THE ASTROPHYSICAL JOURNAL, 222:850–858, 1978 June 15

THE EXCITATION AND EVOLUTION OF DENSITY WAVES*

PETER GOLDREICH AND SCOTT TREMAINE
California Institute of Technology
Received 1977 August 16; accepted 1978 January 4

ABSTRACT

We study the linear oscillations of a thin self-gravitating gas sheet. The unperturbed velocity field of the sheet is a parallel shear flow. A Coriolis acceleration is included to simulate the effects of rotation. The sheet exhibits Lindblad resonances, and it can sustain both short and long wavelength density waves.

We derive equations which govern the excitation and evolution of density waves in all regions of space, including the Lindblad resonances and the forbidden region around corotation. These equations are solved in the tight winding limit.

An initial disturbance in the form of a wave packet of short leading waves evolves as follows. The packet propagates toward corotation, is reflected at the boundary of the forbidden region, and becomes a packet of long leading waves. It then travels back to the Lindblad resonance, where it is reflected and becomes a packet of long trailing waves. Subsequently, this packet moves toward corotation and is reflected again at the boundary of the forbidden region. The packet is now made up of short trailing waves and propagates away from corotation indefinitely.

For sufficiently stable disks, the forbidden region around corotation is wide and density waves are almost completely reflected at its boundaries. For marginally stable disks, some of the incident wave tunnels through the forbidden region and the reflected wave is amplified.

The excitation of density waves by an arbitrary external potential is considered. In our model sheet, the sole effect of a barlike potential is to excite the long trailing wave at the Lindblad resonances. The amplitude of the excited wave is calculated.

Subject headings: galaxies: internal motions — galaxies: structure — hydrodynamics — stars: stellar dynamics

I. INTRODUCTION

Our aim is to describe the behavior of density waves in a simple analog of a rotating disk. In § II, we describe the model and set up the linear perturbation equations. We state results obtained from previous studies of these equations in a form appropriate to our model. In § III, the equations are rewritten in terms of the comoving coordinates of the unperturbed flow. The resulting equations are Fourier transformed and then solved in the tight winding limit. The excitation of a wave packet by an external potential and its subsequent propagation are calculated. In § IV, we discuss our results and their implications for spiral structure in galaxies.

It is important to bear in mind that this paper is about density waves in a special model gas sheet. It is not about density waves in galaxies. Of course, we hope that the behavior of density waves in our model will mimic their behavior in galaxies. This hope remains to be justified. The incentive for studying density waves in the model sheet is analytic simplicity. We are able to solve initial value problems describing the excitation and subsequent evolution of wave packets. Our solutions are valid everywhere including the Lindblad and corotation resonances.

II. THE MODEL

The model we analyze is a thin gas sheet whose unperturbed velocity field is a parallel shear flow. To simulate the effects of rotation, we include a Coriolis acceleration. This model approximates the local dynamics of a differentially rotating disk (Goldreich and Lynden-Bell 1965; Julian and Toomre 1966). Toomre (1969) describes its advantages and limitations. Most important for our purposes are that the model exhibits Lindblad resonances and that its linear perturbations include density waves. Although it is a fluid system, its behavior is similar in many respects to that of a stellar system such as a disk galaxy.

a) Basic Equations

The continuity and Euler equations read

$$\frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma V) = 0 \tag{1}$$

* Contribution 2958 of the Division of Geological and Planetary Sciences, California Institute of Technology, Pasadena, CA 91125.

and

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = -\frac{1}{\Sigma}\nabla P - 2\boldsymbol{\Omega}_p \times \boldsymbol{V} - \nabla\Psi\,, \tag{2}$$

where Σ is the surface density, P is the two-dimensional pressure, $\boldsymbol{V}$ is the two-dimensional velocity, $\nabla = \boldsymbol{i}\partial/\partial x + \boldsymbol{j}\partial/\partial y$, $\boldsymbol{\Omega}_p = \Omega_p\boldsymbol{k}$ is the Coriolis parameter, and Ψ is the gravitational potential. The unperturbed pressure P_0 and surface density Σ_0 are taken to be homogeneous, and the unperturbed velocity is chosen to be

$$\boldsymbol{V}_0 = 2Ax\boldsymbol{j}\,, \tag{3}$$

where $A > 0$ is the Oort constant. The Coriolis acceleration acting on the unperturbed flow is balanced by the unperturbed gravitational acceleration $-\nabla\Psi_0$. *Note.*—With our sign convention, $\Omega_p < 0$.

The continuity and Euler equations governing linear perturbations are

$$\frac{\partial\sigma}{\partial t} + 2Ax\frac{\partial\sigma}{\partial y} + \Sigma_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0\,, \tag{4}$$

$$\frac{\partial u}{\partial t} + 2Ax\frac{\partial u}{\partial y} - 2\Omega_p v = -\frac{\partial\chi}{\partial x}\,, \tag{5}$$

$$\frac{\partial v}{\partial t} + 2Ax\frac{\partial v}{\partial y} + 2Bu = -\frac{\partial\chi}{\partial y}\,, \tag{6}$$

where the Oort constant $B \equiv \Omega_p + A$. The perturbation variables are denoted by lowercase symbols: $\Sigma = \Sigma_0 + \sigma$, $P = P_0 + p$, $\boldsymbol{V} = \boldsymbol{V}_0 + u\boldsymbol{i} + v\boldsymbol{j}$, and $\Psi = \Psi_0 + \psi$. The accelerations due to pressure and gravity are written as $-\nabla\chi$, with

$$\chi \equiv a^2\frac{\sigma}{\Sigma_0} + \psi + \psi_{\rm ex} \tag{7}$$

and $p = a^2\sigma$, where a is the sound speed; ψ is the potential due to perturbations of the disk, and $\psi_{\rm ex}$ is an external potential. Poisson's equation

$$\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right)\psi = 4\pi G\sigma\delta(z) \tag{8}$$

completes the set of linear equations for the perturbation variables σ, u, v, ψ, and χ. Henceforth, we drop the subscript from the unperturbed variables.

The remainder of this section is devoted to examining the free oscillations ($\psi_{\rm ex} = 0$) of the disk. We state several well-known results without proof in order to familiarize the reader with their appearance when applied to our model disk. These results were first obtained in studies of stellar disks, but they are easily modified to apply to gas disks (Lin and Shu 1968).

b) Gravitational Stability

Elementary plane waves of the form $\exp[i(k_x x - \omega t)]$, with k_x and ω constants, are solutions of equations (4)–(8) provided that k_x and ω satisfy the dispersion relation

$$\omega^2 = \kappa^2 - 2\pi G\Sigma|k_x| + k_x^2a^2\,, \tag{9}$$

where $\kappa \equiv 2(B\Omega_p)^{1/2}$ is the epicyclic frequency. The necessary criterion for stability is

$$\frac{1}{Q} \equiv \frac{\pi G\Sigma}{\kappa a} < 1 \tag{10}$$

(Toomre 1964). This criterion is also sufficient (Kalnajs 1965). We consider only stable ($Q > 1$) disks in this paper.

c) WKB Density Waves

Consider stationary perturbations of the form $\exp\left[i\left(\int^x k_x(s)ds + k_y y\right)\right]$, where $k_y > 0$ is a constant. For $\pi G\Sigma k_y/\kappa^2 \ll 1$ and $k_y a/\kappa \ll 1$, equations (4)–(8) admit WKB solutions of this form, provided $k_x(x)$ and k_y are related by

$$(2Axk_y)^2 = \kappa^2 - 2\pi G\Sigma|k_x| + k_x^2a^2 \tag{11}$$

and $|k_x|/k_y \gg 1$. Equation (11) may be recast in a more familiar form by setting $k_y = m/R$ and $2Ax = R(\Omega - \Omega_p)$. Here, we identify R as the disk radius, m as the number of spiral arms, and Ω as the local angular velocity. Our Coriolis parameter plays the role of the pattern speed. With these definitions, equation (11) is identical to the dispersion relation for tightly wound spiral density waves (Lin and Shu 1966). Corotation is at $x = 0$, and the Lindblad resonances occur at $\pm x_L$, where

$$x_L \equiv \kappa/(2Ak_y)\,. \tag{12}$$

The dispersion relation (eq. [11]) yields two solutions for $|k_x|$:

$$\frac{|k_x|}{k_c} = 1 \pm \left\{1 - Q^2\left[1 - \left(\frac{x}{x_L}\right)^2\right]\right\}^{1/2} = 1 \pm Q\left\{\frac{x^2 - x_F{}^2}{x_L{}^2}\right\}^{1/2}, \tag{13}$$

where $|x| = x_F \equiv (Q^2 - 1)^{1/2}x_L/Q$ is the boundary of the forbidden region, $k_c \equiv \pi G\Sigma/a^2$, and the two solutions ($\pm$) are called the short and long waves. The short wave propagates for $|x| > x_F$, and the long wave propagates for $x_F < |x| < x_L$. The existence of the short wave for $|x| > x_L$ is a special feature of gas disks which is not shared by stellar disks. We define waves as leading or trailing spirals according to whether $k_x > 0$ or $k_x < 0$.

The group velocity of the waves in the x-direction is c_g, where

$$\frac{c_g}{a} = \mp \operatorname{sgn}(xk_x)\left[1 - \left(\frac{x_F}{x}\right)^2\right]^{1/2}, \tag{14}$$

and the upper and lower signs apply to the short and long waves (Toomre 1969). Inside corotation ($x < 0$), $c_g > 0$ for the long trailing and short leading waves and $c_g < 0$ for the short trailing and long leading waves. Outside corotation ($x > 0$), the sign of c_g is reversed for each wave type.

Because a localized wave packet propagates radially, the x component of its wave vector k_x evolves with time. From equations (13) and (14), we have

$$\frac{dk_x}{dt} = \frac{\partial k_x}{\partial x}c_g = -2Ak_y\,. \tag{15}$$

Thus, the density waves wind up as they propagate (Toomre 1969). A special feature of our model is that the winding rate is exactly the shear rate of the unperturbed flow.

The normalized surface density perturbation $\theta(x, y) \equiv \sigma(x, y)/\Sigma$ due to a steady wave train may be written as

$$\theta(x, y) = |\theta(x)| \exp\left[i\left(\int^x k_x(s)ds + k_y y\right)\right]\cdot \tag{16}$$

As a consequence of the conservation of the flux of wave action, the amplitude $|\theta(x)|$ satisfies (Toomre 1969; Shu 1970)

$$\frac{|\theta(x)|\,|k_c - |k_x(x)||^{1/2}}{k_x(x)} = \text{constant}\,. \tag{17}$$

This result may be derived from the second order WKB solutions of equations (4)–(8). It implies that both the energy flux and the angular momentum flux of a density wave are conserved.

III. EXCITATION AND EVOLUTION OF WAVES

At this point we abandon the WKB method so successfully exploited in the past, especially by Lin and Shu (1966). Instead, we rely on techniques developed by Goldreich and Lynden-Bell (1965) and by Julian and Toomre (1966) for use in similar problems.

a) The Solution of the Basic Equations

The basic equations (4)–(8) are expressed in the comoving coordinates of the unperturbed flow

$$(x, y' = y - 2Axt, z, t) \tag{18}$$

as

$$\frac{\partial\sigma}{\partial t} + \Sigma\left[\left(\frac{\partial}{\partial x} - 2At\frac{\partial}{\partial y'}\right)u + \frac{\partial v}{\partial y'}\right] = 0\,, \tag{19}$$

$$\frac{\partial u}{\partial t} - 2\Omega_p v = -\left(\frac{\partial}{\partial x} - 2At\frac{\partial}{\partial y'}\right)\chi\,, \tag{20}$$

$$\frac{\partial v}{\partial t} + 2Bu = -\frac{\partial\chi}{\partial y'}\,, \tag{21}$$

$$\left[\left(\frac{\partial x}{\partial x} - 2At\frac{\partial}{\partial y'}\right)^2 + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z^2}\right]\psi = 4\pi G\sigma\delta(z)\,. \tag{22}$$

Equations (19)–(22) admit solutions whose spatial dependence is of the form exp $[i(k_x'x + k_y y')]$. Substituting this solution into equations (19)–(22) and changing to a new time variable

$$\tau \equiv 2At - k_x'/k_y\,, \tag{23}$$

we obtain

$$\frac{d\tilde{\sigma}}{d\tau} + \frac{i\Sigma k_y}{2A}(\tilde{v} - \tau\tilde{u}) = 0\,, \tag{24}$$

$$\frac{d\tilde{u}}{d\tau} - \frac{\Omega_p}{A}\tilde{v} = \frac{ik_y}{2A}\tau\tilde{\chi}\,, \tag{25}$$

$$\frac{d\tilde{v}}{d\tau} + \frac{B}{A}\tilde{u} = -\frac{ik_y}{2A}\tilde{\chi}\,, \tag{26}$$

$$k_y^2(1 + \tau^2)\tilde{\psi} - \frac{\partial^2\tilde{\psi}}{\partial z^2} = -4\pi G\tilde{\sigma}\delta(z)\,. \tag{27}$$

Here $\tilde{\sigma}$, $\tilde{u}$, $\tilde{v}$, $\tilde{\chi}$, and $\tilde{\psi}$, which denote the amplitudes of the Fourier components, are functions of τ, k_x', and k_y.

The next step is to reduce equations (23)–(27) to a single second order differential equation for $\tilde{\sigma}$. We multiply equation (26) by τ and add the resulting equation to (25) to obtain

$$\frac{d}{d\tau}(\tilde{u} + \tau\tilde{v}) - \frac{B}{A}(\tilde{v} - \tau\tilde{u}) = 0\,. \tag{28}$$

We combine this equation with the continuity equation and integrate once to find

$$ik_y(\tilde{u} + \tau\tilde{v})/2B = -\tilde{\sigma}/\Sigma\,. \tag{29}$$

This equation expresses the conservation of vorticity. We have set the constant of integration to zero, because the perturbed vorticity is zero when there are only gravitational forces.

We eliminate $\tilde{u}$ and $\tilde{v}$ from equations (24), (25), and (29) to obtain

$$\frac{d}{d\tau}\left[\frac{1}{(1+\tau^2)}\frac{d\tilde{\theta}}{d\tau}\right] + \left[\frac{2B}{A(1+\tau^2)^2} + \frac{B\Omega_p}{A^2(1+\tau^2)}\right]\tilde{\theta} = -\left(\frac{k_y}{2A}\right)^2\tilde{\chi}\,, \tag{30}$$

where

$$\tilde{\theta} \equiv \tilde{\sigma}/\Sigma\,. \tag{31}$$

We express $\tilde{\chi}$ in terms of $\tilde{\theta}$ and $\tilde{\psi}_{\rm ex}$ by solving Poisson's equation and using equation (7). The solution of Poisson's equation (eq. [27]) gives

$$\tilde{\psi} = -\frac{2\pi G\Sigma}{k_y(1+\tau^2)^{1/2}}\tilde{\theta} \tag{32}$$

for $z = 0$ so that

$$\tilde{\chi} = \left(-\frac{2\pi G\Sigma}{k_y(1+\tau^2)^{1/2}} + a^2\right)\tilde{\theta} + \tilde{\psi}_{\rm ex}\,. \tag{33}$$

Using equation (33) to eliminate $\tilde{\chi}$ from equation (30), we find

$$\frac{d}{d\tau}\left[\frac{1}{(1+\tau^2)}\frac{d\tilde{\theta}}{d\tau}\right] + \left[\frac{2B}{A(1+\tau^2)^2} + \frac{B\Omega_p}{A^2(1+\tau^2)} - \frac{\pi G\Sigma k_y}{2A^2(1+\tau^2)^{1/2}} + \left(\frac{k_y a}{2A}\right)^2\right]\tilde{\theta} = -\left(\frac{k_y}{2A}\right)^2\tilde{\psi}_{\rm ex}\,. \tag{34}$$

Equation (34) is almost identical to equations (72), (94), and (97) of Goldreich and Lynden-Bell (1965). The only differences are in the terms involving the surface density and sound speed and occur because the thin-disk limit was not used in the earlier paper.

The evolution equation (34) may be written in a more revealing form by changing the dependent variable from $\tilde{\theta}$ to $\tilde{\psi}$ with the aid of equation (32). We find

$$\frac{d^2\tilde{\psi}}{dt^2} + S^2(t)\tilde{\psi} = \frac{-2\pi G\Sigma k_y\tilde{\psi}_{\rm ex}}{(1+\tau^2)^{1/2}}\,, \tag{35a}$$

where the effective spring rate

$$S^2(t) = \kappa^2 - 2\pi G\Sigma k(t) + k^2(t)a^2 + 12A^2(1 + \tau^2)^{-2} + 8\Omega_p A(1 + \tau^2)^{-1} \quad (35b)$$

with

$$k(t) = k_y(1 + \tau^2)^{1/2} . \quad (35c)$$

The relation between τ and t is given by equation (23). The first three terms in the expression for $S^2(t)$ are the familiar trio which govern unsheared oscillations ($k_x \neq 0$, $k_y = 0$; cf. eq. [9]). The minimum value of the trio is reached when $k(t) = k_{\min} = \kappa/(Qa)$ and equals $\kappa^2(1 - Q^{-2})$. For sufficiently large $|\tau|$ the $k^2(t)a^2$ term dominates and $S^2(t)$ is positive. However, even for stable sheets ($Q > 1$), $S^2(t)$ may be negative for small t. When $S^2(t) < 0$, the homogeneous solutions of equation (35a) exhibit transient growth. Numerical solutions of equations analogous to equation (35a) by Goldreich and Lynden-Bell (1965) demonstrate that sheared wavelets can achieve impressive transient amplification as $k(t)$ swings past its minimum value k_y. Similar results were reported by Julian and Toomre (1966) from their study of star sheets. In § III*f* we relate this old amplifier to the new amplifier analyzed by Mark (1974, 1976).

The general solution of equation (34) may be written in terms of its homogeneous solutions $\tilde{\theta}_+(\tau)$, $\tilde{\theta}_-(\tau)$ as

$$\tilde{\theta}(k_x', \tau) = \alpha\tilde{\theta}_+(\tau) + \beta\tilde{\theta}_-(\tau) - \frac{1}{C}\left(\frac{k_y}{2A}\right)^2 \int_{-\infty}^{\tau} d\delta[\tilde{\theta}_+(\tau)\tilde{\theta}_-(\delta) - \tilde{\theta}_-(\tau)\tilde{\theta}_+(\delta)]\tilde{\psi}_{\rm ex}(k_x', \delta) , \quad (36)$$

where α and β are arbitrary constants and C is a constant determined by $C \equiv [\tilde{\theta}_-(\tau)d\tilde{\theta}_+(\tau)/d\tau - \tilde{\theta}_+(\tau)d\tilde{\theta}_-(\tau)/d\tau]/(1 + \tau^2)$. Henceforth, we assume that the homogeneous solutions are normalized such that $C = 2i$.

We restrict our attention to a single value of k_y. This corresponds to fixing the number of spiral arms $m = k_y R$ in the analogous disk model. We define reduced dependent variables from which the y or y' dependence is removed by

$$\theta(x, t) \equiv \theta(x, y, t) \exp(-ik_y y) , \quad (37a)$$

$$\theta'(x, t) \equiv \theta(x, y', t) \exp(-ik_y y') . \quad (37b)$$

Similar definitions apply to σ, u, v, χ, and $\psi_{\rm ex}$. To obtain the general solution for $\theta(x, t)$, we use the relations

$$\theta'(x, t) = \int_{-\infty}^{\infty} dk_x'\tilde{\theta}[k_x', \tau(t, k_x')] \exp(ik_x'x) , \quad (38)$$

$$\theta(x, t) = \theta'(x, t) \exp[ik_y(y' - y)] = \theta'(x, t) \exp(-2ik_y Axt) , \quad (39)$$

$$\tilde{\psi}_{\rm ex}(k_x', \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\psi_{\rm ex}[x, t(\tau, k_x')] \exp(ik_y\tau x) , \quad (40)$$

and equations (18), (23), and (36), from which it follows that

$$\theta(x, t) = k_y \int_{-\infty}^{\infty} d\tau[\alpha\tilde{\theta}_+(\tau) + \beta\tilde{\theta}_-(\tau)] \exp[-ik_y\tau x] + \frac{ik_y^3}{16\pi A^2}\int_{-\infty}^{\infty} d\tau \exp(-ik_y\tau x)$$
$$\times \int_{-\infty}^{\tau} d\delta[\tilde{\theta}_+(\tau)\tilde{\theta}_-(\delta) - \tilde{\theta}_-(\tau)\tilde{\theta}_+(\delta)] \int_{-\infty}^{\infty} ds\tilde{\psi}_{\rm ex}[s, t - (\tau - \delta)/2A] \exp[ik_y\delta s] . \quad (41)$$

Equations (34) and (41) are the central equations of this investigation. They govern the excitation and evolution of spiral density waves of arbitrary opening angle, and they remain valid at the corotation and Lindblad resonances.

b) Tightly Wound Spirals

The requirements for the validity of the tight winding approximation as applied to perturbations of our model sheet are

$$\pi G\Sigma k_y/\kappa^2 \ll 1 \quad (42)$$

and

$$ak_y/\kappa \ll 1 \quad (43)$$

(cf. § II*c*). When these requirements are met, it is possible to derive analytic expressions for the homogeneous solutions of equation (34). For $|\tau| \ll \min[\kappa^2/(\pi G\Sigma k_y), \kappa/ak_y]$, equation (34) with $\tilde{\psi}_{\rm ex} = 0$ reduces to equation (30) with $\tilde{\chi} = 0$. The exact solutions of the latter equation are $\tilde{\theta} = (\tau \mp 2i\Omega_p/\kappa) \exp(\pm i\kappa\tau/2A)$. For $|\tau| \gg 1$, the homogeneous WKB solutions of equations (34) are $\tilde{\theta} \approx \tau\xi^{-1/2} \exp(\pm i \int^{\tau} \xi(\delta)d\delta)$, where

$$\xi(\tau) = \frac{1}{2A}[\kappa^2 - 2\pi G\Sigma k_y|\tau| + (ak_y\tau)^2]^{1/2} . \quad (44)$$

These WKB solutions are valid if the minimum value of ξ is much larger than $[k_y a/A]^{1/2}$, or equivalently, if

$$b \equiv \frac{\kappa^2(Q^2-1)}{4k_y aAQ^2} \gg 1 \tag{45}$$

(cf. § III*e*). If inequalities (42), (43), and (45) are satisfied, the regions in which the above solutions are valid overlap, and the appropriately normalized global solutions are

$$\tilde{\theta}_\pm(\tau) = \frac{1}{\xi^{1/2}}\left(\tau \mp \frac{2i\Omega_p}{\kappa}\right)\exp\left[\pm i\int^\tau d\delta\,\xi(\delta)\right]. \tag{46}$$

c) *Steady Wave Trains*

To investigate the physical properties of the homogeneous solutions $\tilde{\theta}_\pm(\tau)$, we must evaluate

$$\theta_\pm(x) = k_y\int_{-\infty}^{\infty} d\tau\,\tilde{\theta}_\pm(\tau)\exp(-ik_y\tau x) \tag{47}$$

(cf. eq. [41]). Note that $\theta_+(x) = \theta_-{}^*(-x)$. The integral may be evaluated by the method of stationary phase. Since $\xi(\tau) > 0$, $\theta_\pm(x) = 0$ for $\pm x < 0$, and we have

$$\theta_\pm(x) = \frac{2A}{a}\left(\frac{2\pi}{k_y}\right)^{1/2}\underset{\text{all }k_x(x)}{\text{SUM}}\left\{\frac{k_x(x)}{|k_c - |k_x(x)||^{1/2}}\exp\left[i\int^x k_x(s)ds\right]\right\} \quad \text{for } \pm x > 0\,, \tag{48}$$

and

$$\theta_\pm(x) = 0 \quad \text{for } \pm x < 0\,.$$

The SUM is over all $k_x(x)$ for a given x where $k_x(x)$ is given by equation (13). Thus, the above solution is a sum of WKB density waves each of which satisfies the dispersion relation (eq. [13]) and the action conservation equation (17). Like the WKB density waves of § II, equation (48) is invalid near x_L and x_F although equations (46) and (47) are valid everywhere.

The novel feature of our solution is that it shows that all four wave types (short leading, long leading, long trailing, and short trailing) are part of a single wave train. The direction of propagation is specified by the sign of c_g, or equivalently, dk_x/dt (cf. eqs. [14] or [15]). In terms of $|x|$ the wave train travels the following route: $\infty \to x_F \to x_L \to x_F \to \infty$. Here sgn $x = \pm 1$ for $\theta_\pm(x)$. At the first reflection $|x| = x_F$, $k = k_c$ and the wave changes from short leading to long leading. The next reflection is at $|x| = x_L$, and the wave is transformed from long leading to long trailing. The final reflection is at $|x| = x_F$ with $k_x = -k_c$, and the wave changes from long trailing to short trailing. We know that each reflection is total because the action flux is conserved.

d) *Wave Packets*

We now turn to the inhomogeneous solutions of equation (34) and consider the excitation by the transient potential

$$\psi_{\text{ex}}(x, t) = \psi(\gamma)\exp[i\gamma x - \tfrac{1}{2}(t/T)^2]\,, \tag{49}$$

where $\kappa^{-1} \ll T \ll x_L/a$ and $\psi(\gamma)$ and γ are constants. The initial state ($t = -\infty$) is taken to be free of density waves so $\alpha = \beta = 0$ in equation (41). With the above expression for ψ_{ex}, two of the three integrations in equation (41) are routine and we find

$$\theta(x, t) = +\frac{ik_y{}^2\psi(\gamma)}{8A^2}\int_{-\gamma/k_y}^{\infty} d\tau[\tilde{\theta}_+(\tau)\tilde{\theta}_-(-\gamma/k_y) - \tilde{\theta}_-(\tau)\tilde{\theta}_+(-\gamma/k_y)]\exp\left[-ik_y\tau x - \frac{1}{2}\left(\frac{2At - \gamma/k_y - \tau}{2AT}\right)^2\right]. \tag{50}$$

The above integral is evaluated by the method of stationary phase and yields

$$\theta(x, t) = \pm\frac{i(2\pi)^{1/2}\psi(\gamma)}{4Aak_y}(\gamma \mp 2i\Omega_p k_y/\kappa)\underset{k_x(x)<\gamma}{\text{SUM}}\left\{\frac{k_x(x)}{|[|k_x(x)| - k_c]x(\gamma)|^{1/2}}\right\}$$
$$\times\exp\left\{i\left[\int_{\pm x(\gamma)}^{x} k_x(s)ds \pm \gamma x(\gamma) + \frac{\pi}{4}\,\text{sgn}\,c_g\right] - \frac{1}{2}\left[\frac{t}{T} - \frac{[\gamma - k_x(x)]}{2ATk_y}\right]^2\right\} \quad \text{for } \pm x > 0\,. \tag{51}$$

Here $x(\gamma) = x(-\gamma) > 0$ is defined such that $k_x[x(\gamma)] = \gamma$. Equation (51) describes two wave packets $\theta_1(x, t)$ and $\theta_2(x, t)$ which are confined to $x > 0$ and $x < 0$ and are related by $\theta_1(x, t) = \theta_2{}^*(-x, t)$.

The wave packets are localized near where

$$k_x(x) = \gamma - 2Ak_y t\,. \tag{52}$$

This relation implies that the wave packets are excited where $k_x(x) = \gamma$, and that they travel with the WKB group velocity (cf. eq. [14]). Comparison of equations (17) and (51) shows that the wave packets satisfy the WKB amplitude relation. The wave packets travel the same routes as the steady wave trains described in § III*c*.

e) *Bar Excitation*

The excitation of density waves by the potential of a bar has often been proposed as a generating mechanism for spiral structure in galaxies. Feldman and Lin (1973) and Lin and Lau (1975) demonstrated that the short trailing wave is excited near the corotation resonance by a bar potential. Their investigations are based on the application of the WKB method to tightly wound spirals in realistic galactic disks. Bar driving at corotation is proportional to $d(\Sigma\Omega/\kappa^2)/dr$. The analog of this gradient in our model sheet is $d(\Sigma\Omega_p/\kappa^2)/dr = 0$. Thus, it is not surprising that our calculations in § III*c* imply that in the tight winding limit a barlike potential does not excite any waves near corotation.

In our model the sole effect of a barlike potential is to excite the long trailing wave at the Lindblad resonances. To see this, note from equation (52) that a potential whose x dependence is of the form $\exp(i\gamma x)$ excites density waves at $\pm x(\gamma)$, where $k_x[\pm x(\gamma)] = \gamma$. The analog of a barlike potential in our geometry is $\psi_{\rm ex}(x, t) = \psi_{\rm ex}(x)$, where $\psi_{\rm ex}(x)$ is real and varies on a scale of order k_y^{-1}. Thus, the Fourier decomposition of $\psi_{\rm ex}(x)$ contains significant power for $|\gamma| \lesssim k_y$ and very little power for $|\gamma| = k_c \gg k_y$. For $|\gamma| \lesssim k_y$, $|x(\gamma)/x_L - 1| \lesssim k_y/k_c \ll 1$ (cf. eq. [13]). Thus the excitation occurs near the Lindblad resonances and the waves propagate away from these resonances as long trailing waves.

To determine the wave driving at the Lindblad resonances by a barlike potential

$$\psi_{\rm ex}(x) \equiv \int_{-\infty}^{\infty} d\gamma \psi(\gamma) \exp(i\gamma x) ,$$

we set $T = \infty$ and $x(\gamma) = x_L$ in equation (51) and then integrate the resulting expression for $\theta(x)$ over γ to obtain

$$\theta(x) = \pm \frac{(2\pi x_L)^{1/2}}{2\kappa a} \underset{k_x(x)<0}{\mathrm{SUM}} \left(\frac{k_x(x)}{|k_x(x) - k_c|^{1/2}}\right)\left(\frac{d\psi_{\rm ex}}{dx} \pm \frac{\Omega_p}{Ax_L}\psi_{\rm ex}\right)_{x=\pm x_L}$$
$$\times \exp\left\{i\left[\int_{\pm x_L}^{x} k_x(s)ds + \frac{\pi}{4}\,\mathrm{sgn}\, c_g\right]\right\} \quad \text{for } \pm x > 0 . \tag{53}$$

f) *Amplification at Corotation*

Mark (1974, 1976) has demonstrated that a density wave incident upon the forbidden zone surrounding corotation splits into a reflected and a transmitted wave, somewhat as hinted at already by Toomre's (1969) Figures 3 and 4. The action flux of the reflected wave exceeds that of the incident wave. This process does not involve a net transfer of action between the waves and the underlying disk, because a transmitted wave with the opposite sign of action density propagates away from the other side of the forbidden zone. Mark established his results for stellar disks, and Lin and Lau (1975) obtained similar results for gas disks. These results were derived by the application of WKB connection formulae to density waves across the forbidden zone. We confirm these results by application of the techniques developed in this paper.

The amplifier works well only in disks that hover on the brink of gravitational instability. Therefore, we lift the restriction on disk stability (cf. eq. [45]) and investigate the homogeneous solutions of equation (35a).

The coefficient of $\tilde{\psi}$ has a maximum at $\tau = 0$ and two minima at $\pm\tau_{\min}$ where

$$\tau_{\min} \approx \frac{\pi G\Sigma}{a^2 k_y} = \frac{k_c}{k_y} . \tag{54}$$

Note that the WKB waves have $|k_x| = k_c$ at $x = \pm x_F$. The above approximation for $\tau_{\min}$ is valid in the limit of tight winding. In the neighborhood of $\pm\tau_{\min}$, equation (35a) may be written as

$$\frac{d^2\tilde{\psi}}{d\eta^2} + \left(\frac{\eta^2}{4} + b\right)\tilde{\psi} = 0 , \tag{55}$$

where

$$\eta = \left(\frac{k_y a}{A}\right)^{1/2} (\tau \mp \tau_{\min}) \tag{56}$$

and

$$b = \frac{\kappa^2}{4Ak_y a}\frac{(Q^2 - 1)}{Q^2} . \tag{57}$$

Note: $b > 0$ for stable disks.

The solutions of equation (55) are parabolic cylinder functions (Abramowitz and Stegun 1964) as already noted by Mark (1976) in his analogous context. The complex solutions $E(-b, -\eta)$ and $E^*(-b, \eta)$ are of greatest interest to us. They satisfy the relation

$$(1 + e^{-2\pi b})^{1/2} E(-b, \eta) - e^{-\pi b} E^*(-b, \eta) = iE^*(-b, -\eta)\,. \tag{58}$$

The leading term in the asymptotic expansion of $E(-b, \eta)$ for $\eta \gg b$ is

$$E(-b, \eta) = \left(\frac{2}{\eta}\right)^{1/2} \exp\left[i\left(\frac{\eta^2}{4} + b \ln \eta + \frac{1}{2}\varphi_2 + \frac{\pi}{4}\right)\right], \tag{59}$$

where $\varphi_2 = \arg \Gamma(\frac{1}{2} - ib)$. The asymptotic expansion of $E(-b, \eta)$ for $\eta \ll -b$ follows from equations (58) and (59).

We can now calculate the amplification at corotation. Consider a simple example in which a steady long trailing wave train is launched at a Lindblad resonance $x = \pm x_L$. From § IIIc, we know that for this wave train $\tilde{\psi}(\tau) = 0$ for $\tau < 0$. For very stable disks ($b \gg 1$), $\tilde{\psi}(\tau) \propto \tilde{\psi}_\pm(\tau)$ for $\tau \geq 0$, where the upper and lower signs apply for the waves launched at $x = \pm x_L$. On the other hand, if $b \ll 1$, the WKB solutions are invalid in the interval $|\tau - \tau_{\min}| \lesssim (A/k_y a)^{1/2}$ and the solutions $\tilde{\psi}_\pm(\tau)$ for $\tau < \tau_{\min}$ go over to linear combinations of $\tilde{\psi}_+(\tau)$ and $\tilde{\psi}_-(\tau)$ for $\tau > \tau_{\min}$. The connection formulae are determined by equations (46), (58), and (59). From § IIIc we know that $\tau_{\min} \approx k_c/k_y$ corresponds to $k_x(x) = -k_c$ and $|x| = x_F$. Thus a long trailing wave incident upon the forbidden region splits into a reflected wave and a transmitted wave. The ratios of the magnitudes of the action fluxes of the reflected and transmitted waves to those of the incident wave are

$$\Gamma_r = (1 + e^{-2\pi b}) \tag{60}$$

and

$$\Gamma_f = e^{-2\pi b}\,. \tag{61}$$

The conservation of action is expressed by $\Gamma_r - \Gamma_f = 1$. Our Γ_r and Γ_f coefficients agree with those derived by Mark. Note that short leading waves incident upon corotation are amplified by the same factor as the long trailing waves. In the tight winding limit, the amplifier is efficient only if

$$Q - 1 \lesssim 2Ak_y a/\kappa^2 \ll 1\,. \tag{62}$$

The above discussion demonstrates the connection between the corotation amplifier analyzed by Mark (1974, 1976) and the swing amplifier described by Goldreich and Lynden-Bell and by Julian and Toomre (1966). In the tight winding limit, both amplifiers are weak; $\Gamma_r \leq 2$ and $\Gamma_f \leq 1$ for stable disks. The former is due to the near gravitational instability that occurs when $k(t)$ drifts past $k_{\min} \equiv \kappa/(Qa)$ and which is reflected by the presence of a soft spot in the effective spring rate near $k_{\min}$ (cf. eqs. [35]). By contrast, the really large swing amplifications found for relatively open spiral waves from equations (35) can occur only when $k(t)$ drifts past its minimum value k_y. They owe their existence to the term $+8\Omega_p A(1 + \tau^2)^{-1}$ which can make the effective spring rate negative near $\tau = 0$ even for $Q > 1$.

IV. DISCUSSION

We have presented analytic solutions which describe the excitation and evolution of tightly wound density waves. Our solutions are valid at all points including the corotation and Lindblad resonances. Furthermore, it is readily proved that all of the perturbation variables are nonsingular even at the resonance points.

Although our techniques are applied most easily to tightly wound waves, they are not restricted to this limiting case. With a very modest amount of numerical work, it would be possible to solve equations (34) and (41) without the tight winding approximation and thus describe density waves of arbitrary opening angle. This work would bridge the gap between the WKB results of Lin, Mark, Shu, and Toomre and the extensive numerical computations of unstable modes of more realistic model galaxies by Bardeen (1975).

Perhaps our most important result is the calculation of the driving of density waves by a barlike potential. We show that, for our model in the tight winding limit, driving of the long trailing waves occurs at the Lindblad resonances. As these waves propagate, they become more and more tightly wound in the trailing sense. Thus, a barlike potential excites trailing density waves in agreement with observations of spiral structure in galaxies.

We are indebted to Professor A. Toomre for encouraging us to carry out this investigation and for his wise counsel. This research was supported by NSF grants AST 76-24281 and AST 76-80801. S. Tremaine was supported by a Richard Chace Tolman Fellowship and a National Research Council of Canada Postdoctoral Fellowship.

REFERENCES

Abramowitz, M., and Stegun, I. A. 1964, *Handbook of Mathematical Functions* (Washington: Government Printing Office).

Bardeen, J. M. 1975, in *Proc. IAU Symposium No. 69, Dynamics of Stellar Systems*, ed. A. Hayli (Dordrecht: Reidel), p. 297.

Feldman, S. I., and Lin, C. C. 1973, *Studies Appl. Math.*, **52**, 1.
Goldreich, P., and Lynden-Bell, D. 1965, *M.N.R.A.S.*, **130**, 125.
Julian, W. H., and Toomre, A. 1966, *Ap. J.*, **146**, 810.
Kalnajs, A. J. 1965, Ph.D. thesis, Harvard University.
Lin, C. C., and Lau, Y. Y. 1975, *SIAM J. Appl. Math.*, **29**, 352.
Lin, C. C., and Shu, F. H. 1966, *Proc. Nat. Acad. Sci. USA*, **55**, 229.
Lin, C. C., and Shu, F. H. 1968, in *Astrophysics and General Relativity*, Vol. **2**, ed. M. Chretien (New York: Gordon & Breach), p. 235.
Mark, J.W-K. 1974, *Proc. IAU Symposium No. 58, Formation and Dynamics of Galaxies*, ed. J. R. Shakeshaft (Dordrecht: Reidel), p. 417.
———. 1976, *Ap. J.*, **205**, 363.
Shu, F. H. 1970, *Ap. J.*, **160**, 99.
Toomre, A. 1964, *Ap. J.*, **139**, 1217.
———. 1969, *Ap. J.*, **158**, 899.

PETER GOLDREICH and SCOTT TREMAINE: Division of Geological and Planetary Sciences, California Institute of Technology, Pasadena, CA 91125