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ON MAPPING THE MAGNETIC FIELD DIRECTION IN MOLECULAR CLOUDS BY POLARIZATION MEASUREMENTS

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ABSTRACT

We predict that interstellar radio-frequency lines possess a few percent linear polarization, provided that (1) the radiative transition rate is at least comparable to the collision rate, (2) the optical depth is moderate and anisotropic, and (3) the number of extrema of the velocity component along the line of sight through the source is small. If the Zeeman splitting exceeds both the collisional frequency and the radiative transition rate, then the polarization is aligned either perpendicular to or parallel to the projection of the magnetic field on the plane of the sky.
Subject headings: interstellar: magnetic fields — polarization — radiative transfer

I. INTRODUCTION

It is not generally appreciated that many interstellar radiofrequency lines may possess measurable amounts of linear polarization. Our aim is to describe the circumstances under which linear polarization will arise. Analysis of a simple model problem suffices to illustrate all of the important physics.

We consider a hypothetical molecule which possesses two levels between which radiative and collisional transitions of frequency ν_r can occur. The upper and lower levels have total angular momenta $F_a = 1$ and $F_b = 0$, respectively. We investigate line formation in a medium in which the macroscopic velocity differences are much greater than the thermal velocities of the molecules, a condition generally met in molecular clouds.

We assume that the Zeeman splitting $\mu\mathfrak{B}/h$ is smaller than the Doppler width $\Delta\nu$, so that circular polarization does not arise. In this *Letter* we make the additional assumption that the Zeeman splitting is much larger than the collision rate C and the spontaneous and stimulated radiative transition rates $A_{a,b}$ and $\mathfrak{R}$. We reserve discussion of the more difficult case, in which this condition is not satisfied, to a later and more detailed paper. The known densities and expected magnetic field strengths in interstellar clouds are such that $\mu\mathfrak{B}/h$ is much greater than C, $A_{a,b}$, and $\mathfrak{R}$ for most centimeter and millimeter wavelength lines.

Rotation endows molecules with minimum magnetic moments comparable to the nuclear magneton (Townes and Schawlow 1955). The Zeeman splitting of a level with magnetic moment equal to one nuclear magneton in a magnetic field of 1 microgauss is $\sim 10^{-3}$ Hz. For comparison, the collision rate, where $n_{H_2} \approx 10^4$ cm^{-3}, is $\sim 10^{-6}$ s^{-1}, and the spontaneous emission rate of a $\lambda = 1$ mm transition of a molecule with a dipole moment of 1 debye is $\sim 3 \times 10^{-4}$ s^{-1}.

The plan of this *Letter* is as follows. In § II we set up the basic equations of radiative transfer and statistical equilibrium. The radiative transfer is solved with the aid of the Sobolev approximation in § III. Asymptotic solutions for the magnitude of the polarization in the limits of high and low optical depth are given and compared with the results of numerical computations in § IV. Section V is devoted to a discussion of the main results and their potential for application.

II. BASIC EQUATIONS

In the limit $\mu\mathfrak{B}/h \gg C$, $A_{a,b}$, and $\mathfrak{R}$, the magnetic field direction is the natural choice for the quantization axis. We denote by $n_\pm$, n_0, and n_b the number densities of molecules in the $m_a = \pm 1$, $m_a = 0$, and $m_b = 0$ sublevels. In the absence of circular polarization, $n_+ = n_- \equiv n_\pm$. The emission and absorption coefficients are completely determined by the sublevel number densities, the reduced dipole matrix element of the transition d, and the angle γ between the propagation direction $\hat{n}$ and magnetic field direction $\hat{b}$.

The equation of radiative transfer in component form is

$$\frac{dI_q}{ds} = -k_q\phi_\nu[I_q - S_q]\,, \quad q = \perp, \parallel\,, \tag{1}$$

where $I_\perp$ and $I_\parallel$ are the specific intensities of radiation polarized along $\hat{e}_\perp = (\hat{b} \times \hat{n})/\sin\gamma$ and $\hat{e}_\parallel = (\hat{b} - \hat{n}\cos\gamma)/\sin\gamma$, s is the path length along $\hat{n}$, and ϕ_ν is the normalized profile function. The absorption coefficients and source functions are

$$k_\perp = (2\pi)^3 \frac{d^2\nu_r}{hc}(n_b - n_\pm)\,,$$

$$k_\parallel = (2\pi)^3 \frac{d^2\nu_r}{hc}(n_b - n_\pm\cos^2\gamma - n_0\sin^2\gamma)\,, \tag{2}$$

$$S_\perp = \frac{h\nu_r^3}{c^2}\frac{n_\pm}{(n_b - n_\pm)}\,,$$

$$S_\parallel = \frac{h\nu_r^3}{c^2}\frac{(n_\pm\cos^2\gamma + n_0\sin^2\gamma)}{(n_b - n_\pm\cos^2\gamma - n_0\sin^2\gamma)}\,. \tag{3}$$

The Einstein coefficients are related to d by

$$A_{a,b} = \frac{64\pi^4 d^2 \nu_r^3}{3hc^3}, \qquad B_{a,b} = \frac{32\pi^4 d^2}{3h^2 c}. \tag{4}$$

The rate equations which govern the sublevel populations are written as

$$\frac{dn_\pm}{dt} = -A_{a,b}n_\pm + \Re_\pm(n_b - n_\pm) - C'(n_\pm - n_0) + C[n_b \exp(-h\nu_r/kT) - n_\pm],$$

$$\frac{dn_0}{dt} = -A_{a,b}n_0 + \Re_0(n_b - n_0) - 2C'(n_0 - n_\pm) + C[n_b \exp(-h\nu_r/kT) - n_0],$$

$$2n_\pm + n_0 + n_b = n = \text{constant}. \tag{5}$$

Here C and C' are collisional transition rates, and T is the kinetic temperature. The net rates of stimulated absorption and emission are

$$\Re_\pm = \frac{B_{a,b}}{6}\int_0^\infty d\nu\phi_\nu \int \frac{d\Omega}{4\pi}(I_\perp + I_\parallel \cos^2\gamma,)$$

$$\Re_0 = \frac{B_{a,b}}{3}\int_0^\infty d\nu\phi_\nu \int \frac{d\Omega}{4\pi} I_\parallel \sin^2\gamma. \tag{6}$$

III. SOBOLEV APPROXIMATION

It is extremely difficult to solve the coupled equations of radiative transfer and statistical equilibrium in a non-stationary medium. Fortunately, in the limit that the thermal velocities are much smaller than the macroscopic velocity differences, a simple approximate method for solving the equations of radiative transfer has been developed by Sobolev (1960) and extended by Castor (1970) and Lucy (1971). In the strictest sense, this approximation is valid only if the radial velocity is monotonic along each line of sight through the medium. For the moment, we assume that this requirement is satisfied. If it is violated, the linear polarization is generally decreased.

The beauty of the Sobolev approximation is that it reduces the profile-averaged specific intensity $\int I_q(\nu, r, \hat{n})\phi_\nu d\nu$ at each point r to a function of the external continuum radiation field incident along $\hat{n}$ and two locally determined quantities, namely, the source function at r and the probability that a photon emitted at r in direction $\hat{n}$ will escape without further interaction.

The escape probability in direction $\hat{n}$ at r is given by

$$\beta_q(r, \hat{n}) = [1 - \exp(-\tau_q)]/\tau_q. \tag{7}$$

Here $\tau_q(\nu, r, \hat{n})$ is the total optical depth at frequency ν along a ray which passes through r. The frequency ν is related to the velocity $v(r)$ by $\nu = \nu_r(1 - \hat{n}\cdot v/c)$. The optical depth varies inversely with $|\hat{n}\cdot\nabla v(r)|$ and may be written as

$$\tau_q(\nu, r, \hat{n}) = \frac{ck_q(r, \hat{n})}{\nu_r|\Lambda_i(r)n_i^2|}, \tag{8}$$

where the Λ_i are the eigenvalues of the symmetric part of the strain rate tensor, and the n_i are the components of $\hat{n}$ along the principal axes of this tensor.

The profile-averaged specific intensity inside the cloud is

$$\int_0^\infty d\nu\phi(\nu)I_q(\nu, \hat{n}, r) = S_q(r, \hat{n}) - [S_q(r, \hat{n}) - \tfrac{1}{2}B]\beta_q(r, \hat{n}), \tag{9}$$

where the external radiation field is assumed to be entirely due to the cosmic blackbody radiation B. The net rates of stimulated absorption and emission are obtained from equations (6)–(9).

Radio astronomers measure the difference between the true specific intensity and the cosmic blackbody radiation. Thus the appropriate definition of polarization is

$$P = \frac{I_\perp - I_\parallel}{I_\perp + I_\parallel - B}, \tag{10}$$

where the excess emergent specific intensity is given by the standard relation

$$I_q - \tfrac{1}{2}B = (S_q - \tfrac{1}{2}B)[1 - \exp(-\tau_q)]. \tag{11}$$

IV. ASYMPTOTIC EXPRESSIONS FOR THE POLARIZATION

The radiative rates $\Re_\pm$ and $\Re_0$ may be evaluated analytically in the limits of low and high optical depth under conditions in which all of the Λ_i have the same sign. These conditions correspond to pure expansion ($\Lambda_i > 0$) or pure contraction ($\Lambda_i < 0$) together with an arbitrary rotation described by the antisymmetric part of the strain rate tensor. Explicit expressions for P are obtained by inserting the analytic formulae for $\Re_\pm$ and $\Re_0$ into the steady state rate equations. This is a somewhat arduous task. Only the results are quoted below.

The angles defining $\hat{n}$ and $\hat{b}$,

$$\hat{n} = (\sin\theta\sin\lambda, \sin\theta\cos\lambda, \cos\theta),$$

$$\hat{b} = (\sin\alpha\sin\beta, \sin\alpha\cos\beta, \cos\alpha), \tag{12}$$

are referred to the principal axes of the symmetric part of the strain rate tensor. The polar axis is chosen to correspond to the principal axis of the eigenvalue Λ_3:

$$\cos\gamma \equiv \hat{n}\cdot\hat{b} = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\lambda - \beta). \tag{13}$$

We define the function $f(\alpha, \beta)$, which appears in the formulae for P, by

$$f(\alpha, \beta) = \frac{3[\sin^2\alpha(\sin^2\beta\Lambda_1 + \cos^2\beta\Lambda_2) + \cos^2\alpha\Lambda_3]}{\Lambda_1 + \Lambda_2 + \Lambda_3}, \tag{14}$$

and the "mean" optical depth by

$$\mathrm{TAU} = \frac{24\pi^3 d^2 (n_b - n_a)}{h|\Lambda_1 + \Lambda_2 + \Lambda_3|}, \qquad (15)$$

where $n_a \equiv (2n_\pm + n_0)/3$.

The asymptotic expressions for P read,

$$P = \frac{3}{40} \sin^2 \gamma \, \mathrm{TAU} \, [1 - f(\alpha, \beta)] \Big/ \left\{1 + \frac{(C + 3C')}{A_{a,b}} [1 - \exp(-h\nu_r/kT_{\mathrm{BB}})]\right\}, \qquad (16)$$

for TAU $\ll 1$, and

$$P = \frac{\sin^2 \gamma}{2\,\mathrm{TAU}} [1 - f(\alpha, \beta)] \Big/ \left[1 + \frac{10}{3} \frac{(C + 3C')}{A_{a,b}} \frac{(n_b - n_a)}{n_b}\right], \qquad (17)$$

for TAU $\gg 1$. There is an extra requirement for the validity of equation (16), namely $3|\Lambda_i - \Lambda_j|/|\Lambda_1 + \Lambda_2 + \Lambda_3| \ll 1$ must be satisfied.

Several general features of the expressions for P are worth noting. The maximum value of P occurs at $\gamma = \pi/2$. Anisotropic τ is a requirement for $P \neq 0$; $f(\alpha, \beta) = 1$ for $\Lambda_1 = \Lambda_2 = \Lambda_3$. Collisional excitation is isotropic, so P is small for large values of $(C + 3C')/A_{a,b}$. In addition, $P \propto$ TAU for TAU $\ll 1$, and $P \propto \mathrm{TAU}^{-1}$ for TAU $\gg 1$. The largest values of P are attained for TAU ≈ 1.

To determine P for TAU near unity, the rate equations (5) are integrated numerically to find the steady state sublevel number densities. The values of $\mathfrak{R}_\pm$ and $\mathfrak{R}_0$ are obtained from the Sobolev approximation as indicated in § III. The specific example analyzed is a spherically symmetric expansion with constant radial velocity, which might represent the outer regions of an expanding circumstellar envelope. For simplicity, the magnetic field is taken to be radial. The results are displayed in Figure 1. The agreement with the asymptotic formula for TAU $\gg 1$ is seen to be good. The maximum value of $P \approx 0.07$.

V. DISCUSSION

To our knowledge, there are no reports of linearly polarized radio-frequency lines other than those associated with maser emission from OH, H_2O, and SiO. Goldreich, Keeley, and Kwan (1973*a*, *b*) attributed the linear polarization in the maser lines to the same basic mechanism described in this *Letter*. A special feature of maser line formation is that, since the optical depth is negative, the radiative transfer may be anisotropic even for $|\tau| \gg 1$. The 21 cm line presumably has $P \ll 1$ because $C/A_{a,b} \gg 1$ in all clouds for which TAU $= O(1)$. It is not clear what magnitude of polarization to expect for lines formed in molecular clouds. Certainly $C/A_{a,b} \lesssim 1$ in the source regions of many millimeter wavelength lines. The major uncertainty is the scale of the spatial variations of the velocity. If the radial velocity is monotonic, or nearly so, along the line of sight through the clouds, then

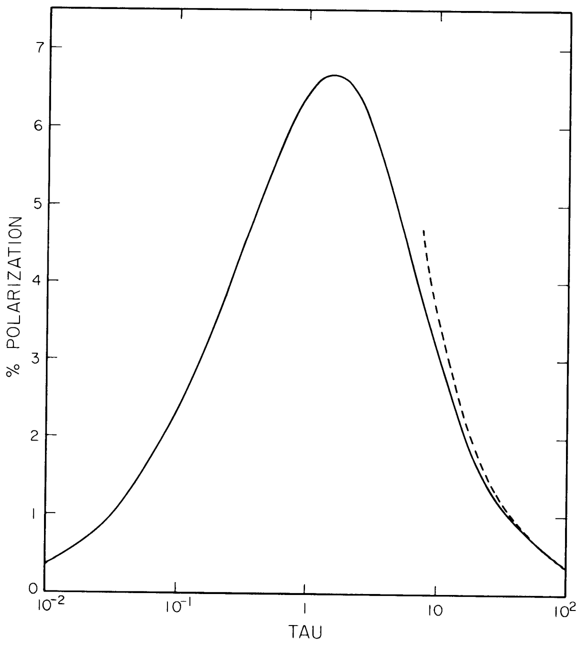

FIG. 1.—Polarization as a function of optical depth TAU. The solid line gives our numerical results, while the dashed one corresponds to the asymptotic expression (17). The values of the parameters used are: $\Lambda_1 = \Lambda_2 = 1 \times 10^{-9}\,\mathrm{s}^{-1}$, $\Lambda_3 = 0$, $\lambda_r = c/\nu_r = 0.3$ cm, $C/A_{a,b} = 0.212$, $C'/A_{a,b} = 0$, $T = 10$ K, $T_{\mathrm{BB}} = 2.7$ K. The angles are: $\theta = \pi/2$, and $\lambda = \alpha = \beta = 0$, which imply $\gamma = \pi/2$. $P \propto \sin^2 \gamma$ is an excellent approximation for all TAU.

lines having $P \geq 0.01$ should be common. Perhaps the best sources to observe for linearly polarized lines are expanding circumstellar envelopes, since their kinematics is favorable. However, interferometers are needed to resolve these objects.

Observations of linearly polarized lines would yield information on the magnetic field direction and optical depth in the source. It is to be expected that the magnitude of the polarization, and perhaps its direction as well, may vary across the line. Such behavior would yield information on the mechanism of line formation.

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