\mathrm{corr},\lambda}$ when $\mathbf{w}^-$ is not infinitesimally small, it is necessary to account for backreaction: −ve waves alter +ve waves, which react back on the −ve ones. As −ve waves cross a plane at fixed z', the +ve waves at that plane are changing on their cascade time scale τ_λ^+. Hence, over times separated by τ_λ^+, the −ve waves crossing z' are cascaded by entirely different +ve waves. This implies that $\tau^-_{\mathrm{corr},\lambda} \sim \tau_\lambda^+$.[6] Because the +ve waves are strained at rate w_λ^-/λ, it follows that $\tau_\lambda^+ \sim \lambda/w_\lambda^-$. Thus

$$\varepsilon^+ \sim \frac{(w_\lambda^+)^2}{\tau_\lambda^+} \sim \frac{(w_\lambda^+)^2 w_\lambda^-}{\lambda} . \tag{9}$$

Invoking Kolmogorov's hypothesis of the scale (i.e. λ) independence of the energy fluxes given by equations (3) and (9), we obtain the inertial–range scalings,

$$w_\lambda^\pm \sim \frac{(\varepsilon^\pm)^{2/3}}{(\varepsilon^\mp)^{1/3}} \lambda^{1/3} ; \qquad \Lambda_\lambda \sim \frac{(\varepsilon^-)^{1/3}}{(\varepsilon^+)^{2/3}} V_A \lambda^{2/3} . \tag{10}$$

The +ve wave cascade shares some characteristics with both weak and strong balanced MHD cascades. In the weak cascades, the cascade time is longer than the waveperiod, and a wave experiences multiple, randomly-phased perturbations during its cascade time. In the strong cascades, the cascade time is comparable to (or shorter than) the waveperiod, and a wave suffers a coherent strain as it cascades. Furthermore, weak turbulence submits to perturbation theory but strong turbulence does not. In the +ve wave cascade:

- The period of a +ve wave is shorter than its cascade time.
- A +ve wave is coherently strained as it cascades.[7]
- The +ve wave cascade is non-perturbative.

We contend that the +ve wave cascade is strong because the second and third items have dynamical significance whereas the first does not. The dimensionless parameter that indicates whether the +ve waves are strongly cascaded is

$$\hat{\chi}_\lambda^- \equiv \frac{w_\lambda^- \tau^-_{\mathrm{corr},\lambda}}{\lambda} , \tag{11}$$

and not χ_λ^-. Strong cascades correspond to $\hat{\chi}_\lambda^- \sim 1$, and weak ones to $\hat{\chi}_\lambda^- < 1$. For the −ve wave cascade, $\hat{\chi}_\lambda^+ = \chi_\lambda^+ \sim 1$, since the correlation time of +ve waves in the frame of the −ve ones is Λ_λ/V_A. We call the criterion $\hat{\chi}_\lambda^\pm \sim 1$ "modified critical balance," to distinguish it from critical balance (which would incorrectly imply $\chi_\lambda^\pm \sim 1$).

4. SUMMARY

We have deduced the behavior of imbalanced strong MHD turbulence. Its salient properties are:

1. The +ve and −ve waves carry unequal energy fluxes, $\varepsilon^+ \neq \varepsilon^-$, while they both undergo *strong* cascades.
2. In the inertial–range, the r.m.s. Elsasser amplitudes are proportional to the one–third power of the transverse scale: $w_\lambda^\pm \propto \lambda^{1/3}$. This is similar to the balanced, strong cascade. Moreover, their ratio, $w_\lambda^+/w_\lambda^- \sim \varepsilon^+/\varepsilon^-$ is independent of λ.
3. The parallel scales of the +ve and −ve waves are equal. The common parallel scale of eddies of transverse scale λ, is $\Lambda_\lambda \propto \lambda^{2/3}$, similar to the balanced, strong cascade.
4. The cascade times of the (larger amplitude) +ve waves are longer than their waveperiods by the constant factor $(\varepsilon^+/\varepsilon^-) \geq 1$, independent of scale: unlike the imbalanced, weak cascade, there is no tendency for the cascade to strengthen at small λ. When $\varepsilon^+ = \varepsilon^-$, the the turbulence corresponds to the balanced strong cascade of Goldreich & Sridhar (1995).

[6] We are implicitly assuming that most of the change in w_λ^- is accumulated on scales comparable to λ. We justify this assumption quantitatively in the Appendix.

[7] The correlation time of the strain induced by the −ve waves, $\tau^-_{\mathrm{corr},\lambda}$, is comparable to the cascade time, τ_λ^+, of the +ve waves.

APPENDIX

BACKREACTION AND THE CORRELATION TIME OF THE −VE WAVES

Let $\tau_\lambda^+ \sim \Lambda'_\lambda/V_A$ denote the time scale over which w_λ^+ varies at fixed z'. Below we prove that (i) the correlation time of the −ve waves, at fixed z', is $\tau^-_{\mathrm{corr},\lambda} \sim \tau_\lambda^+$; (ii) $\Lambda'_\lambda/\Lambda_\lambda \sim \varepsilon^+/\varepsilon^-$.

During the time interval $\tau^+_{\lambda_*} \sim \Lambda'_{\lambda_*}/V_A$, −ve waves cascade from transverse scales $\lambda_{\max}$ to λ_*. We approximate this cascade as taking place in discrete steps of duration $\tau_\lambda^- \sim \Lambda_\lambda/V_A$ in each of which λ decreases by a constant factor of order unity. Thus $\lambda_{\max}$ is related to λ_* by

$$\int_{\lambda_*}^{\lambda_{\max}} \frac{d\lambda}{\lambda}\,\Lambda_\lambda \;\sim\; \Lambda'_{\lambda_*} \tag{A1}$$

If Λ'_λ increases with λ, equation (A1) implies that

$$\Lambda'_{\lambda_*} \;\sim\; \Lambda_{\lambda_{\max}}\,; \tag{A2}$$

most of the cascade time is spent near scales $\sim \lambda_{\max}$.

The +ve waves encountered by −ve waves, at the same z' and λ, but separated in time by $\tau^+_{\lambda_*}$, differ by $(\delta w_\lambda^+/w_\lambda^+) \sim \Lambda'_{\lambda_*}/\Lambda'_\lambda$.[8] Thus over each step of the cascade from $\lambda_{\max}$ to λ_*,

$$\delta w_\lambda^- \;\sim\; \frac{\delta w_\lambda^+}{\lambda}\frac{\Lambda_\lambda}{V_A}\,w_\lambda^- \;\sim\; \frac{\Lambda'_{\lambda_*}}{\Lambda'_\lambda}\,w_\lambda^-\,, \tag{A3}$$

where we have used, $\chi_\lambda^+ \sim (w_\lambda^+\Lambda_\lambda/V_A\lambda) \sim 1$. Then the mean square fractional variation of $w^-_{\lambda_*}$ accumulated during the the entire cascade amounts to

$$\left(\frac{\delta w^-_{\lambda_*}}{w^-_{\lambda_*}}\right)^2 \;\sim\; \int_{\lambda_*}^{\lambda_{\max}} \frac{d\lambda}{\lambda}\left(\frac{\Lambda'_{\lambda_*}}{\Lambda'_\lambda}\right)^2\left(\frac{w_\lambda^-}{w^-_{\lambda_*}}\right)^2. \tag{A4}$$

Provided $\Lambda'_\lambda/w_\lambda^-$ increases with λ,

$$\frac{\delta w^-_{\lambda_*}}{w^-_{\lambda_*}} \;\sim\; 1\,. \tag{A5}$$

Although most of the cascade time is spent near $\lambda_{\max}$ (see equation A2), the accumulated change in $w^-_{\lambda_*}$ comes from scales near λ_*. Moreover, since $w^-_{\lambda_*}$ varies by order unity, at fixed z', during the time interval, τ_λ^+, the correlation time of the −ve waves is,

$$\tau^-_{\mathrm{corr},\lambda} \;\sim\; \tau_\lambda^+ \;\sim\; \frac{\lambda}{w_\lambda^-}\,, \tag{A6}$$

which proves item (i). To prove (ii), we note that

$$\frac{\Lambda'_\lambda}{\Lambda_\lambda} \;\sim\; \frac{\tau_\lambda^+}{\tau_\lambda^-} \;\sim\; \frac{w_\lambda^+}{w_\lambda^-} \;\sim\; \frac{\varepsilon^+}{\varepsilon^-} \;\geq\; 1\,, \tag{A7}$$

where we have used equations (A6) and (10).

[8] We reserve the symbol δ to denote differences accrued over the time interval $\tau^+_{\lambda_*}$.

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